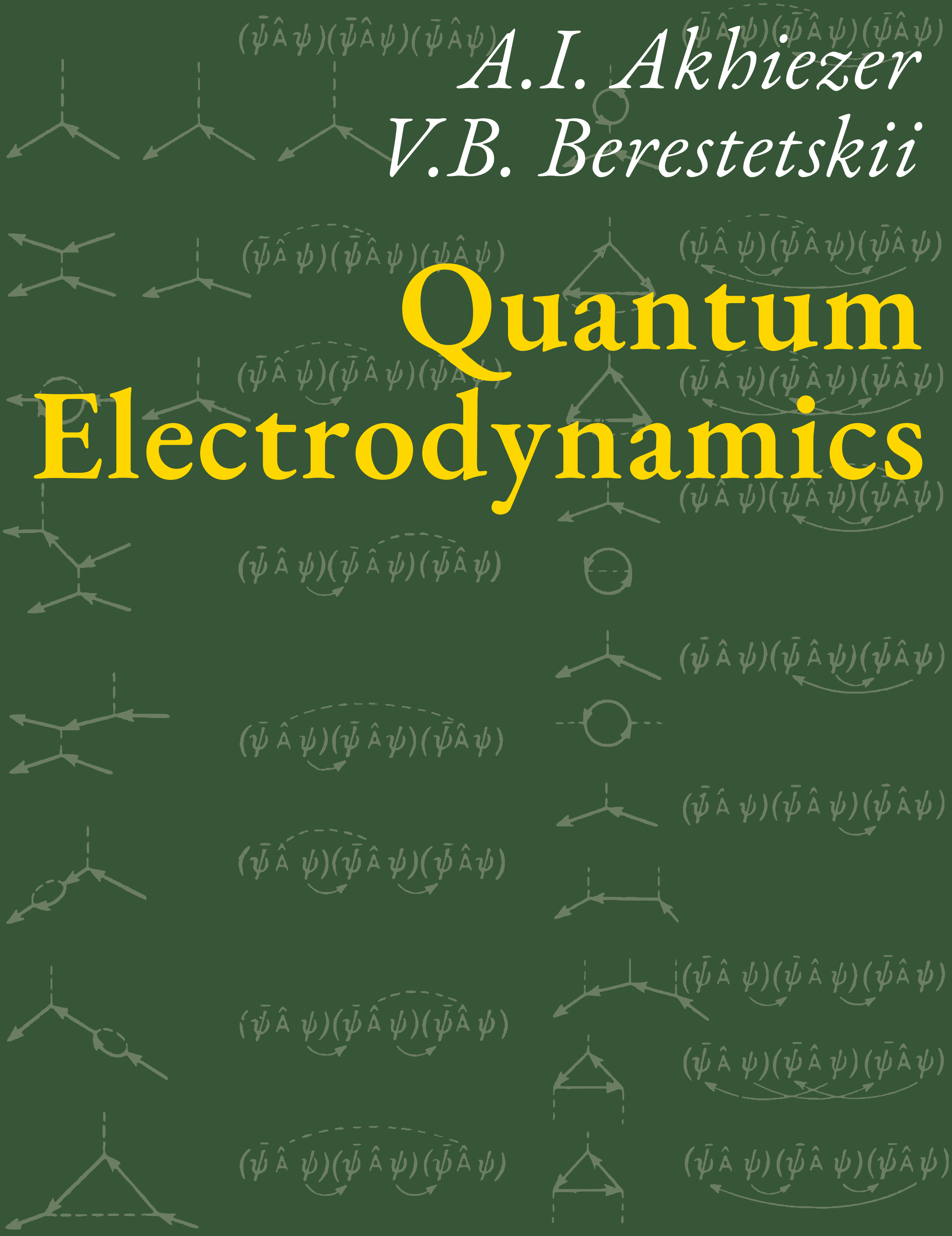

A.I. Akhiezer
V.B. Berestetskii

# Quantum Electrodynamics

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# QUANTUM ELECTRODYNAMICS

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Authorized English Edition Revised and Enlarged by the Authors

*Translated from the Second Russian Edition by*

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*Vancouver, B.C., Canada*

**INTERSCIENCE PUBLISHERS 1965**
a division of John Wiley & Sons, New York · London · Sydney

English Translation and All Additions to the Russian Edition

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Library of Congress Catalog Card Number 63-17766

Printed in the United States of America

# Translator's Note

This translation follows in the main the second Russian edition of 1959, but in addition incorporates several new sections (in particular §§ 9.4, 26.2, 26.6, 48.4, 48.5, 51.5, 52.5) provided by the authors specially for this edition, as well as a number of minor revisions, and corrections of numerous typographical errors in the formulas. The translator is grateful to the authors for their cooperation.

G. VOLKOFF

*Vancouver, Canada*

# Preface to the Second Edition

In preparing the second edition we have subjected the book to extensive revision. The principal aim and the contents of the book have not been altered; the book is devoted to the systematic presentation of electromagnetic processes only. Only some general theorems and methods go beyond the framework of electrodynamics proper. In the second edition the space devoted to these has been increased (reflection properties, Green's functions, functional methods, etc.).

In the presentation of the principles of quantum electrodynamics the theory of renormalizations has been subjected to the most extensive revision. Without claiming complete mathematical rigor we have attempted to present the concept of renormalization from a single simple physical point of view, avoiding purely prescription-like methods of eliminating divergences, and making maximum use of the general properties of quantum mechanical systems. In connection with this the structure of the book has been somewhat altered. The investigation of the scattering matrix together with the theory of radiation corrections has been segregated into a separate chapter (Chapter VII). The investigation of electrodynamic processes in the first nonvanishing approximation, which involves neither the removal of divergences nor renormalizations, is carried out in Chapters V and VI, while higher order approximations are considered in Chapter VIII.

In presenting specific effects we have aimed, within reasonable limits, at the greatest possible degree of completeness of results and of detail of calculations. The number of different electrodynamic phenomena described has been increased. In particular, the theory of processes involving polarized particles, the method of impact parameters, etc., have been included.

We wish to express our sincere gratitude to V. Aleksin, V. Bar'iakhtar, V. Boldyshev, D. Volkov, S. Peletminskii, R. Polovin, and P. Fomin, who have aided us significantly in the preparation of the second edition of this book.

A. I. Akhiezer, V. B. Berestetskii

# From the Preface to the First Edition

At the present time a number of particles is known which correspond to various quantum fields interacting with each other. However, of the many types of physical interactions existing in nature the only one, apart from gravitation, that has been studied in sufficient detail is the electromagnetic interaction. The theory of the latter interaction is the subject of quantum electrodynamics, to which the systematic exposition of this book is devoted.

Since the electromagnetic interaction is the fundamental one in the case of electrons and photons, quantum electrodynamics enables us to explain and to predict a wide range of phenomena related to the behavior of these particles. As for the application of quantum electrodynamics to other particles (nucleons and mesons), it is considerably restricted because of the essential role played by other types of interactions (nuclear or meson interactions) in the case of those particles. Therefore, problems relating to mesons are not treated in this book, and the interaction of nucleons with the electromagnetic field is treated only in the limit of low velocities.

The formulation of the fundamental equations of quantum electrodynamics, and even the very possibility of separating the interacting fields into the electromagnetic and electron-positron fields, is based on the fact that the interaction between these fields is a weak one. This circumstance finds its expression in the smallness of the constant $\alpha = e^2/\hbar c$ which characterizes the interaction. Therefore the interaction between the fields is treated in quantum electrodynamics as a small perturbation, and the mathematical method employed in quantum electrodynamics is perturbation theory in which all quantitative results are expressed in terms of power series in $\alpha$.

Since both the electromagnetic and the electron-positron fields are systems with an unlimited number of degrees of freedom, the application of perturbation theory gives rise to divergent expressions characteristic of the present theory, which are absent only in the first nonvanishing approximation of perturbation theory. The development of quantum electrodynamics in recent years has permitted the establishment of principles for regularizing divergent expressions, so that it has then become possible to calculate higher order approximations (the so-called radiation corrections).

This progress is to a great extent associated with the new invariant formulation of perturbation theory. Invariant perturbation theory has made it possible to present the results in a compact and relativistically invariant form, and this has allowed the rules for regularization to be formulated. On the other hand, the use of invariant perturbation theory has significant practical advantages over the earlier methods even in the case of first-order calculations. Therefore the whole presentation in this book is based on invariant perturbation theory.

Although it is a completely satisfactory theory within a definite field of physical phenomena, modern quantum electrodynamics has the important drawback that in order to remove the divergences which arise in the theory additional concepts must be introduced which are neither contained in the fundamental formulation of the theory, nor reflected in its basic equations. This state of affairs is apparently due to profound causes. They are implicit in the fact that it is frequently impossible to construct a closed theory of a limited set of phenomena (in the present case, the purely electromagnetic ones) without taking into account a wider class of interactions existing in nature.

We wish to express our gratitude to Academician L. D. Landau and Professor I. Ia. Pomeranchuk and to the participants in the seminars directed by them for discussions of a number of problems presented in this book.

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CHAPTER I

# Quantum Mechanics of the Photon

## § 1. The Photon Wave Function

### 1.1. *Introduction*

The corpuscular properties of light were historically the first fundamental fact which served as a basis for the development of quantum theory. The Planck-Einstein relation between the energy $w$ of a quantum of light, the photon, and the frequency $\omega$ of the electromagnetic field corresponding to it is expressed as $w = \hbar\omega$ and is historically the first relation containing the quantum constant $\hbar$.

The systematic quantum mechanics of the atom was developed before that of the photon and this situation has a profound physical reason. Atomic particles, the electrons and the nuclei, have rest masses different from zero. For them there exists a range of energies, small compared to their rest energy, in which relativistic effects may be neglected. Since the rest mass of the photon is zero, there exists for it no nonrelativistic energy region, and the quantum mechanics of the photon must necessarily be relativistic from the outset.

As the counterpart of a particle, quantum mechanics introduces a field of one or more wave functions which determine the probability distribution and the expectation values of the various physical quantities related to the partticle. The wave functions satisfy a certain system of differential equations which determine the nature of the motion of the particle.

In order to transpose this description into the relativistic domain we must, first of all, take into account the requirements of the principle of relativity. They amount to the requirement that the field equations must be invariant under Lorentz transformations. However, this by itself is not sufficient for a unique determination of the form of the field equations which would give expression to the individual properties of a given particle. In the case of photons the setting up of the equations is facilitated by the existence of a classical analog, viz., of the equations of the classical electromagnetic field. It is natural to use Maxwell's equations as the quantum mechanical equations of motion for the photon and it follows that the wave properties of the photon will coincide with the properties of the electromagnetic field. As will be shown later, this, together with the relation $w = \hbar\omega$, suffices for the formulation of a theory of photons and of their interaction with charged particles.

Our first problem is the study of photons in the absence of electric charges. Although properties of particles become apparent only as a result of their interaction with other particles, such an investigation is necessary as a preparatory stage for the study of interactions.

### 1.2. *The Photon Wave Function in k Space*

The electromagnetic field is described by the electric field vector $\boldsymbol{E}$ and the magnetic field vector $\boldsymbol{H}$, which in the absence of charges satisfy Maxwell's equations for a vacuum:

$$\begin{aligned} \operatorname{curl} \boldsymbol{E} &= -\frac{\partial \boldsymbol{H}}{\partial t}, \\ \operatorname{div} \boldsymbol{H} &= 0, \\ \operatorname{curl} \boldsymbol{H} &= \frac{\partial \boldsymbol{E}}{\partial t}, \\ \operatorname{div} \boldsymbol{E} &= 0. \end{aligned} \tag{1.1}$$

(Here and subsequently we make use of the system of units in which the velocity of light is equal to unity, $c = 1$.)

In accordance with the foregoing we shall interpret the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ as quantities describing the quantum mechanical photon state. In order to give such a "corpuscular" interpretation to the system of equations (1.1) we compare this system with the Schrödinger equation in ordinary quantum mechanics. This may be done conveniently if we first subject equations (1.1) to a Fourier transformation with respect to the space coordinates $\boldsymbol{r}$, i.e., if we go over into $\boldsymbol{k}$ space. By writing $\boldsymbol{E}$ and $\boldsymbol{H}$ in the form[1]

$$\begin{aligned} \boldsymbol{E} &= \int \boldsymbol{E}_k e^{i\boldsymbol{kr}} d\boldsymbol{k}, \\ \boldsymbol{H} &= \int \boldsymbol{H}_k e^{i\boldsymbol{kr}} d\boldsymbol{k}, \end{aligned} \tag{1.2}$$

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[1] Following the authors' usage, the scalar product of two three-dimensional vectors will be denoted by boldface letters standing next to each other *without* an

we can easily see that in virtue of (1.1) the Fourier components $\boldsymbol{E}_k$ and $\boldsymbol{H}_k$ satisfy the following system of equations:

$$\begin{aligned} \dot{\boldsymbol{H}}_k &= -i[\boldsymbol{k}, \boldsymbol{E}_k], \\ \boldsymbol{k}\boldsymbol{H}_k &= 0, \\ \dot{\boldsymbol{E}}_k &= i[\boldsymbol{k}, \boldsymbol{H}_k], \\ \boldsymbol{k}\boldsymbol{E}_k &= 0, \end{aligned} \tag{1.3}$$

where dots denote differentation with respect to time. (For the sake of brevity we shall omit here and subsequently the argument $t$ in the functions $\boldsymbol{E}_k \equiv \boldsymbol{E}(\boldsymbol{k}, t)$ and $\boldsymbol{H}_k \equiv \boldsymbol{H}(\boldsymbol{k}, t)$.) To this system we must also add the condition that the fields be real:

$$\begin{aligned} \boldsymbol{E}_{-k} &= \boldsymbol{E}_k^*, \\ \boldsymbol{H}_{-k} &= \boldsymbol{H}_k^*. \end{aligned} \tag{1.4}$$

Instead of the two vectors $\boldsymbol{E}_k$ and $\boldsymbol{H}_k$ we can use the vectors $\boldsymbol{E}_k$ and $\dot{\boldsymbol{E}}_k$ by eliminating $\boldsymbol{H}_k$ with the aid of (1.3):

$$\boldsymbol{H}_k = i\left[\frac{\boldsymbol{k}}{k^2}, \dot{\boldsymbol{E}}_k\right]: \tag{1.5}$$

It is convenient to eliminate the necessity of taking into aecount the requirement of reality. In order to do this we introduce a transformation which automatically guarantees that the relations (1.4) will be satisfied:

$$\begin{aligned} \boldsymbol{E}_k &= N(k)(\boldsymbol{f}_k + \boldsymbol{f}_{-k}^*), \\ \dot{\boldsymbol{E}}_k &= -ikN(k)(\boldsymbol{f}_k - \boldsymbol{f}_{-k}^*), \end{aligned} \tag{1.6}$$

where $N(k)$ is a certain normalizing factor, which, as we shall see later, it is convenient to choose equal to

$$N(k) = \sqrt{\frac{k}{2(2\pi)^3}}\,. \tag{1.7}$$

The substitution (1.6) means that instead of the two functions $\boldsymbol{E}_k$ and $\dot{\boldsymbol{E}}_k$, which in virtue of (1.4) are actually defined in a half-space, we introduce the single function $\boldsymbol{f}_k$, which is independently defined over the whole $k$-space.

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intervening dot, and either with or without parentheses: thus $\boldsymbol{kr}$ or $(\boldsymbol{kr})$, but not $\boldsymbol{k}\cdot\boldsymbol{r}$. Similarly, the vector product will be denoted by the square brackets $[\boldsymbol{A}, \boldsymbol{B}]$ with or without a comma, but not by $\boldsymbol{A}\times\boldsymbol{B}$.

If we make use of (1.6), the expansion (1.2) will assume the following form:

$$\begin{aligned} &\boldsymbol{E} = \mathfrak{E}+\mathfrak{E}^*, \qquad \boldsymbol{H} = \mathfrak{H}+\mathfrak{H}^*, \\ &\mathfrak{E} = \int N(k)\boldsymbol{f}_k e^{i\boldsymbol{kr}}\, d\boldsymbol{k}, \\ &\mathfrak{H} = \int N(k)\left[\frac{\boldsymbol{k}}{k}, \boldsymbol{f}_k\right] e^{i\boldsymbol{kr}}\, d\boldsymbol{k}. \end{aligned} \tag{1.8}$$

We can easily obtain the equation satisfied by $\boldsymbol{f}_k$. By eliminating $\boldsymbol{H}_k$ from (1.3) we obtain

$$\left(\frac{\partial^2}{\partial t^2}+k^2\right)\boldsymbol{E}_k = 0,$$

which may be rewritten in the following form:

$$\left(\frac{\partial}{\partial t}+ik\right)\left(\frac{\partial}{\partial t}-ik\right)\boldsymbol{E}_k = 0.$$

This second-order equation may be transformed into a first-order equation for $\boldsymbol{f}_k$. Indeed, by utilizing the relation

$$\left(\frac{\partial}{\partial t}-ik\right)\boldsymbol{E}_k = -2iN(k)k\boldsymbol{f}_k,$$

which follows from (1.6), we obtain

$$i\frac{\partial \boldsymbol{f}_k}{\partial t} = k\boldsymbol{f}_k. \tag{1.9}$$

The substitution of expressions (1.5) and (1.6) into (1.3) yields

$$\boldsymbol{k}\boldsymbol{f}_k = 0. \tag{1.10}$$

Equations (1.9) and (1.10) may be combined into one equation. On multiplying (1.10) by $\boldsymbol{k}/k$ and on adding it to (1.9), we obtain

$$i\frac{\partial \boldsymbol{f}_k}{\partial t} = \mathrm{w}\boldsymbol{f}_k, \tag{1.11}$$

where

$$\begin{aligned} &(\mathrm{w}\boldsymbol{f}_k)_\alpha = w_{\alpha\beta} f_{\boldsymbol{k}\beta}, \\ &w_{\alpha\beta} = k\left(\delta_{d\beta}-\frac{k_\alpha k_\beta}{k^2}\right). \end{aligned} \tag{1.11'}$$

It follows from (1.11) that

$$\frac{\partial}{\partial t}(kf_k) = 0.$$

Therefore (1.10) will always hold if it holds at some initial time.

Equation (1.11) together with the initial condition (1.10) are equivalent to the system of Maxwell's equations.

Equation (1.11) has the form of a Schrödinger equation in which w is the Hamiltonian operator. (Here and subsequently we shall make use of the system of units in which $\hbar = 1$.) The eigenvalues of this operator are equal to $k$. This would amount to nothing other than the quantum relation between the energy and the frequency of the photon, if we could identify the Hamiltonian operator formally introduced in the foregoing with the physical operator for the photon energy. We shall show later that such an identification can be justified, and that the function $f_k$ can be interpreted as the photon wave function in $k$-space in the usual quantum mechanical sense of this word. We shall also be able to define in $k$-space operators for other physical quantities referring to the photon, for example, the operators for the momentum, the angular momentum, etc.

## 1.3. *Energy*

We shall show that the operator w which we have just introduced can indeed be interpreted as the operator for the photon energy. In order to do this we introduce the expression for the energy $\bar{w}$ of the electromagnetic field corresponding to a given photon as

$$\bar{w} = \tfrac{1}{2}\int (\boldsymbol{E}^2+\boldsymbol{H}^2)\,d\boldsymbol{r}. \tag{1.12}$$

(We make use here of Heaviside units for $\boldsymbol{E}$ and $\boldsymbol{H}$.) This expression represents a space integral of quantities quadratic in the field vectors. On the other hand, space integrals of expressions quadratic in the wave function are interpreted in quantum mechanics as expectation values of the corresponding physical quantities. Since we are considering the field corresponding to a single photon, it would appear that the natural generalization is the interpretation of expression (1.12) as the expectation value of the photon energy.

We show that $\overline{w}$ may be represented in the form

$$\overline{w}=\int f_k^* \mathrm{w} f_k dk, \tag{1.13}$$

where w is determined by formula (1.11).

In order to do this we substitute expansion (1.2) into (1.12):

$$\overline{w}=\tfrac{1}{2}\int\{E_k E_{k'}+H_k H_{k'}\}\, e^{i(k+k')r} dk\, dk' dr.$$

After carrying out the integration over $r$ with the aid of the relation

$$\int e^{i(k+k')r} dr=(2\pi)^3\delta(k+k'),$$

where $\delta\,(k+k')$ is a three-dimensional $\delta$-function, and on utilizing (1.5), we obtain

$$\overline{w}=4\pi^3\int\left(E_k E_{-k}+\frac{1}{k^2}\dot{E}_k\dot{E}_{-k}\right)dk.$$

We express $E_k$ and $\dot{E}_k$ in terms of $f_k$ in accordance with (1.6). This yields

$$\overline{w}=16\pi^3\int N^2 f_k^* f_k\, dk.$$

If we choose for $N(k)$ the value (1.7), we obtain relation (1.13).

We consider a monochromatic solution of equation (1.11):

$$f_k\equiv f(k,t)=f_0(k)\, e^{-i\omega t}, \tag{1.14}$$

where $\omega$ is the eigenvalue of the photon energy operator w. It is obvious that $f_0(k)$ differs from zero only when $k=\omega$, and the expression for the energy (1.13) assumes the form

$$\overline{w}=\omega\int f_k^* f_k\, dk.$$

We demand that this equation be identical with the quantum relation $\overline{w}=\omega$ and then obtain the condition for the normalization of the photon wave function

$$\int f_k^* f_k dk=1. \tag{1.15}$$

Thus, if the condition for the normalization of the wave function (1.15) is satisfied, the energy of the electromagnetic field becomes identical with the photon energy.

### 1.4. *Normalization of the Photon Wave Function*

In practice it is more convenient to deal with a Fourier series rather than with a Fourier integral, i.e., with a discrete rather than with a continuous spectrum. For this we must introduce a certain normalizing volume $\Omega$ and assume that the whole field is confined within it. Certain boundary conditions, which select from all the solutions of Maxwell's equations a discrete spectrum of solutions that satisfy these boundary conditions, can be specified over the boundary of the region $\Omega$. It is simplest to choose the volume in the form of a cube of side $a$ and to postulate that the boundary conditions amount to the conditions of triple periodicity of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ with the period $a$ along each one of the cartesian axes. If these conditions are satisfied, then those fields are selected for which the components $k_i$ of the propagation vector satisfy the conditions

$$k_i = \frac{2\pi}{a} n_i ,$$

where $n_i$ are integers. Thus, $k$-space is divided into cells of volume

$$\Delta = \frac{(2\pi)^3}{\Omega} . \tag{1.16}$$

One allowed value of $\boldsymbol{k}$ corresponds to each such cell. Therefore, the number of values of $\boldsymbol{k}$ lying within a certain range $d\boldsymbol{k}$ is equal to

$$dn = \frac{d\boldsymbol{k}}{\Delta} . \tag{1.16$'$}$$

Integration over $k$-space is replaced by summation over these cells; i.e.,

$$\int F(\boldsymbol{k})\, d\boldsymbol{k} = \Delta \cdot \sum_{\boldsymbol{k}} F(\boldsymbol{k}),$$

where the summation is carried out over all the allowed discrete values of $\boldsymbol{k}$.

The expansion (1.8) assumes the form

$$\mathfrak{E} = \Delta \sum_{\boldsymbol{k}} N(k) \boldsymbol{f}_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} ; \tag{1.17}$$

i.e., the coefficients in the expansion of $\mathfrak{E}$ in a Fourier series are the quantities $\Delta \cdot N \cdot \boldsymbol{f}_{\boldsymbol{k}}$.

The normalization condition (1.15) can now be written in the following form:

$$\sum_k f_k^* f_k = \frac{1}{\Delta}. \tag{1.18}$$

If the volume $\Omega$ chosen is sufficiently large, then the relation (1.16) will depend neither on the shape of the volume $\Omega$, nor on the nature of the boundary conditions. In other words, relation (1.16) is a universal asymptotic relation valid for $ka \gg 1$, where $a$ is a quantity characterizing the linear dimensions of the region.

Later we shall show that in the solution of any physical problem, the quantities $\Delta$ and $\Omega$ appear only in the intermediate calculations and disappear from the final result.

The transition from the Fourier integral to the Fourier series may be carried out only partially, with respect to only one of the coordinates. We assume, for example, that the boundary conditions are such that $k$-space is divided into spherical shells of thickness $\delta$. Then, on making use of spherical polar coordinates, we obtain

$$\int F(\mathbf{k})\,d\mathbf{k} = \int F(\mathbf{k})k^2\,dk\,do \to \delta \sum_k k^2 \int F(\mathbf{k})\,do.$$

The expansion (1.8) in this case takes the form

$$\mathfrak{E} = \delta \sum_k k^2 N(k) \int f_{\mathbf{k}} e^{i\mathbf{kr}}\,do, \tag{1.19}$$

while the normalization condition takes the form

$$\sum_k k^2 \int f_{\mathbf{k}}^* f_{\mathbf{k}}\,do = \frac{1}{\delta}. \tag{1.20}$$

The division of $k$-space into spherical shells can be carried out by choosing the volume $\Omega$ in the form of a sphere of large radius $R$ and postulating that the field vanishes over the surface of the sphere. It can be shown that in this case

$$\delta = \frac{\pi}{R}. \tag{1.21}$$

The number of different possible values of $k$ in the range $dk$ is evidently equal to

$$dn = \frac{dk}{\delta}. \tag{1.21'}$$

If we assume that the two-dimensional space of solid angles is also divided into cells $\delta_0$, then the expansion (1.19) is transformed into a three-dimensional series. In this case formulas (1.17) and (1.19) become identical if we postulate that in the latter formula the cell $\Delta$ is chosen equal to $\Delta = k^2\, \delta\delta_0$.

## § 2. Photon States of Definite Momentum

### 2.1 *Photon Momentum Operator*

As is well known, the momentum of the electromagnetic field $\bar{p}$ is given by the following expression:

$$\bar{p} = \int [EH]\, dr. \tag{2.1}$$

We shall show that when the normalization condition for the photon wave function (1.15) is satisfied, the vector $\bar{p}$ can be interpreted as the expectation value of the photon momentum.

In order to do this we express $\bar{p}$ in terms of $f_k$. Substitution of expansion (1.2) into (2.1) yields

$$\bar{p} = \int [E_k H_{k'}] e^{i(k+k')r}\, dk\, dk'\, dr$$

$$= (2\pi)^3 \int [E_k H_{-k}]\, dk = -(2\pi)^3 i \int \frac{k}{k^2} (E_k \dot{E}_{-k})\, dk.$$

On expressing $E_k$ and $\dot{E}_k$ in terms of $f_k$, in accordance with (1.6), we obtain

$$\bar{p} = (2\pi)^3 \int N^2 (f_k^* f_k - f_{-k}^* f_{-k} + f_{-k}^* f_k^* - f_k f_{-k}) \frac{k}{k}\, dk.$$

On replacing $k$ by $-k$ in the integrand, we can easily see that the integral of the last two terms vanishes, while the integrals of the first two terms are identical. By using the definition (1.7) of $N(k)$ we obtain finally

$$\bar{p} = \int f_{k\alpha}^* k f_{k\alpha}\, dk, \tag{2.2}$$

where the subscript $\alpha = 1, 2, 3$ denotes the components of the vector $f_k$ and summation over $\alpha$ is implied.

It follows from (2.2) that we are justified in identifying the operator indicating multiplication by the propagation vector $\mathbf{p} = k$ with the photon momentum operator, and $k$-space with momentum space. The quantity $f_k^* f_k$ may be interpreted as the probability density of the photon having momentum $k$, while relation (2.2) may be interpreted as the usual quantum mechanical expression for the expectation value. Thus, the normalization condition (1.15) acquires a simple and natural meaning.

### 2.2. *Impossibility of Introducing a Photon Wave Function in the Coordinate Representation*

It might appear that by subjecting the function $f_k$ to the inverse Fourier transformation

$$\frac{1}{(2\pi)^3} \int f_k e^{ikr}\, dk = F,$$

it should be possible to obtain the photon wave function $F$ in the coordinate representation (121); cf. also (148).[1] Because of the normalization condition (1.15) for $f_k$, the function $F$ will also be normalized in the usual manner:

$$\int F^* F\, dr = 1.$$

However, the quantity $F^*F$ will not have the meaning of the probability density of finding the photon at a given point of space. Indeed, the presence of a photon can be established only as a result of its interaction with charges. This interaction is determined by the values of the electromagnetic field vectors $E$ and $H$ at the given point, but the latter are not determined by the value of the wave function $F$ at that point but by its values over all space. Indeed, the Fourier components of the field vectors expressed in terms of $f_k$ contain in accordance with (1.6) and (1.7) the factor $\sqrt{k}$. Formally this can be written in the form $\mathfrak{E} = \sqrt[4]{-\Delta} \cdot F$, where $\Delta$ is the Laplacian operator; but $\sqrt[4]{-\Delta}$ is in fact an integral operator, and therefore the relationship between $E$ and $F$ is not a local one, but an integral one; in other words, $F(r, t)$ is determined not by the value of the field $E(r, t)$ at the same point, but by the field distribution over a certain region whose effective dimensions are in order of magnitude equal to a wavelength. As a result of this, the localization of a photon in a region smaller in order of magnitude than a wavelength has no meaning, and the concept of probability density for the localization of a photon does not exist.

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[1] References to the literature appear in parentheses. The complete list of references is given at the end of the volume.

This result is closely associated with the behavior of particle density under Lorentz transformations, since it is impossible to construct a bilinear expression in the electromagnetic field vectors which would form a four-vector satisfying an equation of continuity.

We note that in practice it is often sufficient (instead of the equation of continuity for the probability density) to utilize the equation of continuity for the energy density

$$\frac{\partial s_0}{\partial t} + \operatorname{div} s = 0, \tag{2.3}$$

where $s_0 = \frac{1}{2}(E^2 + H^2)$ is the energy density, and $s = [EH]$ is the energy flux density. However, these quantities do not form a four-vector.

### 2.3. *Plane Waves*

We return to the photon wave function in momentum space $f_k = f(k, t)$. Equation (1.9) defines its time dependence

$$f(k, t) = f_0(k)\, e^{-ikt}.$$

The time-independent function $f_0(k)$ is restricted only by the transversality condition (1.10). By utilizing the method described in subsection 1.4 we shall treat $k$-space as discrete and shall consider the photon state for which $f_0(k)$ differs from zero only at point $p$. This is a state of definite momentum, and the wave function corresponding to it will be an eigenfunction of the momentum operator $\mathbf{p}$. By taking into account the normalization condition (1.18), we can write the eigenfunction of the momentum operator in the form

$$f(k, t) = \frac{ie}{\sqrt{\Delta}}\, \delta_{k,p}\, e^{-i\omega t}, \tag{2.4}$$

where $\omega = |p|$ and $e$ is the photon polarization vector whose absolute value is equal to unity and which is perpendicular to $p$: $|e|^2 = 1$, $ep = 0$. (The phase factor $i$ is introduced into (2.4) for convenience.)

For a given $p$ two linearly independent vectors $e$ are possible. We take the $z$-axis directed along $p$. Then for these two vectors we can choose the unit vectors $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$ corresponding to the $x$-axis and the $y$-axis:

$$\begin{gathered}
\chi_{1x} = 1, \quad \chi_{1y} = \chi_{1z} = 0, \\
\chi_{2y} = 1, \quad \chi_{2x} = \chi_{2z} = 0, \\
\boldsymbol{\chi}_1 \boldsymbol{\chi}_2 = 0.
\end{gathered} \tag{2.5}$$

If $\boldsymbol{e} = \boldsymbol{\chi}_\mu$ $(\mu = 1,2)$, then the photon is linearly polarized.

We can also choose the following two mutually perpendicular unit vectors $\boldsymbol{\chi}_1'$ and $\boldsymbol{\chi}_2'$:

$$\begin{gathered}
\chi_{1x}' = \frac{1}{\sqrt{2}}, \quad \chi_{1y}' = \frac{i}{\sqrt{2}}, \quad \chi_{1z}' = 0, \\
\chi_{2x}' = \frac{1}{\sqrt{2}}, \quad \chi_{2y}' = -\frac{i}{\sqrt{2}}, \quad \chi_{2z}' = 0, \\
\boldsymbol{\chi}_1'^* \boldsymbol{\chi}_2' = 0.
\end{gathered} \tag{2.6}$$

In this case if $\boldsymbol{e} = \boldsymbol{\chi}_\mu'$, the photons will be circularly polarized.[1]

Thus, states of definite momentum are twofold-degenerate. In order to specify a state uniquely we have to specify its type of polarization. Therefore, the function $f(\boldsymbol{k}, t)$ should be provided with two subscripts (quantum numbers) $\boldsymbol{p}$ and $\mu$ $(\mu = 1, 2)$. The quantities $p_x, p_y, p_z$, and $\mu$ form a complete set of quantum numbers for the photon. (Of course, the energy is thereby also determined, since $\omega = p$.) The system of functions $f_{p\mu}(\boldsymbol{k}, t)$ is a complete orthonormal system in terms of which an arbitrary transverse function $f(\boldsymbol{k}, t)$ can be expanded:

$$\begin{gathered}
f(\boldsymbol{k}, t) = \sum_{p,\mu} a_{p\mu} f_{p\mu}(\boldsymbol{k}, t) = \sum_{\mu} \frac{i}{\sqrt{\Delta}} a_{k\mu} \boldsymbol{e}_\mu e^{-ikt}, \\
a_{k\mu} = \int f f_{p\mu}^* d\boldsymbol{k} = -i\sqrt{\Delta}(f\boldsymbol{e}_\mu^*) e^{ikt}.
\end{gathered} \tag{2.7}$$

By making use of (2.4) it is possible in accordance with (1.16) and (1.17) to represent the electric and magnetic fields corresponding to a photon state of definite momentum $p$ and polarization $\mu$, in the following form:

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[1] We shall refer to $\boldsymbol{\chi}_1'$ as right circular and to $\boldsymbol{\chi}_2'$ as left circular polarization. (In optics the converse definition is usually adopted).

$$\mathfrak{E}_{p\mu} = i\sqrt{\frac{\omega}{2\Omega}}\chi_\mu e^{i(pr-\omega t)},$$
$$\mathfrak{H}_{p\mu} = i\sqrt{\frac{\omega}{2\Omega}}\left[\frac{p}{p}\chi_\mu\right]e^{i(pr-\omega t)}. \tag{2.8}$$

In quantum mechanics a plane wave is frequently normalized in such a way as to correspond to unit flux. Similarly, we can normalize the photon field to unit energy flux density. The field corresponding to such normalization has the following form:

$$\mathfrak{E} = \frac{1}{\sqrt{2}}ee^{i(pr-\omega t)}, \qquad \mathfrak{H} = \left[\frac{p}{p}\mathfrak{E}\right]. \tag{2.9}$$

Indeed, in the expression for the energy flux density $s = [EH]$ we may omit the terms $[\mathfrak{E}\mathfrak{H}]$ and $[\mathfrak{E}^*\mathfrak{H}^*]$, which vanish when averaged over time, and we can take

$$s = [\mathfrak{E}\mathfrak{H}^*] + [\mathfrak{E}^*\mathfrak{H}]. \tag{2.10}$$

On substituting (2.9) into this expression we obtain $s = p/p$.

The expansion (2.7) corresponds to the expansion of an arbitrary electric field satisfying Maxwell's equations (1.1) into plane polarized waves:

$$\mathfrak{E} = \sum_{p,\mu} a_{p\mu}\mathfrak{E}_{p\mu},$$

where the coefficients $a_{p\mu}$ are determined in accordance with (2.7), while $\mathfrak{E}_{p\mu}$ have the form (2.8). Similar expansions hold also for the magnetic field.

### 2.4. *Polarization Density Matrix for the Photon*

We have just given a complete quantum mechanical description of a photon state. In accordance with (2.4) the photon state for a given momentum is characterized by the polarization unit vector $e$. On expanding $e$ in terms of the unit vectors (2.5) or (2.6),

$$e = e_1\chi_1 + e_2\chi_2, \tag{2.11}$$

we can say that the state of polarization is determined by the pair of complex numbers $e_1$ and $e_2$; the quantities $|e_1|^2$ and $|e_2|^2$ represent probabilities of a definite (linear or circular) polarization of the photon, determined by the unit vectors $\chi_1$ and $\chi_2$. Since $e_1$ and $e_2$ are related by the normalization condition

$$|e_1|^2+|e_2|^2 = 1$$

and since, moreover, the common phase factor of $\boldsymbol{e}$ is arbitrary, the polarization is determined by two real parameters and (2.11) can be written in the following form:

$$\boldsymbol{e} = \chi_1 \cos\alpha+\chi_2 \sin\alpha\cdot e^{i\beta}. \tag{2.12}$$

If $\chi_1$ and $\chi_2$ are defined in accordance with (2.5), then $\beta = 0$ denotes linear polarization at an angle $\alpha$ to the $x$-axis, $\beta = \pm\pi/2$ and $\alpha = \pi/4$ denote circular polarization, and arbitrary $\alpha$ and $\beta$ correspond to elliptical polarization.

A more general case is that of an incomplete quantum mechanical description, when a definite wave function cannot be ascribed to the photon. For example, if the photon has been previously scattered by an electron, then there exists only the wave function for the combined photon-electron system whose expansion in terms of the free photon wave functions contains the electron wave functions. We shall confine ourselves to the case when the photon has a definite momentum, i.e., when there exists a wave function (2.4), but the polarization state cannot be specified definitely, since the coefficients $e_\mu$ in (2.11) depend on the parameters characterizing the other system.

Such a photon state is referred to as a state of partial polarization.[1] It can be described by means of a density matrix (c., for example, (119, §12), (120, §5))

$$\varrho_{\mu\nu} = \overline{e_\mu e_\nu^*} \quad (\mu, \nu = 1, 2), \tag{2.13}$$

where the bar denotes averaging over appropriate parameters.

From the definition (2.13) it follows that $\varrho$ is a Hermitian matrix

$$\varrho_{\mu\nu} = \varrho_{\nu\mu}^*.$$

From the normalization condition for $e_\mu$ it follows that

$$\mathrm{Sp}\,\varrho = \varrho_{11}+\varrho_{22} = 1.$$

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[1] Compare the description of partially polarized light in classical electrodynamics, for example, in (118), §50.

Thus, the density matrix depends on three real parameters. It can be written in the following form:

$$\varrho = \tfrac{1}{2}\begin{pmatrix} 1+\xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1-\xi_3 \end{pmatrix},$$

or

$$\varrho = \tfrac{1}{2}\Big(1+\sum_{j=1}^{3}\xi_j\tau_j\Big), \tag{2.14}$$

where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are four matrices forming a complete system of linearly independent two-dimensional matrices. (We note that the matrices $\tau_j$ are identical with the Pauli matrices $\sigma_j$, cf. §9.)

By using (2.14) it is possible to express $\xi_j$ in terms of $\varrho$, viz.,

$$\xi_j = \mathrm{Sp}\,(\varrho \cdot \tau_j). \tag{2.15}$$

The quantities $\xi_j$, called the Stokes parameters, can be directly determined experimentally. For example, let the unit vectors $\chi_1$ and $\chi_2$ in (2.11) correspond to the linear polarizations (2.5). Then $\xi_3$ will be determined by the value of the probability of polarization along the $x$-axis:

$$\varrho_{11} = \tfrac{1}{2}\,(1+\xi_3).$$

In order to determine $\xi_1$ it is necessary to obtain the probability of polarization along an axis making an angle of 45° with the $x$-axis. Indeed, the transformation to such an axis corresponds to the unit vector transformation

$$\chi_1 = \frac{\chi_1' + \chi_2'}{\sqrt{2}}, \qquad \chi_2 = \frac{\chi_1' - \chi_2'}{\sqrt{2}}.$$

By writing (2.11) in the form

$$e = e_1'\chi_1' + e_2'\chi_2'$$

and by defining $\varrho'$ in accordance with (2.13), $\varrho_{\mu\nu}' = \overline{e_\mu' e_\nu'^*}$, we obtain for the probability of linear polarization corresponding to $\chi'$ the following expression:

$$\varrho_{11}' = \tfrac{1}{2}(\varrho_{11} + \varrho_{22} + \varrho_{12} + \varrho_{21}) = \tfrac{1}{2}\,(1+\xi_1).$$

Finally, in order to obtain $\xi_2$ it is necessary to consider circular polarization. Let $\chi_1''$ and $\chi_2''$ be the unit vectors corresponding to the circular polarizations (2.6); i.e.,

$$\chi_1 = \frac{\chi_1''+\chi_2''}{\sqrt{2}}, \quad \chi_2 = \frac{\chi_1''-\chi_2''}{\sqrt{2}}.$$

The probability of polarization corresponding to the unit vector $\chi_1''$ will be expressed by analogy with the foregoing in the following form:

$$\varrho_{11}'' = \tfrac{1}{2}[\varrho_{11}+\varrho_{22}+i(\varrho_{12}-\varrho_{21})] = \tfrac{1}{2}(1+\xi_2).$$

We see that the determination of the polarization of a beam of photons requires the measurement of two linear polarizations (at an angle of 45° to each other) and of one circular polarization. We note that if we had taken the initial matrix (2.13), as determined by the unit vectors (2.6), then $\varrho_{11} = \frac{1}{2}(1+\xi_3)$ would have given us the probability of circular polarization, while $\varrho_{11}' = \frac{1}{2}(1+\xi_1')$ and $\varrho_{11}'' = \frac{1}{2}(1+\xi_2)$ would have given us the probabilities of the linear polarizations.

Since the probabilities must be positive and smaller than unity, it follows from the expressions for $\varrho_{11}$, $\varrho_{11}'$, and $\varrho_{11}''$ that $|\xi_j| \leqslant 1$. On noting that

$$\text{Det } \varrho = \tfrac{1}{4}(1-\xi^2) = \overline{|e_1|^2}\,\overline{|e_2|^2} - \overline{(e_1 e_2^*)}\,\overline{(e_2 e_1^*)} \geqslant 0$$

it may be concluded that

$$\xi^2 \equiv \xi_1^2+\xi_2^2+\xi_3^2 \leqslant 1.$$

As an example we consider some limiting cases. When $\xi_j = 0$, $\varrho = \frac{1}{2}$. In this case the probability of any arbitrary polarization is equal to $\frac{1}{2}$. Such a state is said to be completely unpolarized.

When $\xi^2 = 1$ we can write $\xi_j$ in the following form:

$$\xi_3 = \cos 2\alpha, \quad \xi_1 = \sin 2\alpha \cos\beta, \quad \xi_2 = \sin 2\alpha \sin\beta.$$

In the case of these values of the parameters the matrix $\varrho$ is identical with the matrix obtained on the basis of expression (2.12). Therefore, it describes a state of complete polarization.

In the case of partial polarization the density matrix can be regarded as a linear combination of two density matrices corresponding to a completely unpolarized and a completely polarized state. Indeed, let

$$\xi_j = P\eta_j,$$

where

$$\eta^2 \equiv \eta_1^2+\eta_2^2+\eta_3^2 = 1.$$

Then expression (2.14) can be written in the form

$$\varrho = \frac{1}{2}(1-P)+\frac{P}{2}\Big(1+\sum_j \tau_j\eta_j\Big).$$

The coefficient $P$ is referred to as the degree of polarization of the photon.

## § 3. Angular Momentum. Photon Spin

### 3.1. *Angular Momentum Operator*

We now express the angular momentum $\overline{\boldsymbol{M}}$ of the electromagnetic field corresponding to one photon in terms of the wave function for this photon $f_k$. We identify this angular momentum with the expectation value of the angular momentum of the photon in this state $f_k$. The angular momentum of the field is determined, as is well known, in the following manner:

$$\overline{\boldsymbol{M}} = \int \big[\boldsymbol{r}[\boldsymbol{E}\boldsymbol{H}]\big]\, d\boldsymbol{r}. \tag{3.1}$$

By substituting into this formula the Fourier expansions for the fields, we obtain

$$\overline{\boldsymbol{M}} = \int \big[\boldsymbol{r}[\boldsymbol{E}_k\boldsymbol{H}_{k'}]\big]\, e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}}\, d\boldsymbol{k}\, d\boldsymbol{k}'\, d\boldsymbol{r}.$$

We first carry out the integration over $\boldsymbol{r}$:

$$\int \boldsymbol{r} e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}}\, d\boldsymbol{r} = -i\nabla_{k'}\int e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}}\, d\boldsymbol{r} = -i(2\pi)^3\nabla_{k'}\delta(\boldsymbol{k}+\boldsymbol{k}'),$$

where $\nabla_{k'}$ corresponds to differentiation with respect to the variables $\boldsymbol{k}'$. We next carry out the integration over $\boldsymbol{k}'$:

$$\int d\boldsymbol{k}'\big[\nabla_{k'}\delta(\boldsymbol{k}+\boldsymbol{k}')\,[\boldsymbol{E}_k\boldsymbol{H}_{k'}]\big] = -\int d\boldsymbol{k}'\,\delta(\boldsymbol{k}+\boldsymbol{k}')\big[\nabla_{k'}\,[\boldsymbol{E}_k\boldsymbol{H}_{k'}]\big].$$

On replacing $\boldsymbol{H}$ in the above equation by the expression (1.5) connecting $\boldsymbol{H}_k$ with $\dot{\boldsymbol{E}}_k$, we obtain, after some simple rearrangements,

$$\big[\nabla_{k'}\,[\boldsymbol{E}_k\boldsymbol{H}_{k'}]\big] = -i\left[\boldsymbol{k}'\nabla_{k'}\left(\frac{\boldsymbol{E}_k\dot{\boldsymbol{E}}_{k'}}{k'^2}\right)\right] - i(\boldsymbol{k}'\boldsymbol{E}_k)\operatorname{curl}_{k'}\frac{\dot{\boldsymbol{E}}_{k'}}{k'^2} + i\left[\frac{\dot{\boldsymbol{E}}_{k'}}{k'^2}\boldsymbol{E}_k\right].$$

Thus, we obtain

$$\overline{\boldsymbol{M}} = (2\pi)^3 \int dk\, dk' \delta(k+k') \left\{ \left[ \boldsymbol{k}' \nabla_{k'} \left( \frac{\boldsymbol{E}_k \dot{\boldsymbol{E}}_{k'}}{k'^2} \right) \right] - \frac{1}{k'^2} [\dot{\boldsymbol{E}}_{k'} \boldsymbol{E}_k] + (\boldsymbol{k}' \boldsymbol{E}_k) \operatorname{curl}_{k'} \frac{\dot{\boldsymbol{E}}_{k'}}{k'^2} \right\}.$$

After carrying out the integration over $\boldsymbol{k}'$, and on noting that the last term vanishes in the course of this integration since $\boldsymbol{k}\boldsymbol{E}_k = 0$, we obtain

$$\overline{\boldsymbol{M}} = (2\pi)^3 \int dk \left\{ \left[ \boldsymbol{k} \nabla \left( \frac{1}{k^2} \boldsymbol{E}_k^{(c)} \dot{\boldsymbol{E}}_{-k} \right) \right] - \frac{1}{k^2} [\dot{\boldsymbol{E}}_{-k} \boldsymbol{E}_k] \right\},$$

where $\nabla$ denotes differentiation with respect to $\boldsymbol{k}$; the superscript $c$ on $\boldsymbol{E}_k$ indicates that in carrying out the differentiation this quantity should be regarded constant. Finally, we express $\boldsymbol{E}_k$ and $\dot{\boldsymbol{E}}_k$ in terms of $\boldsymbol{f}_k$ in accordance with (1.6). This gives

$$\overline{\boldsymbol{M}} = \frac{i}{2} \int dk \{ [\boldsymbol{k} \nabla (\boldsymbol{f}_k^{(c)} \boldsymbol{f}_k^* - \boldsymbol{f}_{-k}^{*(c)} \boldsymbol{f}_{-k} + \boldsymbol{f}_{-k}^{*(c)} \boldsymbol{f}_k^* - \boldsymbol{f}_k^{(c)} \boldsymbol{f}_{-k})] - ([\boldsymbol{f}_k^* \boldsymbol{f}_k] - [\boldsymbol{f}_{-k} \boldsymbol{f}_{-k}^*] + [\boldsymbol{f}_k^* \boldsymbol{f}_{-k}^*] - [\boldsymbol{f}_{-k} \boldsymbol{f}_k]) \}.$$

The terms that contain different arguments, i.e., $\boldsymbol{k}$ and $-\boldsymbol{k}$, vanish on integration. For example, the integral $\int [\boldsymbol{f}_{-k} \boldsymbol{f}_k] dk$ on replacement of the variable of integration $\boldsymbol{k}$ by $-\boldsymbol{k}$ changes into $\int [\boldsymbol{f}_k \boldsymbol{f}_{-k}] dk$, i.e., into an expression of opposite sign. The integral $\int [\boldsymbol{k} \nabla (\boldsymbol{f}_k^{(c)} \boldsymbol{f}_{-k})] dk$ on replacement of $\boldsymbol{k}$ by $-\boldsymbol{k}$ changes into $\int [\boldsymbol{k} \nabla (\boldsymbol{f}_{-k}^{(c)} \boldsymbol{f}_k)] dk$, whereas integration by parts leads to the same expression but of opposite sign. Also, on noting that the integrals of terms which depend on the argument $-\boldsymbol{k}$ do not change when $\boldsymbol{k}$ is replaced by $-\boldsymbol{k}$, and, on integrating by parts,

$$\int [\boldsymbol{k} \nabla (\boldsymbol{f}_k^{(c)} \boldsymbol{f}_k^*)] \, dk = - \int [\boldsymbol{k} \nabla (\boldsymbol{f}_k^{*(c)} \boldsymbol{f}_k)] \, dk,$$

we finally obtain the following expression for the angular momentum:

$$\overline{\boldsymbol{M}} = \int dk \, \{ -i \, [\boldsymbol{k} \nabla (\boldsymbol{f}_k^{*(c)} \boldsymbol{f}_k)] - i \, [\boldsymbol{f}_k^* \boldsymbol{f}_k] \}$$

or, in terms of its components,

$$\overline{M}_\alpha = \int dk \, f_\beta^* \{ -i \, [\boldsymbol{k} \nabla]_\alpha f_\beta - i e_{\alpha\beta\gamma} f_\gamma \}, \tag{3.2}$$

where $e_{\alpha\beta\gamma}$ is the unit antisymmetric tensor of the third rank.

### 3.2. *Photon Spin Operator*

We introduce the vector operator **s** defined by the result of the operation of its components on the components of some arbitrary vector $f$ according to the formula

$$(\mathbf{s}_\alpha f)_\beta = -(\mathbf{s}_\beta f)_\alpha = -ie_{\alpha\beta\gamma} f_\gamma . \tag{3.3}$$

On noting that the vector product of two vectors $f$ and $g$ is expressed in terms of **s** in the following manner:

$$[gf]_\alpha = ig_\beta \mathbf{s}_\alpha f_\beta , \tag{3.4}$$

we write the angular momentum defined by formula (3.2) in the form

$$\overline{M} = \int f_\alpha^* \{-i\,[k\nabla] + \mathbf{s}\} f_\alpha \, dk . \tag{3.5}$$

We see that the expression for the expectation value of the angular momentum of the photon (3.5) has the structure of quantum mechanical expectation values, and that for the angular momentum operator for the photon we must take the operator

$$\mathrm{M} = -i\,[k\nabla] + \mathbf{s} . \tag{3.6}$$

We now show that the operator (3.6) is identical with the operator for the infinitesimal rotation of the vector field multiplied by $i$. In order to do this we consider an infinitesimal rotation of $k$-space about the origin (189). As a result of such a rotation the radius vector of each point in $k$-space is changed by the increment

$$\delta \boldsymbol{k} = [\delta \boldsymbol{\alpha} \boldsymbol{k}],$$

where $\delta\boldsymbol{\alpha} = \boldsymbol{\nu}\delta\alpha$ is the vector characterizing the infinitesimal rotation, $\delta\alpha$ is the angle of rotation, and $\boldsymbol{\nu}$ is the unit vector along the axis of rotation. This infinitesimal rotation of $k$-space transforms the vector field $f(k)$ into the field $f'(k)$ related to $f(k)$ by the following expression:

$$f'(k + \delta k) = f(k) + [\delta\boldsymbol{\alpha} f(k)].$$

As a result of this the field at the point $k$ is altered by the amount

$$\delta f(k) = f'(k) - f(k).$$

Since

$$f'(k+\delta k) = f'(k) + (\delta k \nabla) f'(k) = f'(k) + (\delta\boldsymbol{\alpha}[k\nabla]) f(k)$$

[in the last term we may neglect the difference between $f'(k)$ and $f(k)$], we obtain

$$\delta f(k) = -(\delta\boldsymbol{\alpha}[k\nabla]f(k) + [\delta\boldsymbol{\alpha} f(k)]. \tag{3.7}$$

For a spherically symmetric field which can be represented in the form $f(k) = g(k)k$, evidently $\delta f = 0$.

The second term in (3.7) can be transformed by utilizing the operator s introduced in (3.3):

$$[\delta\boldsymbol{\alpha} f] = i(\delta\boldsymbol{\alpha}\mathbf{s})f.$$

On the other hand, the infinitesimal rotation operator **J** is defined by the relation

$$\delta f = (\delta\boldsymbol{\alpha}\mathbf{J})f = \delta\alpha(\boldsymbol{\nu}\mathbf{J})f.$$

On comparing this expression with (3.7) we see that

$$\mathbf{J} = -[k\nabla] - i\mathbf{s} = -i\mathbf{M},$$

i.e., that up to the factor $i$, **M** is identical with the infinitesimal rotation operator.

By using expression (3.6) we can easily see that the components of the angular momentum operator satisfy the following commutation relations:

$$\mathrm{M}_x\mathrm{M}_y - \mathrm{M}_y\mathrm{M}_x = i\mathrm{M}_z,$$
$$\mathrm{M}_\alpha\mathbf{M}^2 - \mathbf{M}^2\mathrm{M}_\alpha = 0.$$

These relations hold for any quantum mechanical system and are a consequence of the general relation between the angular momentum operator and the infinitesimal rotation operator.

From these relations it follows, as is well known, that the eigenvalues of the operator $\mathbf{M}^2$ are equal to

$$M^2 = j(j+1),$$

where $2j+1$ is a positive integer; later we shall see that for a photon $j$ is also an integer.

The eigenvalues of the operator $\mathrm{M}_z$ are equal to

$$M_z = M \quad (-j \leqslant M \leqslant j).$$

The operator **M** commutes with the energy. Therefore photon states with definite values of $w$, $M^2$, and $M_z$ are possible (quantum numbers $\omega, j, M$). We now proceed to obtain the wave functions for these states.

### 3.3 *Photon Spin Wave Functions*

Formula (3.6) shows that the angular momentum operator for the photon consists of two terms. The first term is identical with the usual quantum mechanical operator **L** for the orbital angular momentum in the momentum representation:

$$\mathbf{L} = -i[k\nabla_k]. \tag{3.8}$$

The second term **s** may be called the operator for the spin angular momentum. However, the separation of the angular momentum of the photon into an orbital and a spin part has restricted physical meaning. First, the usual definition of spin as the angular momentum of a particle at rest is inapplicable to the photon since the rest mass of the photon is equal to zero. Second, as we shall see later, states with definite values of orbital and spin angular momenta do not in the general case satisfy the condition of transversality. Therefore, only certain superpositions of such states have physical meaning. Nevertheless, from the formal point of view the representation of angular momentum in the form of a sum of two terms is very useful. It enables us to construct wave functions for photon states having definite values of angular momentum from simpler orbital angular momentum and spin eigenfunctions.

The vector subscript $\alpha$ of the photon wave function $f_\alpha(k, t)$ can be regarded as an independent variable which assumes three values: $\alpha = x, y, z$. In accordance with this we introduce the notation

$$f_\alpha(k, t) = f(k, \alpha, t).$$

The function $f(k, \alpha, t)$ is a scalar in the generalized momentum and spin space defined by the variables $k_x, k_y, k_z, \alpha$. The different components of the vector $f$ are now values of the scalar $f$ at different points of the spin subspace. The operator **L** acts only on the variables $k$, while the operator **s** in accordance with its definition (3.3) acts only on the variable $\alpha$. Therefore, the operators **L** and **s** commute.

The eigenfunctions of the operators $\mathbf{L}^2$ and $\mathbf{L}_z$ corresponding to the eigenvalues $\mathbf{L}^2 = l(l+1)$, $\mathbf{L}_z = m$ will be denoted by $Y_{lm}$. They are functions of the variable $\boldsymbol{n} = k/k$ and satisfy the equations

$$\mathbf{L}^2 Y_{lm} = l(l+1) Y_{lm},$$

$$\mathbf{L}_z Y_{lm} = m Y_{lm}.$$

As is well known, $Y_{lm}$ represent spherical harmonics. We give the explicit expression for the spherical harmonics with the choice of the phase factor which will be used in all subsequent discussions (41), (119) (the factor $-1$ is associated only with positive odd values of $m$)

$$Y_{lm}(\vartheta,\varphi)=\frac{(-1)^l}{2\,l!}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\,e^{im\varphi}\frac{1}{\sin^m\vartheta}\frac{d^{l-m}}{d(\cos\vartheta)^{l-m}}\sin^{2l}\vartheta \tag{3.9'}$$

They are normalized by the condition

$$\int Y^*_{lm}Y_{l'm'}\,do=\delta_{ll'}\delta_{mm'}. \tag{3.9}$$

We now obtain the eigenfunctions of the spin operator. We denote by $\boldsymbol{\chi}_{s\mu}$ the eigenfunction of the operators $\mathbf{s}^2$ and $\mathrm{s}_z$, which corresponds to the eigenvalues $s^2=s(s+1)$ and $s_z=\mu$ of these operators.

The argument of the function $\boldsymbol{\chi}_{s\mu}$ is the spin variable $\alpha$. Therefore, this function can also be represented in the form of the vector $\boldsymbol{\chi}_{s\mu}$. If we write $\boldsymbol{\chi}$ in the form of the column vector,

$$\boldsymbol{\chi}=\begin{pmatrix}\chi_x\\ \chi_y\\ \chi_z\end{pmatrix},$$

then by utilizing (3.3) we can easily obtain an explicit representation for the operators $\mathrm{s}_\alpha$ and $\mathbf{s}^2$ in the form of the following matrices:

$$\mathrm{s}_x=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\qquad \mathrm{s}_y=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},$$
$$\mathrm{s}_z=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\qquad \mathbf{s}^2=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}. \tag{3.10}$$

It follows from (3.10) that the quantity $s$ may assume only the single value $s=1$. In other words, the photon spin is equal to unity. Therefore, in the future we shall sometimes omit the subscript $s$ on the function $\chi_{s\mu}$ and denote it simply by $\chi_\mu$.

The magnitude $\mu$ of the spin component may assume the three values

$$\mu=0,\pm1.$$

The functions $\chi_\mu$ satisfy the following equations:

$$s^2\chi_\mu = 2\chi_\mu,$$
$$s_z\chi_\mu = \mu\chi_\mu.$$

By utilizing (3.10) we can easily obtain the solutions of these equations:

$$\chi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \chi_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}; \quad \chi_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \tag{3.11}$$

The functions (3.11) are orthogonal to one another since they are eigenfunctions of the Hermitian operator $s_z$ corresponding to different eigenvalues. Their normalization has been chosen so that

$$\sum_a \chi_\mu^*(a)\,\chi_{\mu'}(a) = \delta_{\mu\mu'}$$

or, in vector form,

$$\chi_\mu\chi_{\mu'}^* = \delta_{\mu\mu'}. \tag{3.12}$$

The unit vectors $\chi_\mu$ provide three basic unit vectors along which any arbitrary vector $f$ may be decomposed:

$$f = \sum_{\mu=-1}^{1} f^\mu\chi_\mu. \tag{3.13}$$

We shall refer to $f^\mu$ as the contravariant components of the vector $f$ along these basic unit vectors. By utilizing the expressions (3.11) for $\chi_\mu$ we can easily establish the relationship between $f^\mu$ and the cartesian components of the vector $f$:

$$\begin{aligned} f^0 &= f_z, \\ f^{\pm 1} &= \mp\frac{1}{\sqrt{2}}(f_x \mp if_y). \end{aligned} \tag{3.14}$$

Along with the contravariant components $f^\mu$ we can also introduce the covariant components of the vector $f_\mu$, which are defined by the condition that the scalar product of two vectors $f$ and $g$,

$$fg = f_z g_z + \tfrac{1}{2}(f_x + if_y)(g_x - ig_y) + \tfrac{1}{2}(f_x - if_y)(g_x + ig_y),$$

should have the following form:

$$fg = \sum_{\mu=-1}^{1} g^\mu f_\mu = \sum_{\mu=-1}^{1} f^\mu g_\mu.$$

This will hold, as may be easily seen, if

$$f_\mu = (-1)^\mu f^\mu, \tag{3.15}$$

i.e.

$$f_0 = f_z;\ f_{\pm 1} = \mp \frac{1}{\sqrt{2}}(f_x \pm i f_y).$$

$$f = \sum_{\mu=-1}^{1} f_\mu \chi_\mu^*.$$

Since the spin operator commutes with the momentum operator, it is possible to speak of states of definite momentum **p** and spin component $\mu$. The components of the polarization vector in (2.4) may be chosen in such a way that

$$e = \chi_\mu.$$

The two possible polarizations correspond to only two values of the component of the spin angular momentum $\mu$. The third value is excluded by the condition of transversality. If the $z$-axis is directed along $p$, then the transversality condition excludes the state $\chi_0$. The two vectors $\chi_1'$ and $\chi_2'$, corresponding to circular polarization (2.6), are equivalent, respectively, to $\chi_1$ and $\chi_{-1}$. Thus, the value of the spin component $\mu = 1$ corresponds to right circular polarization, while $\mu = -1$ corresponds to left circular polarization.

## § 4. Photon States of Definite Angular Momentum and Parity

### 4.1. *Eigenfunctions of the Photon Angular Momentum Operator*

We now proceed to determine the form of the eigenfunctions of the operators $\mathbf{M}^2$ and $\mathrm{M}_z$. We denote them by $Y_{jM}$ and call them vector spherical harmonics or spherical vectors. As we have done previously, we can introduce the spin variables $a$. Then

$$(Y_{jM}(\boldsymbol{n}))_a = Y_{jM}(\boldsymbol{n}, a).$$

These functions satisfy the equations

$$\begin{aligned} \mathbf{M}^2 Y_{jM}(\boldsymbol{n}, a) &= j(j+1) Y_{jM}(\boldsymbol{n}, a), \\ \mathrm{M}_z Y_{jM}(\boldsymbol{n}, a) &= M Y_{jM}(\boldsymbol{n}, a). \end{aligned} \tag{4.1}$$

Instead of solving these equations directly, we can utilize for the determination of $Y_{jM}$ the quantum mechanical rules for the addition of angular momenta (9). Indeed, the problem reduces to the problem (well known in quantum mechanics) of the construction of the wave function of a system consisting of two noninteracting subsystems. In our case such subsystems are the orbital and the spin degrees of freedom of the photon associated respectively with the variables $\boldsymbol{k}$ and $\alpha$.

Since the spin angular momentum is given by $s = 1$, then according to the rules of addition of angular momenta the total angular momentum of the photon may assume a given value $j$ if the orbital angular momentum $l$ is equal to

$$\begin{aligned} l &= j, j\pm 1 \quad \text{for} \quad j \neq 0, \\ l &= 1 \qquad\quad \text{for} \quad j = 0. \end{aligned} \tag{4.2}$$

Thus, in the general case there exist three different wave functions $Y_{jM}$ corresponding to three orbital states. We denote them by $Y_{jlM}$.

The wave function $Y_{jlM}$ is given, as is well known, by a linear combination of products of orbital and spin functions,

$$Y_{jlM}(\boldsymbol{n}, \alpha) = \sum_{m+\mu=M} C^{jM}_{lm1\mu} Y_{lm}(\boldsymbol{n}) \chi_\mu(\alpha), \tag{4.3}$$

for which the angular momentum components $m$ and $\mu$ are related by the addition rule $m+\mu = M$. The coefficients $C^{jM}_{lm1\mu}$ are in the general case determined by the properties of the rotation group (cf., for example, (119) and (41); cf. also § 8).

For a given $l$ and $M$ the coefficients $C^{jM}_{lm1\mu}$ form a matrix of three columns ($j = l, l\pm 1$) and three rows ($\mu = 0, \pm 1$). The rows and columns of this matrix satisfy the orthogonality and normalization condition

$$\sum_\mu C^{jM}_{lm1\mu} C^{j'M}_{lm1\mu} = \delta_{jj'}, \tag{4.4}$$

$$\sum_j C^{jM}_{lm1\mu} C^{jM}_{lm1\mu'} = \delta_{\mu\mu'}. \tag{4.5}$$

The elements of the matrix $C^{jM}_{lm1\mu}$ are found in Table 1.

Formula (4.3) can also be rewritten in vector form:

$$\boldsymbol{Y}_{jlM}(\boldsymbol{n}) = \sum_\mu C^{jM}_{l,m-\mu,1,\mu} Y_{l,\,M-\mu}(\boldsymbol{n}) \boldsymbol{\chi}_\mu. \tag{4.6}$$

A comparison of this formula with (3.13) shows that the contravariant components of the vector spherical harmonic are equal to

$$(\boldsymbol{Y}_{jlM})^\mu = C^{jM}_{l,m-\mu,1,\mu} Y_{l,\,M-\mu}. \tag{4.7}$$

TABLE 1

| $\mu \diagdown j$ | $l+1$ | $l$ | $l-1$ |
|---|---|---|---|
| $-1$ | $\sqrt{\frac{(l-M)(l-M+1)}{(2l+1)(2l+2)}}$ | $\sqrt{\frac{(l+M+1)(l-M)}{2l(l+1)}}$ | $\sqrt{\frac{(l+M)(l+M+1)}{2l(2l+1)}}$ |
| $0$ | $\sqrt{\frac{(l+M+1)(l-M+1)}{(2l+1)(l+1)}}$ | $\frac{M}{\sqrt{l(l+1)}}$ | $-\sqrt{\frac{(l+M)(l-M)}{l(2l+1)}}$ |
| $1$ | $\sqrt{\frac{(l+M)(l+M+1)}{(2l+1)(2l+2)}}$ | $-\sqrt{\frac{(l+M)(l-M+1)}{2l(l+1)}}$ | $\sqrt{\frac{(l-M)(l-M+1)}{2l(2l+1)}}$ |

The covariant and the cartesian components are determined from this formula by means of formulas (3.15) and (3.14).

The vector spherical harmonics $Y_{jlM}$ form an orthogonal system of functions (a difference in the values of any one of the subscripts on two functions indicates that the two functions belong to different eigenvalues of the self-conjugate operators $\mathbf{M}^2$, $\mathrm{M}_z$, or $\mathbf{L}^2$). Taking into account the normalization condition on the coefficients (4.4), we can write

$$\int Y^*_{jlM} Y_{j'l'M'} do = \delta_{jj'} \delta_{ll'} \delta_{MM'} , \tag{4.8}$$

where $Y^*_{jlM}$ denotes a vector whose cartesian components are complex-conjugates of the cartesian components of the vector $Y_{jlM}$. We note that $(Y^*_{jlM})^\mu$ is not identical with $(Y_{jlM})^{\mu *}$, but is equal to

$$(Y^*_{jlM})^\mu = (-1)^\mu (Y_{jlM})^{-\mu *} = (Y_{jlM})^*_\mu .$$

It follows from this that

$$Y^*_{jlM} = Y_{jl-M}(-1)^{j+l+1+M} .$$

## 4.2. *Longitudinal and Transverse Vector Spherical Harmonics*

We have obtained a system of eigenfunctions of the operators of the square of the angular momentum of the photon $\mathbf{M}^2$ and of its component $\mathrm{M}_z$. The state of the photon with definite values of $j$ and $M$ is described by a wave function which in the general case is a linear combination of three vector spherical harmonics:

$$f_{jM} = \sum_{l=j-1}^{j+1} a_l Y_{jlM} .$$

The coefficients of this linear combination are not independent since the photon wave function must satisfy the transversality condition (1.10):

$$f_{jM}\boldsymbol{n} = 0.$$

Therefore there are not three, but two, different photon states with given quantum numbers $j$, $M$. We denote the corresponding wave functions by $f_{jM\lambda}$, where $\lambda$ may assume two values: $\lambda = 1, 0$.

In order to obtain the explicit form of $f_{jM}$, we note that from the three vectors $\boldsymbol{Y}_{jlM}$, which are linearly independent at each point of $\boldsymbol{k}$-space, we can construct three linear combinations $\boldsymbol{Y}_{jM}^{(\lambda)}$ ($\lambda$ assumes the three values: $\lambda = 0, \pm 1$), which are mutually perpendicular, with one of these vectors, to which we assign the value $\lambda = -1$, being longitudinal, i.e., directed along the radius vector $\boldsymbol{k}$, while the other two vectors corresponding to the values $\lambda = 0$, 1 are transverse. We now obtain these combinations.

We utilize for this the well-known formula for the expansion in spherical harmonics of the product $n^{\mu} Y_{jM}$, where $n^{\mu}$ are the components of the unit vector:

$$n^0 = \cos\vartheta, \qquad n^{\pm 1} = \mp \frac{1}{\sqrt{2}} \sin\vartheta e^{\mp i\varphi}$$

($\vartheta$ and $\varphi$ are the polar and the azimuthal angles in the spherical polar system of coordinates.) In our notation this expansion [cf., for example, (16), (41)] has the following form:

$$\boldsymbol{n} Y_{jM} = \sqrt{\frac{j}{2j+1}} \boldsymbol{Y}_{j,\, j-1,\, M} - \sqrt{\frac{j+1}{2j+1}} \boldsymbol{Y}_{j,\, j+1,\, M}. \tag{4.9}$$

Formula (4.9) can also be easily obtained directly, by expanding the vector $\boldsymbol{n} Y_{jM}$ in terms of vector spherical harmonics; the expansion coefficients are equal to $\int \boldsymbol{n} Y_{jM} \boldsymbol{Y}_{jlM}^{*} do$. Consequently, expression (4.9) represents that linear combination of vector spherical harmonics which forms a longitudinal vector. Therefore, we define $\boldsymbol{Y}_{jM}^{(\lambda)}$ for $\lambda = -1$ by

$$\boldsymbol{Y}_{jM}^{(-1)} \equiv \boldsymbol{n} Y_{jM}. \tag{4.10}$$

We now consider the scalar product of the unit vector $\boldsymbol{n}$ with the vector spherical harmonic $\boldsymbol{Y}_{jjM}$

$$\boldsymbol{n} \boldsymbol{Y}_{jjM} = \sum_{\mu} C^{jM}_{j,\, M-\mu,\, 1,\, \mu} n_{\mu} Y_{j,\, M-\mu}.$$

It vanishes, as may be seen by utilizing expression (4.9) and formulas (4.5). Thus, $Y_{jjM}$ is one of the desired transverse vector spherical harmonics. We assign to it the value $\lambda = 0$:

$$Y_{jM}^{(0)} \equiv Y_{jjM}. \tag{4.11}$$

Finally, we define the second transverse vector spherical harmonic $Y_{jM}^{(1)}$ by the relation

$$iY_{jM}^{(1)} = [\boldsymbol{n}Y_{jM}^{(0)}]. \tag{4.12}$$

By utilizing the expansion of $\boldsymbol{n}Y_{jM}$, it is possible to express $Y_{jM}^{(1)}$ in terms of the vector spherical harmonics $Y_{jlM}$:

$$Y_{jM}^{(1)} = \sqrt{\frac{j}{2j+1}}\, Y_{j,j+1,M} + \sqrt{\frac{j+1}{2j+1}}\, Y_{j,j-1,M}. \tag{4.13}$$

On solving equations (4.9) and (4.13) with respect to $Y_{j,j-1,M}$ and $Y_{j,j+1,M}$ we obtain

$$\begin{aligned} Y_{j,j-1,M} &= \frac{1}{\sqrt{2j+1}}\left(\sqrt{j+1}\, Y_{jM}^{(1)} + \sqrt{j}\, Y_{jM}^{(-1)}\right), \\ Y_{j,j+1,M} &= \frac{1}{\sqrt{2j+1}}\left(\sqrt{j}\, Y_{jM}^{(1)} - \sqrt{j+1}\, Y_{jM}^{(-1)}\right). \end{aligned} \tag{4.14}$$

It is also possible to express the vector spherical harmonics $Y_{jM}^{(0)}$ and $Y_{jM}^{(1)}$ in terms of the derivatives of the spherical harmonic $Y_{jM}$. In order to do this we have to use the well-known formula for the differentiation of spherical harmonics, which may be written in the following form:

$$k\nabla Y_{jM} = \frac{1}{\sqrt{2j+1}}\left(j\sqrt{j+1}\, Y_{j,j+1,M} + (j+1)\sqrt{j}\, Y_{j,j-1,M}\right).$$

On comparing this formula with (4.13) we see that

$$Y_{jM}^{(1)} = \frac{k}{\sqrt{j(j+1)}} \nabla_k Y_{jM}. \tag{4.15}$$

and, in accordance with (4.12),

$$Y_{jM}^{(0)} = \frac{-i[k\nabla_k]Y_{jM}}{\sqrt{j(j+1)}} = \frac{1}{\sqrt{j(j+1)}} LY_{jM}, \tag{4.16}$$

where L is the operator (3.8).

We shall also write down the expressions for the components of the vector spherical harmonics $Y_{jM}^{(\lambda)}$ along the unit vectors of spherical polar coordinates. On utilizing formulas (4.10), (4.15), and (4.16) we obtain

$$
\begin{aligned}
&(Y_{jM}^{(-1)})_n = Y_{jM},\\
&(Y_{jM}^{(-1)})_\vartheta = (Y_{jM}^{(-1)})_\varphi = 0,\\
&(Y_{jM}^{(1)})_n = (Y_{jM}^{(0)})_n = 0,\\
&(Y_{jM}^{(1)})_\vartheta = +i(Y_{jM}^{(0)})_\varphi = \frac{1}{\sqrt{j(j+1)}}\,\frac{\partial Y_{jM}}{\partial\vartheta},\\
&(Y_{jM}^{(1)})_\varphi = -i(Y_{jm}^{(0)})_\vartheta = \frac{1}{\sqrt{j(j+1)}}\,\frac{1}{\sin\vartheta}\,\frac{\partial Y_{jM}}{\partial\varphi}.
\end{aligned}
\tag{4.17}
$$

It may be seen from the definitions (4.10), (4.11), and (4.12) that the vector spherical harmonics $Y_{jM}^{(\lambda)}$ are normalized in the same way as $Y_{jlM}$. The functions $Y_{jM}^{(\lambda)}$, like the functions $Y_{jlM}$, are mutually orthogonal for different values of $j$ and $M$. Since, moreover, for different $\lambda$ they are mutually perpendicular at all points, we have

$$\int Y_{jM}^{(\lambda)*}Y_{j'M'}^{(\lambda')}do = \delta_{jj'}\delta_{MM'}\delta_{\lambda\lambda'}.$$

### 4.3. *Parity of Photon States*

Thus, having taken into account the requirement of transversality, we can represent the photon wave function of a state with a definite value of the angular momentum in the form

$$f_{jM} = a_1 Y_{jM}^{(1)} + a_0 Y_{jM}^{(0)},$$

where the coefficients $a_1$ and $a_0$ are now arbitrary. This means that the states $f_{jM}$ are twofold degenerate. We can remove this degeneracy by requiring that each state must also have definite parity, i.e., that it should be an eigenstate of the inversion operator I. The latter is defined for the vector field $f(k)$ in the following manner:

$$\mathrm{I}f(k) = -f(-k).$$

(The minus sign is introduced since under inversion any direction goes over into the opposite one $\mathrm{I}\boldsymbol{n} = -\boldsymbol{n}$.) Since $\mathrm{I}^2 = 1$ the operator I has the two eigenvalues $\pm 1$.

The operator I commutes with the angular momentum operator **M** and, therefore, we can set ourselves the problem of finding those states for which the angular momentum and parity have definite values simultaneously. These states can be easily obtained if we use the well-known property of spherical harmonics

$$Y_{lm}(-\boldsymbol{n}) = (-1)^l Y_{lm}(\boldsymbol{n}),$$

from which it follows in virtue of (4.7) that

$$\mathrm{I}\boldsymbol{Y}_{jlM}(\boldsymbol{n}) = (-1)^{l+1}\boldsymbol{Y}_{jlM}(\boldsymbol{n}).$$

For transverse vector spherical harmonics the following relations hold:

$$\mathrm{I}\boldsymbol{Y}_{jM}^{(\lambda)} = (-1)^{j+1-\lambda}\,\boldsymbol{Y}_{jM}^{(\lambda)}. \tag{4.18}$$

This can be seen from their definitions (4.11) and (4.13). Thus, for given $j$ and $M$ two states differing in parity are possible. We denote the wave functions of these states by $f_{jM\lambda}(\lambda = 0,1)$.

The states corresponding to the value $\lambda = 1$ are said to be states of *electric* type, while the states with $\lambda = 0$ are said to be states of *magnetic* type. These names are associated with the fact that, as we shall see later (cf. § 27), the emission of a photon in the corresponding states is determined by the electric or the magnetic moment of the system of charges.

The vector spherical harmonics $\boldsymbol{Y}_{jM}^{(\lambda)}$ determine only the angular dependence of the photon wave function $f_{jM\lambda}$, which is also a function of the time and of the absolute value of the propagation vector $k$

$$f_{jM\lambda} = a(k, t)\boldsymbol{Y}_{jM}^{(\lambda)}(\boldsymbol{n}).$$

According to (1.9) $a(k, t) = a(k) \exp(-ikt)$, where $a(k)$ is an arbitrary function which is restricted only by the normalization condition,

$$\int_0^\infty |a(k)|^2 k^2\, dk = 1.$$

If we make use of the discrete spectrum of $k$, then in accordance with (1.20),

$$\sum_k |a(k)|^2 k^2 = \frac{1}{\delta}. \tag{4.19}$$

If, simultaneously with the angular momentum and the parity, the photon energy also has a definite value, then $a(k)$ differs from zero only for $k = \omega$, i.e., $a(k) = (i/\omega\sqrt{\delta})\delta_{k\omega}$ (the choice of the phase factor is, of course, arbitrary). Thus, the photon state may be uniquely specified by giving four quantum numbers corresponding to the values of the energy $\omega$, the angular momentum $j$, the component of the angular momentum $M$ and parity $\lambda$. The normalized photon wave function then has the form

$$f(\boldsymbol{k}, t) = \frac{i}{\omega \sqrt{\delta}} \boldsymbol{Y}^{(\lambda)}_{jM} e^{-i\omega t} \delta_{k\omega}. \tag{4.20}$$

We note that for $\lambda = 1$ it is not possible to ascribe a definite value of the orbital angular momentum to the photon state, since according to (4.13) the vector spherical harmonic $\boldsymbol{Y}^{(1)}_{jM}$ is a linear combination of the vector spherical harmonics $\boldsymbol{Y}_{jlM}$ with different values of $l$. Here we have evidence of the fact that, strictly speaking, it is not possible to separate the total angular momentum of the photon into orbital and spin parts, as we have already stated in the preceding section.

If $j = 0$, then according to the rules of addition of angular momenta (4.2) there exists only one vector spherical harmonic $\boldsymbol{Y}_{010}$ which is identical with the longitudinal vector spherical harmonic $\boldsymbol{Y}^{(-1)}_{00} = \boldsymbol{n}Y_{00}$. From this it follows that there are no transverse vector spherical harmonics for $j = 0$. This result has a simple meaning. The state with angular momentum zero represents a spherically symmetric state, but a spherically symmetric vector field can only be longitudinal.

Thus, a photon cannot exist in a state of angular momentum zero.

### 4.4. *Expansion in Spherical Waves*

The wave function of a photon in an arbitrary state may be expanded in terms of the functions (4.20):

$$f = \sum_{\omega, j, M, \lambda} a_{\omega jM\lambda} f_{\omega jM\lambda}.$$

Since the system (4.20) is orthonormal the expansion coefficients will be determined in the following manner:

$$a_{\omega jM\lambda} = \int f f^*_{\omega jM\lambda} dk = \sum_k k^2 \delta \int f f^*_{\omega jM\lambda} do. \tag{4.21}$$

Formula (4.21) can be used, in particular, for expanding states of definite momentum $\boldsymbol{p}$ and polarization $\mu$ in terms of states of definite angular momentum. In such a case

$$a^{(p\mu)}_{\omega jM\lambda} = \boldsymbol{e}_\mu \boldsymbol{Y}^{(\lambda)*}_{jM}\left(\frac{\boldsymbol{p}}{p}\right) \sqrt{\delta_0}\, \delta_{\omega p}, \tag{4.22}$$

where $\delta_0$ is the element of solid angle defined in subsection 1.4. With the aid of (2.7) we can easily obtain the expansion of the wave function (4.20) of a state of definite angular momentum in terms of states of definite momenta. This yields

$$a_{p\mu}^{(\omega jM\lambda)} = (a_{\omega jM\lambda}^{(p\mu)})^* .$$

The probability that a photon which has a definite angular momentum and parity is moving in a given direction $\boldsymbol{n}$ and has a given polarization $\mu$ is evidently equal to $|a_{p\mu}^{(\omega jM\lambda)}|^2$.

The probability of the direction of motion lying within a certain solid angle has more physical meaning. In order to determine this probability we must multiply $|a_{p\mu}|^2$ by the number of different states lying within $do$, which is equal to $do/\delta_0$. If we sum this probability over the polarizations, then in accordance with (4.22) we obtain:

$$\frac{do}{\delta_0}\sum_{\mu} |a_{p\mu}^{(\omega jM\lambda)}|^2 = |Y_{jM}^{(\lambda)}(\boldsymbol{n})|^2\, do \equiv F(x)\, do , \tag{4.23}$$

where $x = \cos\vartheta$.

We note that since the two transverse vector spherical harmonics are related by expression (4.12), expression (4.23) does not depend on $\lambda$; i.e., the angular distribution of the photon is determined only by its angular momentum, and not by its parity.

The function $F(x)$ can be represented in the form of an expansion in terms of Legendre polynomials

$$F(x) = \frac{1}{4\pi}\sum_{n=0}^{j} a_{2n}P_{2n}(x) .$$

The coefficients $a_{2n}$ are given in Table 2. Explicit expressions for the functions $F(x)$ are given in Table 3.

TABLE 2

| $j$ | 1 | | 2 | | | 3 | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert M\rvert$ / $a_{2n}$ | 0 | 1 | 0 | 1 | 2 | 0 | 1 | 2 | 3 |
| $a_0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a_2$ | −1 | 1/2 | 5/7 | 5/14 | −5/7 | 1 | 3/4 | 0 | −5/4 |
| $a_4$ | | | −12/7 | 8/7 | −2/7 | 3/11 | 1/22 | —7/22 | 3/22 |
| $a_6$ | | | | | | −25/11 | 75/44 | −15/22 | 5/44 |

### 4.5. *Expressions for the Electric and Magnetic Fields*

We now obtain expressions for the electric and magnetic fields corresponding to a photon state of definite energy, angular momentum, and parity. As before, we use the notation

$$\boldsymbol{E} = \mathfrak{E}+\mathfrak{E}^*; \quad \boldsymbol{H} = \mathfrak{H}+\mathfrak{H}^*.$$

In accordance with (1.8), (1.19), (4.20), and (4.12) we have

$$\mathfrak{E} = i\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2}\int Y^{(\lambda)}_{jM} e^{ikr} do,$$

$$\mathfrak{H} = -\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2}\int Y^{(1-\lambda)}_{jM} e^{ikr} do.$$

In order to evaluate the integrals occurring in these expressions we use the well-known expansion of a plane wave in terms of spherical harmonics:

$$e^{ikr} = \sum_{lm} g_l(kr)\, Y^*_{lm}\left(\frac{k}{k}\right) Y_{lm}\left(\frac{r}{r}\right), \tag{4.24}$$

where

$$g_l(kr) = (2\pi)^{3/2} i^l \frac{J_{l+1/2}(kr)}{\sqrt{kr}} \tag{4.25}$$

($J_{l+1/2}$ are Bessel functions).

From the asymptotic expansion of the Bessel functions it follows that for $kr \gg 1$

$$g_l(kr) \approx 4\pi i^l \frac{\sin\left(kr - l\dfrac{\pi}{2}\right)}{kr}. \tag{4.26}$$

We note that the function $g_l$ satisfies the normalization condition

$$\int_0^\infty g_l(kr)\, g_l^*(k'r) r^2 dr = \frac{(2\pi)^3\, \delta(k-k')}{k^2}. \tag{4.27}$$

Indeed, in the last integral the important contributions come from large values of $r$. Therefore, we can utilize expression (4.26), and this leads to (4.27).

TABLE 3

*The Functions* $F(x)$, $(x = \cos \vartheta)$

| $M/j$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 | $-\frac{3}{8\pi}(x^2-1)$ | $-\frac{15}{8\pi}(x^4-x^2)$ | $-\frac{21}{64\pi}(25x^6-35x^4+11x^2-1)$ | $-\frac{45}{64\pi}(49x^8-91x^6+51x^4-9x^2)$ | $-\frac{165}{512\pi}(441x^{10}-1029x^8+826x^6-266x^4+29x^2-1)$ |
| 1 | $\frac{3}{16\pi}(x^2+1)$ | $\frac{5}{16\pi}(4x^4-3x^2+1)$ | $\frac{7}{256\pi}(225x^6-305x^4+111x^2+1)$ | $\frac{9}{256\pi}(784x^8-1463x^6+855x^4-153x^2+9)$ | $\frac{11}{1024\pi}(11{,}025x^{10}-26{,}019x^8+21{,}378x^6-7070x^4+813x^2+1)$ |
| 2 | | $-\frac{5}{16\pi}(x^4-1)$ | $-\frac{35}{128\pi}(9x^6-11x^4+3x^2-1)$ | $-\frac{9}{128\pi}(196x^8-371x^6+225x^4-49x^2-1)$ | $-\frac{77}{256\pi}(225x^{10}-549x^8+474x^6-170x^4+21x^2-1)$ |
| 3 | | | $\frac{105}{256\pi}(x^6-x^4-x^2+1)$ | $\frac{63}{256\pi}(16x^8-31x^6+15x^4-x^2+1)$ | $\frac{231}{2048\pi}(225x^{10}-579x^8+514x^6-190x^4+29x^2+1)$ |
| 4 | | | | $-\frac{63}{128x}(x^8-2x^6+2x^2-1)$ | $-\frac{231}{1024\pi}(25x^{10}-69x^8+58x^6-10x^4-3x^2-1)$ |
| 5 | | | | | $\frac{1155}{2048\pi}(x^{10}-3x^8+2x^6+2x^4-3x^2+1)$ |

By utilizing explicit expressions for the components of the vector spherical harmonics we obtain

$$\int Y_{jlM}\left(\frac{k}{k}\right) e^{ikr}\, do = g_l(kr)\, Y_{jlM}\left(\frac{r}{r}\right),$$

$$\int Y_{jM}^{(0)}\left(\frac{k}{k}\right) e^{ikr}\, do = g_j(kr)\, Y_{jM}^{(0)}\left(\frac{r}{r}\right),$$

$$\int Y_{jM}^{(1)}\left(\frac{k}{k}\right) e^{ikr}\, do = \frac{1}{\sqrt{2j+1}}\left\{\sqrt{j}\, g_{j+1}(kr)\, Y_{j,j+1,M}\left(\frac{r}{r}\right) + \sqrt{j+1}\, g_{j-1}(kr)\, Y_{j,j-1,M}\left(\frac{r}{r}\right)\right\}.$$

From this it follows that for states of electric type ($\lambda = 1$),

$$\mathfrak{E} = i\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2}\left\{\sqrt{\frac{j}{2j+1}}\, g_{j+1}(kr)\, Y_{j,j+1,M} + \sqrt{\frac{j+1}{2j+1}}\, g_{j-1}(kr)\, Y_{j,j-1,M}\right\} e^{-i\omega t}, \tag{4.28}$$

$$\mathfrak{H} = -\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2} g_j(kr)\, Y_{jjM}\, e^{-i\omega t},$$

while for states of magnetic type ($\lambda = 0$),

$$\mathfrak{E} = i\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2} g_j(kr)\, Y_{jjM}\, e^{-i\omega t},$$

$$\mathfrak{H} = -\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2}\left\{\sqrt{\frac{j}{2j+1}}\, g_{j+1}(kr)\, Y_{j,j+1,M} + \sqrt{\frac{j+1}{2j+1}}\, g_{j-1}(kr)\, Y_{j,j-1,M}\right\} e^{-i\omega t}. \tag{4.29}$$

We note that formulas (4.28) reduce to (4.29) if we make the substitution $\mathfrak{E} \to -i\mathfrak{H}$, $\mathfrak{H} \to i\mathfrak{E}$, which corresponds to the invariance of Maxwell's equations under the transformation $\boldsymbol{H} \to \boldsymbol{E}$, $\boldsymbol{E} \to -\boldsymbol{H}$.

Since the vector spherical harmonic $Y_{jM}^{(0)} \equiv Y_{jjM}$ is transverse, the magnetic field in states of electric type and the electric field in states of magnetic type are both transverse, i.e., we have

$$(\boldsymbol{r}\mathfrak{H})_{\lambda=1} = (\boldsymbol{r}\mathfrak{E})_{\lambda=0} = 0.$$

But the electric field in states of electric type and the magnetic field in states of magnetic type are expressed in terms of the vector spherical harmonics $Y_{j,\,j+1,\,M}$, which are not transverse. By utilizing (4.14), we can represent $\mathfrak{E}$ and $\mathfrak{H}$ as follows:

$$\mathfrak{E}_{(\lambda=1)} = i\sqrt{\frac{\delta}{2}}\left(\frac{\omega}{2\pi}\right)^{3/2}\frac{1}{2j+1}\left\{\sqrt{j(j+1)}\,(g_{j-1}-g_{j+1})\,Y_{jM}^{(-1)} + (jg_{j+1}+(j+1)g_{j-1})\,Y_{jM}^{(1)}\right\}e^{-i\omega t} \tag{4.30}$$

$$\mathfrak{H}_{(\lambda=0)} = i\mathfrak{E}_{(\lambda=1)}.$$

The first term in (4.30) gives a radial component of the electric field. At large distances, for $kr \gg 1$ (the "wave zone"), the radial component vanishes. Indeed, for $kr \gg 1$, $g_{l-1}(kr) = g_{l+1}(kr)$, in accordance with (4.26). Therefore the following asymptotic expressions hold for the fields:

$$\mathfrak{E}_{(\lambda)} = -i\mathfrak{H}_{(1-\lambda)} = i^{j+1-\lambda}\sqrt{\frac{\delta\omega}{\pi}}\,Y_{jM}^{(\lambda)}\,\frac{\sin\left(kr-\frac{\pi}{2}(j-\lambda)\right)}{r}. \tag{4.31}$$

An arbitrary electromagnetic field may be represented in the form of an expansion in terms of fields corresponding to photon states of definite angular momentum and parity:

$$\mathfrak{E} = \sum_{\omega jM\lambda} a_{\omega jM\lambda}\mathfrak{E}_{\omega jM\lambda},$$

where $a_{\omega jM\lambda}$ are determined by formula (4.21). In particular, the coefficients (4.22) define the expansion of a plane-polarized wave in terms of spherical waves

$$\mathfrak{E}_{p\mu} = \sum_{jM\lambda}\left(e_\mu Y_{jM}^{(\lambda)*}\left(\frac{p}{p}\right)\right)\mathfrak{E}_{pjM\lambda}$$

or

$$ee^{ikr} = \sum_{ljM}\left(eY_{jlM}^{*}\left(\frac{k}{k}\right)\right)g_l(kr)\,Y_{jlM}\left(\frac{r}{r}\right). \tag{4.32}$$

## § 5. Scattering of Photons by a System of Charges

### 5.1. *Incoming and Outgoing Waves*

So far we have been studying states of the free photon, i.e., such states as occur only in the absence of electric charges. However, this is already sufficient to draw some conclusions concerning the behavior of a photon interacting with charges.

We turn to formula (4.31), which gives the asymptotic form of those solutions of Maxwell's equations (1.1) which correspond to a photon of definite energy, angular momentum, and parity. These solutions may be interpreted as a sum of waves coming in to a central point and going out from the central point. Indeed, they may be represented as follows:

$$\begin{aligned}
\mathfrak{E} &= \mathfrak{E}^{\text{in}}+\mathfrak{E}^{\text{out}}, \\
\mathfrak{H} &= \mathfrak{H}^{\text{in}}+\mathfrak{H}^{\text{out}}, \\
\mathfrak{E}^{\text{in}} &= A\boldsymbol{Y}_{jM}^{(\lambda)}(\boldsymbol{n})\frac{e^{-ik(r+t)}}{r}, \\
\mathfrak{E}^{\text{out}} &= B\boldsymbol{Y}_{jM}^{(\lambda)}(\boldsymbol{n})\frac{e^{-ik(r-t)}}{r}, \\
\mathfrak{H}^{\text{in}} &= -[\boldsymbol{n}, \mathfrak{E}^{\text{in}}], \\
\mathfrak{H}^{\text{out}} &= [\boldsymbol{n}, \mathfrak{E}^{\text{out}}],
\end{aligned} \tag{5.1}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$, while $A$ and $B$ denote normalizing factors, such that

$$B = (-1)^{j+1-\lambda}A. \tag{5.2}$$

We note that each of the fields $\mathfrak{E}^{\text{in}}$, $\mathfrak{H}^{\text{in}}$ and $\mathfrak{E}^{\text{out}}$, $\mathfrak{H}^{\text{out}}$ separately satisfy Maxwell's equations (1.1). A similar decomposition into incoming and outgoing waves can be carried out likewise in the exact expressions (4.28) and (4.29). It corresponds to representing the Bessel functions occurring in the radial functions $g_l$ $(kr)$ (4.25) in the form of a sum of Hankel functions $H^{(1)}(kr)$ and $H^{(2)}(kr)$:

$$J_{l+1/2}(kr) = \tfrac{1}{2}[H_{l+1/2}^{(1)}(kr)+H_{l+1/2}^{(2)}(kr)].$$

However, the terms obtained in this manner will not individually satisfy Maxwell's equations (1.1) at $r = 0$, since the radial functions will in this case have singularities at the origin. The requirement that the solution remain finite at $r = 0$ imposes on the amplitudes of the incoming and the outgoing waves the relation (5.2).

We now consider the case when there exists a system of charges within a certain bounded region $r < r_0$ in the neighborhood of the origin. For the sake of brevity we shall refer to it as the scatterer. Outside the scatterer, equations (1.1) evidently hold and, therefore, the asymptotic solutions (5.1) are valid. However, since at $r = 0$ equations (1.1) do not hold, the amplitudes of the incoming and the outgoing waves will no longer be related by expression (5.2). The relation between $A$ and $B$ will in this case be determined by the properties of the scatterer. In the general case, the outgoing wave may correspond to a photon whose energy and angular momentum differ from the energy and the angular momentum characterizing the incoming wave. Therefore we write the relation between the amplitudes in the following form:

$$B_{\omega jM\lambda} = \sum_{\omega' j' M' \lambda'} (\omega jM\lambda|S|\omega' j' M' \lambda') A_{\omega' j' M' \lambda'}.$$

The matrix $S$ occurring in this expression is called the scattering matrix.

We note that this relationship does not cover all the processes which may occur because of the presence of the scatterer. For example, as a result of the interaction two or more photons may be emitted or a photon may be absorbed. Such processes cannot be investigated within the framework of the quantum mechanics of a single photon. Later (in Chapter IV) we shall study in greater detail a more general scattering matrix which includes all possible processes.

Here we shall confine ourselves to the case when the scatterer, first, has spherical symmetry, and, second, scatters photons elastically. In this case conservation of angular momentum, of parity, and of energy holds and, therefore, the scattering matrix must be diagonal and must not depend on $M$. Its elements can be represented in the following form:

$$(\omega jM\lambda|S|\omega' j' M' \lambda') = \delta_{\omega\omega'} \delta_{jj'} \delta_{MM'} \delta_{\lambda\lambda'} S_{j\lambda}(\omega) (-1)^{j+1-\lambda};$$

i.e.,

$$B = (-1)^{j+1-\lambda} S_{j\lambda}(\omega) A, \tag{5.3}$$

where the factor $(-1)^{j+1-\lambda}$ has been introduced in order that in the absence of the scatterer $S_{j\lambda}(\omega)$ should reduce to unity, and (5.3) should reduce to (5.2).

The property of elastic scattering means also that the energy flux of the outgoing wave is in absolute magnitude equal to the energy flux of the incoming wave. On the basis of the definition of energy flux density and of formula (5.1) it can be easily shown that the flux densities in the incoming and the outgoing waves are respectively given by

$$s^{\text{in}} = -2|A|^2|Y|^2\boldsymbol{n}; \qquad s^{\text{out}} = 2|B|^2|Y|^2\boldsymbol{n}.$$

Therefore from the equality $s^{\text{in}} = s^{\text{out}}$ it follows that $|A|^2 = |B|^2$; i.e., $|S_{j\lambda}(\omega)| = 1$ or

$$S_{j\lambda}(\omega) = e^{2i\delta_{j\lambda}(\omega)}, \tag{5.4}$$

where $\delta_{j\lambda}$ is real.

We can eliminate the restriction imposed on the properties of the scatterer if we alter somewhat the formulation of the problem, and, in particular, regard all processes other than elastic scattering as "absorption." In this we shall include both absorption proper, and also scattering processes involving a change in frequency or in angular momentum, or the conversion of a photon into two photons, and all other incoherent processes. We shall take all these absorption processes into account in a summary fashion as a reduction in the intensity of the outgoing wave in comparison with that of the incoming wave. In this case we can retain formula (5.3) if instead of (5.4) we take

$$S_{j\lambda} = \alpha_{j\lambda} e^{2i\delta_{j\lambda}}, \tag{5.5}$$

where

$$0 \leqslant \alpha_{j\lambda} \leqslant 1.$$

### 5.2. *Effective Scattering Cross Section*

We can now construct a formal theory for photon scattering analogous to the theory of scattering of particles by a central field in quantum mechanics.

We represent the photon field at large distances from the scatterer in the form of a plane wave and an outgoing spherical wave

$$\sqrt{2}\,\mathfrak{E} = \boldsymbol{e} e^{ikz} + \boldsymbol{F}(\boldsymbol{n}) \frac{e^{ikr}}{r}, \tag{5.6}$$

where $\boldsymbol{e}$ is the polarization unit vector; the normalization corresponds to unit energy flux density in the incident wave, $\boldsymbol{s}_0 \equiv \boldsymbol{n}_0$ ($\boldsymbol{n}_0$ is a unit vector directed along the $z$-axis, while the factor $\exp(-ikt)$ has been omitted for the sake of brevity). The energy flux through the area $r^2 do$ in the outgoing wave is equal to $ds = |\boldsymbol{F}|^2\, do$. Therefore, in accordance with its definition the differential cross section for elastic scattering is given by

$$d\sigma = |\boldsymbol{F}(\boldsymbol{n})|^2 do. \tag{5.7}$$

We now show that $\boldsymbol{F}$ can be expressed in terms of the elements of the scattering matrix $S_{j\lambda}$. In order to do this we write the general asymptotic expression for a field of some given frequency $k$ in the form of a series of incoming and outgoing waves. In accordance with (5.1) and (5.3) this has the form

$$\mathfrak{E} = \sum_{jM\lambda} a_{jM\lambda} \boldsymbol{Y}_{jM}^{(\lambda)} \frac{1}{r}\{(-1)^{j+1-\lambda} e^{-ikr} + S_{j\lambda} e^{ikr}\}. \tag{5.8}$$

We now choose the coefficients $a_{jM\lambda}$ in such a way as to pick out the incident wave in (5.8). In order to do this we make use of the plane-wave expansion (4.32). By utilizing (4.14) and the asymptotic representation of the radial function (4.26) we obtain

$$\boldsymbol{e}e^{ikz} = \frac{2\pi}{ikr}\sum_{jM\lambda}\left(\boldsymbol{e}\boldsymbol{Y}_{jM}^{(\lambda)*}(\boldsymbol{n}_0)\right)\boldsymbol{Y}_{jM}^{(\lambda)}(\boldsymbol{n})\{(-1)^{j+1-\lambda}e^{-ikr}+e^{ikr}\}. \tag{5.9}$$

We see that if we set

$$a_{jM\lambda} = \frac{2\pi}{ik}\left((\boldsymbol{e}\boldsymbol{Y}_{jM}^{(\lambda)*}(\boldsymbol{n}_0)\right),$$

then (5.8) assumes the form (5.6), with

$$\boldsymbol{F}(\boldsymbol{n}) = \frac{2\pi}{ik}\sum_{jM\lambda}\left(\boldsymbol{e}\boldsymbol{Y}_{jM}^{(\lambda)*}(\boldsymbol{n}_0)\right)(S_{j\lambda}-1)\,\boldsymbol{Y}_{jM}^{(\lambda)}(\boldsymbol{n}). \tag{5.10}$$

If we expand the polarization unit vector of the incident wave $\boldsymbol{e}$ in terms of the unit vectors $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$ corresponding to the $x$- and $y$-axes:

$$\boldsymbol{e} = e_1\boldsymbol{\chi}_1 + e_2\boldsymbol{\chi}_2,$$

and if we expand the vector amplitude of the scattered wave in terms of the unit vectors $\boldsymbol{\chi}_\vartheta$ and $\boldsymbol{\chi}_\varphi$ belonging to spherical polar coordinates $\vartheta$ and $\varphi$:

$$\boldsymbol{F} = F_1\boldsymbol{\chi}_\vartheta + F_2\boldsymbol{\chi}_\varphi,$$

then by utilizing the explicit expressions for the vector spherical harmonics (4.3), (4.11), (4.13), and (4.17), we can represent (5.10) in the following form:

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = R\begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

where $R$ is the following two-dimensional matrix:

$$R = R^{(1)} + R^{(0)},$$

$$R^{(\lambda)} = -i\begin{pmatrix} A_\lambda(\vartheta)\cos\varphi & A_\lambda(\vartheta)\sin\varphi \\ B_\lambda(\vartheta)\sin\varphi & -B_\lambda(\vartheta)\cos\varphi \end{pmatrix} \qquad (\lambda = 0, 1),$$

$$A_0(\vartheta) = \frac{1}{2k}\sum_j \frac{2j+1}{j(j+1)}(S_{j0}-1)\frac{dP_j(\cos\vartheta)}{d\cos\vartheta},$$

$$B_1(\vartheta) = \frac{1}{2k}\sum_j \frac{2j+1}{j(j+1)}(S_{j1}-1)\frac{dP_j(\cos\vartheta)}{d\cos\vartheta},$$

$$A_1(\vartheta) = \frac{1}{2k}\sum_j \frac{2j+1}{j(j+1)}(S_{j1}-1)\left\{j^2P_j(\cos\vartheta) - \frac{dP_{j-1}(\cos\vartheta)}{d\cos\vartheta}\right\},$$

$$B_0(\vartheta) = \frac{1}{2k}\sum_j \frac{2j+1}{j(j+1)}(S_{j0}-1)\left\{j^2P_j(\cos\vartheta) - \frac{dP_{j-1}(\cos\vartheta)}{d\cos\vartheta}\right\}$$

and $P_j$ is a Legendre polynomial. The matrix $R^{(1)}$ determines the scattering of waves of electric type, while $R^{(0)}$ determines the scattering of waves of magnetic type. If the polarization density matrix of the incident wave is given, then the differential cross section can be expressed in the form $d\sigma/do = \mathrm{Sp}\, R\varrho R^+$. The density matrix for the scattered wave $\varrho'$ is related to $\varrho$ by the expression $\varrho' = R\varrho R^+/\mathrm{Sp}\, R\varrho R^+$. The total elastic scattering cross section is given by

$$\sigma_s = \int |F|^2 do = \frac{\pi}{2k^2}\sum_{j\lambda}(2j+1)\,|S_{j\lambda}-1|^2. \tag{5.11}$$

We now determine the effective absorption cross section $\sigma_a$. It is evidently equal to the difference between the energy fluxes of incoming and outgoing waves:

$$\sigma_a = \sum_{iM\lambda}(1-|S_{j\lambda}|^2)|a_{iM\lambda}|^2,$$

where $a_{jM\lambda}$ are the coefficients in formula (5.8); i.e.,

$$\sigma_a = \frac{\pi}{2k^2}\sum_{j\lambda}(2j+1)\,(1-|S_{j\lambda}|^2). \tag{5.12}$$

### 5.3. *The Optical Theorem*

There exists a remarkable relationship between the total effective cross section, i.e., the cross section both for scattering and for absorption, and the forward scattering amplitude for elastic scattering.

We consider the total flux through a sphere of large radius $r$ of the energy associated with the field (5.6). In the absence of absorption, the total flux is equal to zero. The existence of absorption means that there is a net flux equal to $\sigma_a$ entering the sphere (if, as is the case in (5.6), the field is normalized to unit flux density in the plane wave). Thus, we have

$$\int snr^2 do = -\sigma_a. \tag{5.12'}$$

The flux density vector $s$ occurring in this expression which corresponds to the field (5.6) is given by

$$s = n_0 + n\frac{|F|^2}{r^2} + \frac{1}{2}\frac{n}{r}\{(eF^*)e^{-ik(r-z)} + (eF)e^{ik(r-z)}\}$$
$$+\frac{1}{2}\frac{n_0}{r}\{(eF)e^{ik(r-z)} + (eF^*)e^{-ik(r-z)}\}.$$

It is obvious that the first term in this expression will make no contribution to the integral (5.12′). The second term will give the total elastic scattering cross section. The integrals of the remaining terms can be considerably simplified if we first integrate them by parts. We consider, for example, the integral

$$\frac{1}{r}\int (eF)e^{ik(r-z)}r^2 do = -r\int_0^{2\pi} d\varphi \int_0^{\pi} \big(eF(\vartheta, \varphi)\big) e^{ikr(1-\cos\vartheta)} d\cos\vartheta$$
$$= e\frac{1}{ik}\int_0^{2\pi} d\varphi \left\{ e^{2ikr}F(\pi) - F(0) - \int_0^{\pi} e^{ikr(1-\cos\vartheta)} \frac{dF(\vartheta, \varphi)}{d\vartheta} d\cos\vartheta \right\}.$$

On further integration by parts the last integral over $\vartheta$ will give a term of the order of $1/r$, and therefore we can neglect it in comparison with the first term, which does not contain $r$ in the denominator. Further, the oscillating term containing $\exp(2ikr)$ can also be neglected. In order to justify this it is sufficient to average over a small frequency interval $\Delta k \sim 1/r$:

$$\frac{1}{r}\int eF(n)e^{ik(r-z)}r^2 do = -\frac{2\pi}{ik}eF(0).$$

On dealing with the other integrals in a similar manner we obtain

$$\frac{1}{r}\int (n_0 n)\left(eF(n)\right)e^{ik(r-z)}r^2 do = -\frac{2\pi}{ik}eF(0),$$

$$r\int (n_0 n)\left(eF^*(n)\right)e^{-ik(r-z)}do = r\int \left(eF^*(n)\right)e^{-ik(r-z)}do = \frac{2\pi}{ik}eF^*(0).$$

On substituting all these expressions into (5.12) we obtain

$$\sigma_t = \sigma_s + \sigma_a = \frac{2\pi}{ik}e\left(F(0) - F^*(0)\right)$$

or

$$\sigma_t = \frac{4\pi}{k}\operatorname{Im}\left(eF(0)\right). \tag{5.13}$$

This relationship is referred to as the optical theorem.

We see that the total cross section is proportional to the imaginary part of the forward scattering amplitude for elastic scattering (or, more accurately, to the amplitude of elastic scattering without a change in polarization).

The essence of the optical theorem may be interpreted in simple terms as follows. The total cross section is a measure of the weakening of the incident wave. On the other hand, this weakening may be regarded as the result of the interference of the primary wave with the wave scattered forward.

We note that the optical theorem may be applied not only to photons, but to any particles.

### 5.4. *Dispersion Relations*

The scattering amplitude $F$ has definite analytic properties which enable us to set up integral relations between its real and imaginary parts analogous to the relations obtained by Kramers and Kronig for the dielectric constant.

For the sake of simplicity we restrict ourselves to the case of small-angle scattering. We can then neglect the change in the polarization and take $F$ to be a scalar quantity.

We investigate the amplitude $F$ as a function of the frequency $\omega$ for a given change in the momentum of the photon as a result of scattering

$\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$ ($q = k\vartheta$, where $\vartheta$ is the scattering angle), $F = F(\omega, \boldsymbol{q})$. By utilizing the Cauchy formula we can write $F(\omega, \boldsymbol{q})$ in the following form:

$$F(\omega, \boldsymbol{q}) = \frac{1}{2\pi i} \oint \frac{F(\omega', \boldsymbol{q})}{\omega' - \omega} d\omega', \tag{5.14}$$

where the integration is carried out in the plane of the complex variable $\omega'$ along a contour surrounding the point $\omega' = \omega$, and $F(\omega', \boldsymbol{q})$ is a function of the complex variable $\omega'$ for a given parameter $\boldsymbol{q}$. If the function $F(\omega', \boldsymbol{q})$ has no singularities in the upper half-plane, then the contour of integration in the integral (5.14) may be chosen in the form shown in Fig. 1.

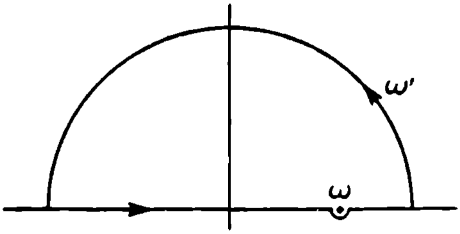

Fig. 1.

Since the integral along the semicircle going around the point $\omega' = \omega$ is equal to half of the total integral (5.14), we have

$$F(\omega, \boldsymbol{q}) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{F(\omega', \boldsymbol{q})}{\omega' - \omega} d\omega' + \frac{1}{\pi i} \int_{R} \frac{F(\omega', \boldsymbol{q})}{\omega' - \omega} d\omega',$$

where the principal value is taken of the first integral along the real axis, while the second integral is taken along a semicircle of infinitely large radius $R \to \infty$. If this latter integral tends to zero as $R \to \infty$, we have

$$F(\omega, \boldsymbol{q}) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{F(\omega', \boldsymbol{q})}{\omega' - \omega} d\omega'. \tag{5.15}$$

We note that as a result of the fact that the field is real $F(-\omega) = F^*(\omega)$.

We shall not investigate in detail here the analytic properties of the function $F(\omega, \boldsymbol{q})$, as can in principle be done on the basis of an investigation of the properties of the series in terms of which the scattering matrix $S$ is expressed (cf. Chapters IV, VII). We shall merely introduce certain simple considerations based on very general assumptions with respect to the properties of the propagation of electromagnetic waves (94). In the presence of a scatterer the field $\boldsymbol{E}$ must satisfy a certain homogeneous equation. It may be expressed in the form of a certain integral relationship (the Huygens-Kirchhoff principle)

$$E(\boldsymbol{r}, t) = \int K(\boldsymbol{r}, t; \boldsymbol{r}' t') E(\boldsymbol{r}', t')\, ds\, dt', \tag{5.16}$$

where $K$ is a function simply related to the Green's function for the equation of the field, and the integration is carried out over $t'$ within the light cone $(t-t')^2 > (\boldsymbol{r}-\boldsymbol{r}')^2$, and over an arbitrary closed surface surrounding the point $\boldsymbol{r}$. This surface may be chosen in the form of an infinite plane perpendicular to the incident wave and situated in the region where the incident wave is not distorted by the presence of the scatterer. Then $E(\boldsymbol{r}', t') \sim \exp(i\boldsymbol{k}\boldsymbol{r}' - i\omega t')$. We choose the point $\boldsymbol{r}$ in the asymptotic region; i.e., we take $E(\boldsymbol{r}, t) = (F/r) \exp[i(\boldsymbol{k}\boldsymbol{r} - \omega t)]$. In this expression we have

$$\frac{1}{r} F = \int K(\boldsymbol{r}, t; \boldsymbol{r}', t) e^{i\omega(t-t') - i(\boldsymbol{k}\boldsymbol{r} - \boldsymbol{k}\boldsymbol{r}')}\, ds\, dt'.$$

We introduce the following notation: $t-t' = \tau$; $\boldsymbol{r} = \boldsymbol{\rho} + \boldsymbol{r}'$; then $r = \varrho + (r'k'/k)$ $(r \gg r')$, and

$$\frac{1}{r} F = \int\limits_{\tau > \varrho} K(\boldsymbol{\rho}, \tau; \boldsymbol{r}, t) e^{i\omega(\tau - \varrho) - i\boldsymbol{q}\boldsymbol{r}'}\, dt'\, ds. \tag{5.17}$$

If $\boldsymbol{q}$ is held constant, then it follows from (5.17) that in the upper half of the $\omega$ plane $F$ has no singularities regarded as a function of $\omega$, since the coefficient of $\omega$ in the exponential is positive. From this it also follows that $F$ falls off exponentially along a semicircle of large radius in the upper half-plane.

Thus, the amplitude $F(\omega, \boldsymbol{q})$ regarded as a function of $\omega$ for fixed values of $\boldsymbol{q}$ possesses properties which guarantee that the dispersion relation (5.15) holds.

Formula (5.15) implies the following relations between the real and the imaginary parts of the scattering amplitude:

$$\operatorname{Re} F(\omega, q) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\operatorname{Im} F(\omega', q)}{\omega' - \omega}\, d\omega',$$
$$\operatorname{Im} F(\omega, q) = -\frac{1}{\pi} \int \frac{\operatorname{Re} F(\omega', q)}{\omega' - \omega}\, d\omega'. \tag{5.18}$$

We now evaluate the difference $F(\omega, q) - F(0, q)$. From (5.15) we obtain

$$F(\omega, q) - F(0, q) = \frac{\omega}{\pi i} \int\limits_{-\infty}^{\infty} \frac{F(\omega', q)}{\omega'(\omega' - \omega)}\, d\omega'.$$

For $F(0, q) = 0$ we obtain the following relations:

$$\operatorname{Re} F(\omega, q) = \frac{\omega}{\pi} \int\limits_{-\infty}^{\infty} \frac{\operatorname{Im} F(\omega', q)}{\omega'(\omega' - \omega)}\, d\omega',$$
$$\operatorname{Im} F(\omega', q) = -\frac{\omega}{\pi} \int\limits_{-\infty}^{\infty} \frac{\operatorname{Re} F(\omega', q)}{\omega'(\omega' - \omega)}\, d\omega'. \tag{5.19}$$

## § 6. The Photon Field Potentials

### 6.1. *Transverse, Longitudinal, and Scalar Potentials*

In the investigation of the interaction between photons and electric charges we shall later need expressions for the potentials of the electromagnetic field corresponding to a photon in some definite state. Therefore, by utilizing the expressions for the electromagnetic field of the photon obtained in the preceding sections we shall now determine the vector potential $A$ and the scalar potential $A_0$ corresponding to the photon. As is well known, $A$ and $A_0$ are related to the fields $E$ and $H$ by the following expressions:

$$E = -\frac{\partial A}{\partial t} - \nabla A_0,$$
$$H = \operatorname{curl} A. \tag{6.1}$$

If we expand $A$ and $A_0$ into Fourier integrals

$$A=\int A_k e^{ikr}dk, \qquad A_0=\int A_{0k}e^{ikr}dk,$$

then equations (6.1) will assume the form

$$\begin{aligned} E_k &= -\dot{A}_k - ikA_{0k}, \\ H_k &= i[kA_k]. \end{aligned} \tag{6.2}$$

Since $E_k$ and $H_k$ are transverse vectors, it is eonvenient to decompose $A_k$ likewise into transverse and longitudinal parts:

$$A_k = A_k^{(\perp)} + nA_k^{(\parallel)}, \tag{6.3}$$

where

$$n=\frac{k}{k}; \quad kA_k^{(\perp)}=0.$$

From (6.2) and (6.3) and the condition of transversality of the field $E_k k=0$ we obtain the following relation between $A_{0k}$ and $A_k^{(\parallel)}$:

$$ikA_{0k}+\dot{A}_k^{(\parallel)}=0. \tag{6.4}$$

By eliminating the magnetic field with the aid of expression (1.5) we find the following simple relations between $A_k^{(\perp)}$, $\dot{A}_k^{(\perp)}$ and $E_k$, $\dot{E}_k$:

$$\begin{aligned} A_k^{(\perp)} &= \frac{1}{k^2}\dot{E}_k, \\ \dot{A}_k^{(\perp)} &= -E_k. \end{aligned} \tag{6.5}$$

By utilizing these relations and also (1.6) we can express $A_k^{(\perp)}$ and $\dot{A}_k^{(\perp)}$ in terms of the photon wave function in momentum space $f_k$:

$$\begin{aligned} A_k^{(\perp)} &= \frac{-i}{\sqrt{2(2\pi)^3\omega}}(f_k-f_{-k}^*), \\ \dot{A}_k^{(\perp)} &= -\sqrt{\frac{\omega}{2(2\pi)^3}}\,(f_k+f_{-k}^*), \end{aligned} \tag{6.6}$$

where $\omega=|k|$.

The quantities $A_{0k}$ and $A_k^{(\parallel)}$ are related only by expression (6.4), while otherwise they are arbitrary. This gives expression to the property of gauge invariance according to which the fields $E$ and $H$ remain unaltered if an arbitrary function $\partial\eta/\partial t$ is added to the scalar potential, while simultaneously $-\nabla\eta$ is added to the vector potential. If, in addition, we impose on the potentials the Lorentz condition

$$\operatorname{div} A + \frac{\partial A_0}{\partial t} = 0,$$

then the following relation will hold between the quantities $A_{0k}$ and $A_k^{(\|)}$:

$$ikA_k^{(\|)} + \dot{A}_{0k} = 0. \tag{6.7}$$

Moreover, the fact that $A$ and $A_0$ are real imposes the following conditions:

$$A_{0k} = A_{0,-k}^*; \qquad A_k^{(\|)} = -A_{-k}^{(\|)*}.$$

In analogy to (6.6) we can introduce the complex function $f_{0k}$ defined in the following manner:

$$A_{0k} = -\frac{i}{\sqrt{2(2\pi)^3\omega}}(f_{0k} - f_{0,-k}^*),$$

$$\dot{A}_{0k} = -\sqrt{\frac{\omega}{2(2\pi)^3}}\,(f_{0k} + f_{0,-k}^*).$$

Then in accordance with (6.4) and (6.7) we obtain

$$A_k^{(\|)} = -\frac{i}{\sqrt{2(2\pi)^3\omega}}(f_{0k} + f_{0,-k}^*),$$

$$\dot{A}_k^{(\|)} = -\sqrt{\frac{\omega}{2(2\pi)^3}}\,(f_{0k} - f_{0,-k}^*).$$

On substituting (6.2) and (6.3) into the equation $\dot{E}_k = i[kH_k]$ and on expressing $A_{0k}$ and $A_k^{(\|)}$ in terms of $f_{0k}$, we find that $f_{0k}$, like $f_k$, satisfies the first-order equation

$$i\frac{\partial f_{0k}}{\partial t} = \omega f_{0k}.$$

With the aid of the functions $f_k$ and $f_{0k}$ the potentials of the field may be written in the following form:

$$
\begin{aligned}
A_0 &= \mathcal{A}_0+\mathcal{A}_0^*,\\
A &= \mathfrak{A}+\mathfrak{A}^*,\\
\mathcal{A}_0 &= -\frac{i}{\sqrt{2(2\pi)^3}}\int\frac{1}{\sqrt{\omega}}f_{0k}e^{ikr}\,dk, \qquad (6.8)\\
\mathfrak{A} &= -\frac{i}{\sqrt{2(2\pi)^3}}\int\frac{1}{\sqrt{\omega}}h_k e^{ikr}\,dk,
\end{aligned}
$$

where

$$h_k = f_k + n f_{0k}. \qquad (6.9)$$

### 6.2. *Longitudinally Polarized "Photon"*

The set of quantities $h_k$, $f_{0k}$, may be utilized in place of the quantities $f_k$ as the photon wave function. The expectation values of the energy, the momentum, and the angular momentum of the photon are related to $h_k$ and $f_{0k}$ by the following expressions:

$$
\begin{aligned}
\overline{w} &= \int\omega(h_k^*h_k - f_{0k}^*f_{0k})\,dk,\\
\overline{p} &= \int k(h_k^*h_k - f_{0k}^*f_{0k})\,dk,\\
\overline{M} &= \int (h_{k\alpha}^*\mathbf{M}h_{k\alpha} - f_{0k}^*\mathbf{L}f_{0k})\,dk,
\end{aligned}
$$

where $\mathbf{M}$ is the total angular momentum operator (3.6), while $\mathbf{L}$ is the orbital angular momentum operator (3.8). These quantities do not depend on the choice of $f_{0k}$. Therefore, the wave function $(h_k, f_{0k})$ does not satisfy the usual normalization condition.

We shall obtain the potentials of the electromagnetic field corresponding to a photon state of definite momentum or angular momentum, if we demand that the wave function $(h_k, f_{0k})$ must be an eigenfunction of the corresponding operators. Since we know the wave functions $f_k$ it remains for us only to determine the functions $f_{0k}$.

We can write the eigenfunctions $f_0(k, t)$ of the momentum operator in a form similar to (2.4):

$$f_{0p}(k, t) = C\frac{i}{\sqrt{\Delta}}e^{-ikt}\delta_{kp}, \qquad (6.10)$$

where $C$ is an arbitrary constant. For states of definite momentum the function $\boldsymbol{h}_k$ in accordance with (6.9), (6.10), and (2.4) has the form

$$\boldsymbol{h}(\boldsymbol{k}, t) = \frac{i}{\sqrt{\Delta}} \boldsymbol{e} e^{-ikt} \delta_{\boldsymbol{kp}}, \tag{6.11}$$

where $\boldsymbol{e}$ may contain an arbitrary longitudinal vector

$$\boldsymbol{e} = \boldsymbol{e}_\perp + C\boldsymbol{n}$$

($\boldsymbol{e}_\perp$ is the transverse unit vector).

It is sometimes useful to give an interpretation of the existence of a longitudinal component of the photon wave function by introducing an additional fictitious photon state with longitudinal polarization. Then, for a given momentum one can speak of three polarizations defined by the vectors $\chi_\mu (\mu = 1, 2, 3)$. Two of these are transverse, and are determined by formulas (2.5) or (2.6), while for the third one we have $\chi_3 = C\boldsymbol{n}$. Since in this case the photon wave function also has a scalar component equal to the longitudinal component, the energy and the momentum of the photon in this fictitious state are both equal to zero.

The eigenfunctions $f_{0k}$ of the angular momentum operator, which in the case of a scalar is identical with the orbital angular momentum operator, are given by spherical harmonics, which we shall write in a form analogous to (4.20):

$$f_{0jM\omega}(\boldsymbol{k}, t) = \frac{iC}{k\sqrt{\delta}} Y_{jM}(\boldsymbol{n}) e^{-ikt} \delta_{k\omega}. \tag{6.12}$$

We note that the parity of the function (6.12) is uniquely determined by its angular momentum and is equal to $I = (-1)^j$. It is identical with the parity of the states of electric type. The vector $\boldsymbol{h}_k$ for states of electric type is in accordance with (6.9), (6.12), and (4.20) equal to

$$\boldsymbol{h}(\boldsymbol{k}, t) = \frac{i}{k\sqrt{\delta}} \left( \boldsymbol{Y}_{jM}^{(1)}(\boldsymbol{n}) + C\boldsymbol{Y}_{jM}^{(-1)}(\boldsymbol{n}) \right) e^{-ikt} \delta_{k\omega}. \tag{6.13}$$

For states of magnetic type $I = (-1)^{j+1}$, and therefore the wave function has no scalar part:

$$f_{0k} = 0, \qquad \boldsymbol{h}_k = \boldsymbol{f}_k. \tag{6.14}$$

On the basis of (6.13) we can say that a state of electric type appears to be a superposition of a transverse photon state ($\lambda = +1$) and a "longitudinal photon" state ($\lambda = -1$).

We also note the following important fact. The free electromagnetic field may, in general, be described without making use of potentials. In such a case no fictitious states corresponding to photons with longitudinal polarization arise, since the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ have the property of transversality, and this property of the fields is relativistically invariant. However, for the investigation of the interaction between photons and electrons the introduction of potentials is necessary, since the potentials of the electromagnetic field occur in that part of the Lagrangian or the Hamiltonian which determines the interaction. At the same time the necessity arises of investigating "longitudinal" (and also "scalar") photons, since it is not possible to make the potentials transverse in a relativistically invariant manner. These questions will be discussed in detail in Chapters III and IV.

### 6.3. *Potentials for Plane and Spherical Waves*

With the aid of the wave functions $(\boldsymbol{h}_k, f_{0k})$ it is possible by utilizing formulas (6.8) to obtain expressions for the potentials of plane waves describing photon states of definite momentum and polarization, and of spherical waves describing photon states of definite angular momentum and parity. We write the potentials in the form

$$\boldsymbol{A} = \mathfrak{A} + \mathfrak{A}^*, \quad A_0 = \mathscr{A}_0 + \mathscr{A}_0^*.$$

It can be easily seen that the quantities $\mathfrak{A}$ and $\mathscr{A}_0$ have the following forms:

In the case of plane waves:

$$\begin{aligned} \mathfrak{A} &= \frac{1}{\sqrt{2\Omega\omega}}(\boldsymbol{e} + C\boldsymbol{n})\, e^{i(\boldsymbol{pr} - \omega t)}, \\ \mathscr{A}_0 &= \frac{1}{\sqrt{2\Omega\omega}}\, C e^{i(\boldsymbol{pr} - \omega t)}, \\ \omega &= |\boldsymbol{p}|; \end{aligned} \tag{6.15}$$

In the case of spherical waves of magnetic type:

$$\mathfrak{A} = \frac{1}{i\omega}\mathfrak{E} = \frac{\sqrt{p}}{4\pi\sqrt{R}}\, g_j(pr)\, \boldsymbol{Y}_{jM}^{(0)}\left(\frac{\boldsymbol{r}}{r}\right) e^{-i\omega t}, \tag{6.16}$$

$$\mathscr{A}_0 = 0;$$

In the case of spherical waves of electric type:

$$\begin{aligned}\mathfrak{A} = \frac{1}{4\pi}\sqrt{\frac{\omega}{R}}\Bigg\{&\sqrt{\frac{j}{2j+1}}\, g_{j+1}(pr)\,\boldsymbol{Y}_{j,j+1,M}\left(\frac{\boldsymbol{r}}{r}\right)\\ &+\sqrt{\frac{j+1}{2j+1}}\, g_{j-1}(pr)\,\boldsymbol{Y}_{j,j-1,M}\left(\frac{\boldsymbol{r}}{r}\right)\\ &-C\Bigg[\sqrt{\frac{j+1}{2j+1}}\, g_{j+1}(pr)\,\boldsymbol{Y}_{j,j+1,M}\left(\frac{\boldsymbol{r}}{r}\right)\\ &-\sqrt{\frac{j}{2j+1}}\, g_{j-1}(pr)\,\boldsymbol{Y}_{j,j-1,M}\left(\frac{\boldsymbol{r}}{r}\right)\Bigg]\Bigg\}e^{-i\omega t},\\ \mathcal{A}_0 = \frac{C}{4\pi}\sqrt{\frac{\omega}{R}}\,& g_j(pr)\,Y_{jM}\left(\frac{\boldsymbol{r}}{r}\right)e^{-i\omega t}.\end{aligned} \tag{6.17}$$

By making different choices for the arbitrary constant $C$ which determines the amount of admixture of the "longitudinal" state, we can make expression (6.17) assume different forms. Thus, for $C = 0$ (86) $\mathfrak{A}$ differs from $\mathfrak{E}$ only by the factor $1/i\omega$. For $C = \left(j/(j+1)\right)^{1/2}$ the expression for $\mathfrak{A}$ reduces to a single term and contains only the function $g_{j-1}$, which is advantageous for a number of applications (cf. § 27 and § 40):

$$\begin{aligned}\mathfrak{A} &= \frac{1}{4\pi}\sqrt{\frac{\omega}{R}}\sqrt{\frac{2j+1}{j+1}}\, g_{j-1}\,\boldsymbol{Y}_{j,j-1,M}\,e^{-i\omega t},\\ \mathcal{A}_0 &= \frac{1}{4\pi}\sqrt{\frac{\omega}{R}}\sqrt{\frac{j}{j+1}}\, g_j\,Y_{jM}\,e^{-i\omega t}.\end{aligned} \tag{6.18}$$

## § 7. System of Photons

### 7.1. *Wave Function for a System of Two Photons*

So far we have been considering a single photon and the electromagnetic field corresponding to it. However, we can generalize the method just presented for introducing the photon wave function in momentum space, and we can investigate also an arbitrary number of photons in a manner analogous to the way a system of particles is treated in quantum mechanics.

We investigate in greater detail a system of two photons. The arguments of the wave function of such a system are the momenta $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and the spin variables $\alpha_1$, $\alpha_2$ for the two particles, and also the time $t$. We denote this wave function by

$$f(\boldsymbol{k}_1, \alpha_1; \boldsymbol{k}_2, \alpha_2; t).$$

The square of the absolute value of the wave function determines the probability that the first photon has the momentum $k_1$ and is polarized along the $\alpha_1$ axis, while the second photon has the momentum $\boldsymbol{k}_2$ and is polarized along the $\alpha_2$ axis.

Instead of representing this wave function as a scalar in the space defined by $\boldsymbol{k}_1, \boldsymbol{k}_2; \alpha_1, \alpha_2$ we can regard it as a tensor function in the space of the momenta $\boldsymbol{k}_1$, $\boldsymbol{k}_2$:

$$f(\boldsymbol{k}_1, \alpha_1; \boldsymbol{k}_2, \alpha_2; t) \equiv f_{\alpha_1 \alpha_2}(\boldsymbol{k}_1, \boldsymbol{k}_2; t) \tag{7.1}$$

in a manner analogous to the way in which we represented the wave function for a single photon in §3 either as a scalar $f(\boldsymbol{k}, \alpha, t)$ or as a vector $f_\alpha(\boldsymbol{k}, t)$.

In first approximation the photons may be regarded as non-interacting particles. (In fact, owing to the interaction between photons and electrons, a weak interaction between photons appears, cf. § 54.) In such a case the energy of the two-photon system is equal to the sum of the energies of the photons, and their wave function satisfies the equation

$$i\frac{\partial f}{\partial t} = \mathrm{w} f,$$

where $\mathrm{w} = \mathrm{w}(\boldsymbol{k}_1)+\mathrm{w}(\boldsymbol{k}_2)$, while $\mathrm{w}(\boldsymbol{k}_1)$ is the Hamiltonian for a photon defined by formula (1.11′).

In addition to this equation the wave function must also satisfy two other conditions. The first of these (the initial condition imposed on the equation for $f$) requires that the polarization of both photons should be transverse. It may be formulated in the form of the following expressions:

$$\begin{aligned} (\boldsymbol{k}_1)_{\alpha_1} f_{\alpha_1 \alpha_2} &= 0, \\ (\boldsymbol{k}_2)_{\alpha_2} f_{\alpha_1 \alpha_2} &= 0. \end{aligned} \tag{7.2}$$

The second condition represents a symmetry condition and follows from the identity of the photons. Photons obey Bose-Einstein statistics, and their wave function must be symmetric with respect to an interchange of the particles. Therefore (omitting for the sake of brevity the argument $t$), we have

$$f(\boldsymbol{k}_1, \alpha_1, \boldsymbol{k}_2, \alpha_2) = f(\boldsymbol{k}_2, \alpha_2, \boldsymbol{k}_1, \alpha_1), \tag{7.3}$$

or, in tensor form,

$$f_{\alpha_1\alpha_2}(\boldsymbol{k}_1, \boldsymbol{k}_2) = f_{\alpha_2\alpha_1}(\boldsymbol{k}_2, \boldsymbol{k}_1).$$

Instead of the variables $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ we can introduce the total momentum of the system (momentum of the center of inertia) $\boldsymbol{K}$:

$$\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$$

and the relative momentum $2\boldsymbol{k}$:

$$\boldsymbol{k} = \frac{\boldsymbol{k}_1 - \boldsymbol{k}_2}{2}.$$

In terms of these variables the wave function of the two-photon system may be written in the following form:

$$f(\boldsymbol{k}_1, \alpha_1, \boldsymbol{k}_2, \alpha_2) = \varphi(\boldsymbol{K}) f(\boldsymbol{k}; \alpha_1, \alpha_2).$$

In virtue of the conditions of transversality (7.2) and of symmetry (7.3) the function $f(\boldsymbol{k}; \alpha_1, \alpha_2) \equiv f_{\alpha_1\alpha_2}(\boldsymbol{k})$ satisfies the following relations:

$$f_{\alpha_1\alpha_2} k_{\alpha_1} = f_{\alpha_1\alpha_2} k_{\alpha_2} = 0, \tag{7.4}$$

$$f(\boldsymbol{k}; \alpha_1, \alpha_2) = f(-\boldsymbol{k}; \alpha_2, \alpha_1). \tag{7.5}$$

If we consider states of definite total momentum $\boldsymbol{K}$, then we can always (with the exception of the case when $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are parallel) transform to a coordinate system in which

$$\boldsymbol{K} = 0, \quad \boldsymbol{k}_1 = \boldsymbol{k}; \quad \boldsymbol{k}_2 = -\boldsymbol{k}.$$

The function $f(\boldsymbol{k}, \alpha_1, \alpha_2)$ represents the two-photon wave function in this coordinate system.

We note that two photons of total momentum equal to zero are experimentally observed in the case of the decay of a neutral system at rest (neutral $\pi$-meson, positronium).

### 7.2. *Even and Odd States of Two Photons*

We now set ourselves the problem of finding the wave functions of the system of two photons of definite angular momentum and parity (113). We take the momentum of the center of inertia $\boldsymbol{K}$ equal to zero.

The angular momentum operator for the system M is the sum of the operators for the orbital and the spin angular momenta:

$$\mathrm{M} = \mathrm{L} + \mathrm{s},$$

where L is the orbital angular momentum operator for relative motion $\mathrm{L} = -i[k\nabla_k]$, and s is the sum of the spin operators for the two photons $\mathrm{s} = \mathrm{s}_1 + \mathrm{s}_2$.

As in the case of a single photon, the eigenfunctions of the operators $\mathrm{L}^2$ and $\mathrm{L}_z$ are given by the spherical harmonics $Y_{lm}(\boldsymbol{n})$.

The eigenvalues of the square of the spin angular momentum are equal to $s^2 = s(s+1)$, where in accordance with the rules for the addition of angular momenta the quantum number $s$ may assume the values $s = 0, 1, 2$. Since for a given value of $s$ the eigenvalue of the operator for the component of the spin assumes the values $\mu = -s, -s+1, \ldots, s$, there exists for a two-photon system a total of nine different spin wave functions which we denote by $\chi_{s\mu}(\alpha_1, \alpha_2)$. They are bilinear combinations of the spin wave functions of the two photons $\chi_{\mu 1}(\alpha_1)$ and $\chi_{\mu 2}(\alpha_2)$.

These combinations may be written in the form of six symmetric functions $\chi_{\mu 1}(\alpha_1)\chi_{\mu 2}(\alpha_2)+\chi_{\mu 1}(\alpha_2)\chi_{\mu 2}(\alpha_1)$ and three antisymmetric functions $\chi_{\mu 1}(\alpha_1)\chi_{\mu 2}(\alpha_2)-\chi_{\mu 1}(\alpha_2)\chi_{\mu 2}(\alpha_1)$. Since states with different values of the quantum number $\mu$ for a given quantum number $s$ must have the same symmetry, we must identify the three antisymmetric functions with the functions $\chi_{s\mu}$ corresponding to the value $s = 1$, while the six symmetric functions must be ascribed to the values $s = 0, 2$. Of these, the four functions for which $\mu_1+\mu_2 \neq 0$ must correspond to the value $s = 2$, while the remaining two functions

$$\chi_0(\alpha_1)\chi_0(\alpha_2) \quad \text{and} \quad \chi_1(\alpha_1)\chi_{-1}(\alpha_2)+\chi_{-1}(\alpha_1)\chi_1(\alpha_2)$$

represent linear combinations of the wave functions $\chi_{20}$ and $\chi_{00}$.

We note that the division of spin states into symmetric and antisymmetric ones has a real meaning. As regards the classification of states according to the value of spin, it has no deep physical meaning, since, as will be seen later, the condition of transversality requires the superposition of functions with $s = 0$ and $s = 2$.

The wave function for a system of two photons with a definite value of the angular momentum $j$ and of its component $M$ along the $z$-axis $f_{jM}(k, \alpha_1, \alpha_2)$ represents, in accordance with the rules for the addition of angular momenta, a linear combination of products of orbital and spin functions $Y_{lm}(\boldsymbol{k}/k)\chi_{s\mu}(\alpha_1, \alpha_2)$.

States of definite angular momentum can be classified with respect to their parity. The inversion operator I acts only on the orbital part of the wave function:

$$\mathrm{I} f_{\alpha_1\alpha_2}(\boldsymbol{k}) = f_{\alpha_1\alpha_2}(-\boldsymbol{k}),$$

i.e.,

$$\mathrm{I} f(\boldsymbol{k}, \alpha_1, \alpha_2) = f(-\boldsymbol{k}, \alpha_1, \alpha_2). \tag{7.6}$$

Since

$$Y_{lm}(-\boldsymbol{n}) = (-1)^l Y_{lm}(\boldsymbol{n}), \quad \boldsymbol{n} = \frac{\boldsymbol{k}}{k},$$

the parity of the state is determined by the quantity $\mathrm{I} = (-1)^l$. On the other hand, the replacement of $\boldsymbol{k}$ by $-\boldsymbol{k}$ is equivalent to an interchange of the momenta of the two photons. From the condition of symmetry of the wave function (7.5) it follows that $Y_{lm}(\boldsymbol{n})\chi_{s\mu}(\alpha_1, \alpha_2) = Y_{lm}(-\boldsymbol{n})\chi_{s\mu}(\alpha_2, \alpha_1)$. We thus see that there exists a connection between the parity of the state and the symmetry of the spin function:

$$\chi_{s\mu}(\alpha_1, \alpha_2) = (-1)^l \chi_{s\mu}(\alpha_2, \alpha_1) \tag{7.7}$$

i.e., even states ($l = 2n$, where $n$ is an integer) are symmetric with respect to the spin variables, while odd states ($l = 2n+1$) are antisymmetric. In other words, the tensor $f_{\alpha_1\alpha_2}$ is symmetric in the case of even states and antisymmetric in the case of odd states. In the former case it has six components, while in the latter case it has three components, which is in agreement with the number of symmetric and antisymmetric spin functions.

### 7.3. *Classification of the States of Two Photons of Definite Angular Momentum*

The state of a single photon of given energy is completely determined by its angular momentum and parity. To each set of the quantum numbers $\omega, j, M, I$ there corresponds one photon state (with the exception that the state for $j = 0$ is missing). In the case of two photons the situation is, generally speaking, different. Let us count the number of states corresponding to a two-photon system when the angular momentum and the parity are specified. We shall denote by $N_{jM}^{(+)}$ the number of even states of given $j$ and $M$, and by $N_{jM}^{(-)}$ the number of odd states.

We begin by considering the odd states. As is well known, an antisymmetric tensor of the second rank may be written in the following form:

$$f_{\alpha_1\alpha_2} = e_{\alpha_1\alpha_2\gamma} F_\gamma, \tag{7.8}$$

where $e_{\alpha_1\alpha_2\gamma}$ is the unit antisymmetric tensor of the third rank, while $F_\gamma$ is some vector. Therefore, the odd states may be described by a vector wave function $\boldsymbol{F}(\boldsymbol{k})$. The properties of the vector $\boldsymbol{F}$ differ significantly from the properties of the single photon vector wave function $\boldsymbol{f}_k$. Indeed, the condition of transversality (7.4) applied to the antisymmetric tensor (7.8) yields

$$e_{\alpha\beta\gamma} n_\beta F_\gamma = 0, \quad \text{i.e.,} \quad [\boldsymbol{nF}] = 0.$$

Thus, the vector $\boldsymbol{F}$ is longitudinal in contrast to the transverse vector $\boldsymbol{f}_k$.

According to the results of § 4 there exist three linearly independent vector functions $\boldsymbol{Y}_{jM}^{(\lambda)}$ corresponding to given values of the quantum numbers $j$ and $M$, with two of these being transverse ($\lambda = 0, 1$), and one of them being longitudinal ($\lambda = -1$). From this it follows that for a given $j$ and $M$ the vector $\boldsymbol{F}$ has the following form:

$$\boldsymbol{F} = \boldsymbol{Y}_{jM}^{(-1)}(\boldsymbol{n}).$$

In accordance with (4.10) this function is odd only when $j$ is even. Thus we have

$$N_{jM}^{(-)} = \begin{cases} 1 & \text{for} \quad j = 2n, \\ 0 & \text{for} \quad j = 2n+1. \end{cases} \tag{7.9}$$

We note that since the tensor wave function for a system of two photons $f_{\alpha_1\alpha_2}$ is bilinear in the components of the photon vector functions $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, then it follows from the antisymmetry property of the tensor $f_{\alpha_1\alpha_2}$ that the vectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ have no components along the same axis. This means that the polarizations of two photons in an odd state are mutually perpendicular. The neutral $\pi$-meson and positronium in the ground state (cf. 33) can serve as examples of odd systems decaying into two photons.

We now proceed to investigate the even states. As we have seen earlier, they correspond to the values of $s = 0, 2$. In accordance with the rules of addition of angular momenta for given values of $j$ and $M$ there are six different wave functions corresponding to different values of $l$:

$$l = j\pm 1, j\pm 2, j \quad \text{for} \quad s = 2$$

and

$$l = j \quad \text{for} \quad s = 0.$$

Since $I = (-1)^l$, then of all these functions we must retain only those for which $l = 2n$. The number of these functions is equal to four when $j = 2n$ and is equal to two when $j = 2n+1$.

In order to obtain $N_{jM}^{(+)}$ it is also necessary to take into account the transversality condition (7.4). In order to do this we consider any one of the tensor wave functions $f_{\alpha_1\alpha_2}$ corresponding to given values of $j$ and $M$. We construct the vector $G_\alpha = f_{\alpha\beta} n_\beta$ which represents the longitudinal component of the wave function. Since the vector $\boldsymbol{n}$ is spherically symmetric, the vector $\boldsymbol{G}$, like $f_{\alpha_1\alpha_2}$, is an eigenfunction of the angular momentum operator with the same eigenvalues $j$ and $M$. But for given values of $j$ and $M$ there exist three linearly independent vector spherical harmonics $Y_{jM}^{(\lambda)}$. Therefore, of the six tensor functions $f_{\alpha_1\alpha_2}$, only three can have longitudinal components different from zero, while the remaining three satisfy the condition of transversality. The parity of the former is identical with the parity of the vector spherical harmonics corresponding to them. Therefore, the number of even, linearly independent tensor functions which have a longitudinal component different from zero is equal to two when $j = 2n$ and is equal to one when $j = 2n-1$. On subtracting this number from the total number of even functions we obtain the number of even states which satisfy the condition of transversality:

$$\begin{aligned} N_{jM}^{(+)} &= 2, \quad j = 2n, \\ N_{jM}^{(+)} &= 1, \quad j = 2n+1. \end{aligned} \tag{7.10}$$

The cases of $j = 0$ and $j = 1$ require separate consideration. When $j = 0$ the rules for the addition of angular momenta allow only two states (both even),

$$l = 0,\ s = 0; \qquad l = 2,\ s = 2.$$

On the other hand, when $j = 0$ there exists only one vector spherical harmonic $Y_{010}^{(-1)}$ (even). This means that of the two tensors $f_{\alpha_1\alpha_2}$ only one satisfies the transversality condition. Thus, there exists only one even state of the system with angular momentum equal to zero. In this case, as may be easily checked, the wave function of the system satisfying the conditions of symmetry and transversality is given by the function

$$f_{\alpha_1\alpha_2} = \text{const}(n_{\alpha_1} n_{\alpha_2} - \delta_{\alpha_1\alpha_2}). \tag{7.11}$$

It may be concluded from (7.11) that the polarizations of the photons in this state are parallel. Indeed, let the $z$-axis be directed along $\boldsymbol{k}$, then we have $f_{xy} = 0$.

When $j = 1$ the rules for the addition of angular momenta allow only one even state: $l = 2$; $s = 2$. But the corresponding tensor function has a longitudinal component, since of the three vector spherical harmonics $Y_{1M}^{(\lambda)}$ only $Y_{1M}^{(0)}$ is even. This means that there are no even states satisfying the transversality condition when $j = 1$. In accordance with (7.9) when $j = 1$ there are also no odd states. Thus, a system of two photons, generally speaking, cannot be in a state of angular momentum equal to unity, while a system having an angular momentum equal to unity cannot decay into two photons.

TABLE 4

| Angular momentum, $j$ | No. of even states, $N_{jM}^{(+)}$. | No. of odd states, $N_{jM}^{(-)}$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | — | — |
| $2n$ | 2 | 1 |
| $2n+1$ | 1 | — |

In the accompanying table a summary is given of the classification of the states of a two-photon system.

### 7.4. *Wave Function for a System of an Arbitrary Number of Photons*

A system of an arbitrary number $N$ of photons can be described by the wave function $f(\boldsymbol{k}_1, a_1; \boldsymbol{k}_2, a_2; \boldsymbol{k}_3, a_3; \dots; \boldsymbol{k}_N, a_N)$ which depends on the photon momenta $\boldsymbol{k}_i$ and the variables $a_i$ which determine the photon polarizations (each variable $a_i$ assumes three values: $a_i = 1,2,3$). The wave function must be symmetric with respect to interchanges of particles; i.e.,

$$f(\dots; \boldsymbol{k}_i, a_i; \dots; \boldsymbol{k}_j, a_j; \dots) = f(\dots; \boldsymbol{k}_j, a_j; \dots; \boldsymbol{k}_i, a_i; \dots),$$

and, moreover, it must satisfy the transversality conditions

$$\sum_{a_i} f(\dots \boldsymbol{k}_i, a_i \dots)\,(\boldsymbol{k}_i)_{a_i} = 0 \qquad (i = 1, 2, \dots, N),$$

where the summation is taken over the three values of the variable

$a_i$ (the number of transversality conditions is equal to the number of photons).

If the state of an individual photon is characterized by the set of quantum numbers $r$ (for example, $\boldsymbol{p}, \mu$, or $\omega, j, M, \lambda$), then the state of the system will be completely determined by specifying the "occupation numbers" $N_r$, i.e., by indicating for each individual state $r$ the number of photons $N_r$ occupying this state. The wave function of the system with a specified distribution of occupation numbers can be expressed in terms of the individual photon wave functions $f_r(\boldsymbol{k}, a)$:

$$f_{N_{r_1}, N_{r_2}\dots}(\boldsymbol{k}_1, a_1; \dots; \boldsymbol{k}_i, a_i; \dots) = \left(\frac{N_{r_1}!\, N_{r_2}! \dots}{N!}\right)^{1/2} \sum f_{r_1}(\boldsymbol{k}_1, a_1) f_{r_2}(\boldsymbol{k}_2, a_2) \dots$$

where $N = \sum\limits_{r_i} N_{r_i}$, and the summation is taken over all the permutations of the subscripts $r$. The function $f_N$ is normalized by the condition

$$\sum_{a_1 \dots a_N} \int |f_{N_{r_1}} \dots (\boldsymbol{k}_1, a_1; \dots; \boldsymbol{k}_N, a_N)|^2 d\boldsymbol{k}_1 \dots d\boldsymbol{k}_N = 1.$$

An arbitrary state of a system of $N$ photons can be represented in the form of a superposition of states with definite distributions of occupation numbers:

$$f_N(\boldsymbol{k}_1, a_1; \dots; \boldsymbol{k}_N, a_N) = \sum_{N_{r_1} \dots N_{r_N}} \Phi(N_{r_1}, \dots, N_{r_i}, \dots) f_{N_{r_1}}, \dots (\boldsymbol{k}_1, a_1; \dots). \tag{7.12}$$

When photons interact with charges the number of photons can change, since photons can be emitted and absorbed. Therefore, in the general case, we have to consider states with an indefinite number of particles. Such states are described by a set of wave functions $f_N$

$$\left\{\begin{array}{l} a_0 \\ a_1 f_1(\boldsymbol{k}_1, a) \\ a_2 f_2(\boldsymbol{k}_1, a_1, \boldsymbol{k}_2, a_2) \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \\ a_N f_N(\boldsymbol{k}_1, a_1; \dots; \boldsymbol{k}_N, a_N) \end{array}\right\} \tag{7.13}$$

where the square of the absolute value of $a_N$ determines the probability that $N$ photons are present ($\sum|a_N|^2 = 1$).

Such a column is equivalent to a certain functional with the aid of which we can describe a system containing an indefinite number of photons. This functional is determined in the following manner (64). Let $\Omega_N(\varphi) = \frac{1}{\sqrt{N!}}\int f_N(\xi_1 \ldots \xi_N)\varphi(\xi_1)\ldots\varphi(\xi_N)d\xi_1\ldots d\xi_N$, where $\varphi(\xi_n)$ is an arbitrary function of the variables $k_n$ and $a_n$ which are for brevity denoted by $\xi_n$ (integration over $\xi_n$ also includes summation over $a_n$). Then, if we specify the functional $\Omega(\varphi) = \sum_{N=0}^{\infty} a_N\Omega_N(\varphi)$, this is equivalent to specifying the column (7.13). Indeed,

$$a_N f_N(\xi_1 \ldots \xi_N) = \sqrt{N!}\,\frac{\delta^N\Omega(\varphi)}{\delta\varphi(\xi_1)\delta\varphi(\xi_2)\ldots\delta\varphi(\xi_N)}\bigg|_{\varphi=0},$$

where $\delta/\delta\varphi$ denotes a variational derivative.[1]

As an example we consider the case of $N$ photons all having the same momentum and polarization $\xi$. In this case $a_N = 1$, $a_i = 0\,(i \neq N)$, and therefore

$$\Omega(\varphi) = \frac{1}{\sqrt{N!}}\varphi^N(\xi).$$

As is well known in quantum mechanics, in order to describe a system of identical particles obeying Bose-Einstein statistics we can go over to a representation in which the role of the wave function is played by the quantities $\Phi$ in (7.12), while the independent variables are the occupation numbers $N_r$. This method of description by means of a wave function in occupation number space is called the method of second quantization. The transition to this representation is carried out by

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[1] The variational derivative may either be obtained from the expression for the variation of the functional

$$\delta\Omega = \int \frac{\delta\Omega}{\delta\varphi(\xi)}\delta\varphi(\xi)\,d\xi$$

where $\delta\varphi$ is the variation of the function $\varphi$, or it may be defined as

$$\frac{\delta\Omega}{\delta\varphi(\xi)} \equiv \lim_{\eta\to 0}\frac{1}{\eta}\{\Omega[\varphi(\xi)+\eta\delta(\xi'-\xi)]-\Omega[\varphi(\xi)]\}.$$

means of introducing operators for the creation and annihilation of a particle in the state $r$ (cf., for example, (119); (27)). These operators, which we denote by $c_r^+$ and $c_r$, are defined by the following expressions:

$$\begin{aligned} c_r\Phi(\ldots, N_r, \ldots) &= \sqrt{N_r}\Phi(\ldots, N_r-1, \ldots), \\ c_r^+\Phi(\ldots, N_r, \ldots) &= \sqrt{N_r+1}\Phi(\ldots, N_r+1, \ldots). \end{aligned} \tag{7.14}$$

In Chapter III we shall introduce these operators and study their properties in detail from another point of view.

## § 8. *L*-Vectors and Spherical Harmonics

### 8.1. *Irreducible Tensors*

In this section we shall investigate the basic properties of quantities which transform under spatial rotations in accordance with the irreducible representations of the rotation group. Such quantities are, for example, the wave functions of a particle with a given value of angular momentum, and with different values of its components along the $z$-axis. In future we shall encounter other quantities of a similar type. For brevity we shall call them $L$-vectors. An $L$-vector has $2L+1$ components. (The transformation matrix for an $L$-vector corresponding to a rotation of the coordinate system will therefore have $2L+1$ rows and columns.) Since $(2L+1)$ is an integer, $L$ may be either integral or half-integral.

We note that if $L$ is an integer, then the $L$-vector is equivalent to an ordinary three-dimensional tensor. In the general case, a tensor of the $L$th rank has $3^L$-components, whose transformations are reducible. However, by means of symmetrization, antisymmetrization and simplification it is always possible to select sets of a smaller number of components which transform under rotations independently of the others. Thus, for example, from the nine components of the second rank tensor $T_{ik}$ it is possible to pick out its trace (scalar), and its antisymmetric part, consisting of three quantities which transform like the components of a vector (a tensor of the first rank), after which there remain five independent components $T_{ik}^0$ of a traceless symmetric tensor:

$$T_{ik}^0 = \tfrac{1}{2}(T_{ik}+T_{ki})-\tfrac{1}{3}T_{ll}\delta_{ik}.$$

Linear combinations of the components $T_{ik}^0$ form an $L$-vector ($L = 2$).

It may be easily shown that the number of independent components of an $L$-th rank tensor in $n$-dimensional space $T^0_{ikl}\dots$ $(i, k, l, \dots = 1, 2, \dots, n)$, which is symmetric with respect to any pair of subscripts $(T^0_{ikl}\dots = T^0_{kil}\dots, T^0_{ikl}\dots = T^0_{lki}\dots)$, and which vanishes on contraction with respect to any two subscripts $(T^0_{iil}\dots = T^0_{ikk}\dots = 0)$, is given by

$$N = \frac{(n+L-1)!}{(n-1)!L!} - \frac{(n+L-3)!}{(n-1)!(L-2)!}.$$

In three-dimensional space $N = 2L+1$; when $n = 2, N = 2$; when $n = 4$, $N = (L+1)^2$.

We denote the components of the $L$-vector $F$ by $F^M (M = -L, -L+1, \dots, L)$. The set of basis vectors, i.e., the set of unit vectors in the $(2L+1)$-dimensional $L$-vector space, can be so selected that the invariant bilinear form in the components of the $L$-vectors $F$ and $G$ (with the same values of $L$), i.e., their scalar product, will have the following form:

$$(F, G) = \sum_{M=-L}^{L} (-1)^M F^M G^{-M}. \tag{8.1}$$

For $L = \frac{1}{2}$ the quantities $F^M$ are identical with the components of a spinor, while for $L = 1$ they are identical with the contravariant components of a vector defined in § 3 in terms of the basis vectors $\chi_M$; i.e.,

$$F^0 = F_z, \quad F^{\pm 1} = \mp \frac{1}{\sqrt{2}} (F_x \mp i F_y). \tag{8.2}$$

In the general case, we shall refer to $F^M$ as the contravariant components of the $L$-vector. If we also define the covariant components $F_M$ by means of the following formula:

$$F^M = (-1)^M F_{-M}, \tag{8.2'}$$

then the scalar product of two $L$-vectors may be written in the following form:

$$(F, G) = \sum_M F^M G_M.$$

### 8.2. *The Algebra of L-Vectors*

The following problem is a basic one in the algebra of $L$-vectors: to construct an $L$-vector $H^M$ from products of the components of the $l$-vector $F^m$ and of the $l'$-vector $G^{m'}$. This problem is solved in the theory of the representations of the rotation group; here we quote only the results.

By way of a preliminary remark we note that if $L = l = l' = 1$, then the problem reduces to the determination of the vector product of two three-dimensional vectors, which, as is well known, is solved as follows:

$$H_i = e_{ijk} F_j G_k,$$

where $i, j, k$ denote the cartesian components of the vectors, while $e_{ijk}$ is the unit antisymmetric tensor of the third rank.

Similarly, in the general case, we have

$$H_M = C^M_{mm'} F_m G_{m'}, \tag{8.3}$$

where the coefficients $C^M_{mm'}$ form a "third-rank tensor" whose components, just as in the case of the tensor $e_{ijk}$, do not depend on the choice of the system of coordinates. Since the values of $C^M_{mm'}$ depend on $L$, $l$, $l'$, they are usually represented in greater detail as $C^M_{mm'} \equiv C^{LM}_{lml'm'}$. In (8.3) summations are implied over $m$ and $m'$ respectively from $-l$ to $l$ and from $-l'$ to $l'$.

Formula (8.3) expresses the quantum mechanical rule for the addition of angular momenta $l+l' = L$. The coefficients $C^{LM}_{lml'm'}$ are called the vector addition coefficients or the Clebsch-Gordan coefficients.

We shall not quote here the general expressions for $C^{LM}_{lml'm'}$. For the case $l' = 1$ they are given in § 4, while for the case $l' = \frac{1}{2}$ they will be given in § 11. We note that $C^{LM}_{lml'm'}$ differ from zero only when the following rules for the addition of angular momenta are satisfied:

$$M = m+m', \qquad |l-l'| \leqslant L \leqslant l+l'.$$

Formula (8.1) is a special case of (8.3) for $l = l'$ and $L = 0$. The formula for ordinary vector multiplication corresponds to the special case of (8.3) for $l = l' = L = 1$. If $\boldsymbol{H} = [\boldsymbol{FG}]$, then

$$H_M = -i\sqrt{2}\, C^{1M}_{1m1m'} F_m G_{m'}.$$

The coefficients $C^{LM}_{lml'm'}$ have the following orthogonality properties:

$$\begin{aligned} &\sum_{mm'} C^{LM}_{lml'm'} C^{L'M'}_{lml'm'} = \delta_{LL'} \delta_{MM'}, \\ &\sum_{LM} C^{LM}_{lml'm'} C^{LM}_{lm_1l'm_1'} = \delta_{mm_1} \delta_{m_1m_1'}. \end{aligned} \tag{8.4}$$

From (8.3) and (8.4) the following formula can be obtained which expresses the product of the components of the $l$- and $l'$-vectors $F$ and $G$ in terms of the components of the $L$-vector $H$ (the Clebsch-Gordan series)

$$F_m G_{m'} = \sum_L C^{LM}_{lml'm'} H_M. \tag{8.5}$$

If $l'$ is the smallest of the three quantities $L$, $l$, $l'$ then for given $l'$, $l$, and $m$ the quantities $C^{LM}_{lml'm'}$ form a square $(2l'+1)$-rowed matrix $C$, whose rows and columns are orthogonal and normalized in accordance with (8.4). ($L$ can assume values from $l-l'$ to $l+l'$, while $m'$ can assume values from $-l'$ to $l'$.)

We note that the number of conditions (8.4) is equal to $2(2l'+1)$, while the number of elements of the matrix $C$ is equal to $(2l'+1)^2$. Therefore, the number of independent elements $N_{l'}$ is equal to

$$N_{l'} = (2l'+1)^2 - 2(2l'+1).$$

If $l' = \frac{1}{2}$, then $N_{l'} = 0$; i.e., the coefficients $C^{LM}_{lm\frac{1}{2}m'}$ are completely determined by the properties of orthonormality.

We note the connection between the Pauli matrices $\sigma^\mu$ ($\sigma^\mu$ are related to $\sigma_x$, $\sigma_y$, $\sigma_z$ by expressions (8.2)) and the vector addition coefficients ($\alpha$, $\beta$ are spinor indices)

$$\sigma^\mu_{\alpha\beta} = -\sqrt{3}\, C^{\frac{1}{2}\alpha}_{1\mu\frac{1}{2}\beta}.$$

The coefficients $C^{LM}_{lml'm'}$ also have a number of symmetry properties with respect to interchanges of subscripts or superscripts. These symmetry properties may be most easily formulated if we introduce the coefficients $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$, which are defined in the following manner [(119) § 97; also E. P. Wigner quoted by (44)]:

$$C^{LM}_{lml'm'} = (2L+1)^{1/2}(-1)^{l-l'+M} \begin{pmatrix} l & l' & L \\ m & m' & -M \end{pmatrix}. \tag{8.6}$$

It is obvious that $m_1+m_2+m_3 = 0$; $l_1+l_2+l_3 = 0$. Let $\begin{pmatrix} l_i & l_j & l_k \\ m_i & m_j & m_k \end{pmatrix}$ be the coefficient obtained from $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ by a permutation of columns. Then we have

$$\begin{pmatrix} l_i & l_j & l_k \\ m_i & m_j & m_k \end{pmatrix} = \varepsilon \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{8.7'}$$

where $\varepsilon = 1$ if the permutation is even, and $\varepsilon = (-1)^{l_1+l_2+l_3}$ if the permutation is odd.

Further, we have

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{l_1+l_2+l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{8.7''}$$

The orthogonality conditions (8.4) may now be rewritten in the following form:

$$\begin{gathered} (2l_3+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \delta_{m_3 m_3'} \delta_{l_3 l_3}, \\ \sum_{l_3} (2l_3+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \end{gathered} \tag{8.8}$$

(A summation is implied whenever a lower index is repeated.)

We also note that when $l_3 = 0$ we have

$$C^{00}_{lml'-m'} = \begin{pmatrix} l & l' & 0 \\ m & -m' & 0 \end{pmatrix} = (2l+1)^{-1/2} \delta_{ll'} \delta_{mm'} (-1)^{l-m}. \tag{8.9}$$

It is convenient to make use of the vector addition coefficients in the symmetric form (8.5) for carrying out the summation over the components of the $l_i$-vectors (the $m_i$ indices). We state the basic summation formulas (161), (25):

$$\begin{gathered} (-1)^{l_1+l_2+l_3+m_1+m_2+m_3} \begin{pmatrix} j_1 & l_2 & l_3 \\ \mu_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -m_1 & \mu_2 & m_3 \end{pmatrix} \\ \times \begin{pmatrix} l_1 & l_2 & j_3 \\ m_1 & -m_2 & \mu_3 \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}, \\ \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ m_1 & m_2 & -\mu_3 \end{pmatrix} = \sum_{l_3} (-1)^{j_3+l_3+\mu_1+m_1} (2l_3+1) \\ \times \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ m_1 & \mu_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & l_2 & l_3 \\ \mu_1 & m_2 & -m_3 \end{pmatrix}. \end{gathered} \tag{8.10}$$

The quantities $\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$, which do not depend on the $m_i$ indices, are called Racah coefficients. Sometimes the following notation is also used for them:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+l_1+j_2+l_2} W(j_1 j_2 l_2 l_1;\ j_3 l_3).$$

Relations (8.10) may be illustrated by the following prescription for the vector addition of angular momenta. Suppose that we have to add the three angular momenta $j_1+j_2+l_1 = l_2$. This addition may be carried out in two stages in two different ways (Fig. 2):

$$\left.\begin{aligned} &1)\ j_1+j_2 = j_3, \quad j_3+l_1 = l_2, \\ &2)\ j_2+l_1 = l_3, \quad j_1+l_3 = l_2. \end{aligned}\right\} \tag{8.11}$$

Formulas (8.10) establish the relations between the coefficients in these two methods of addition.

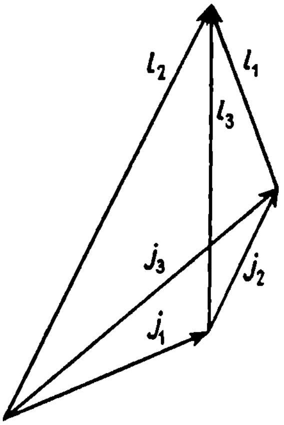


Fig. 2.

Evidently the Racah coefficients differ from zero only if the relations (8.11) hold, i.e., if the lines corresponding to the values of $j_i$, $l_i$ appearing in that equation can form a triangle.

The Racah coefficients possess the following symmetry and orthogonality properties:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_\alpha & j_\beta & j_\gamma \\ l_\alpha & l_\beta & l_\gamma \end{Bmatrix} = \begin{Bmatrix} l_\alpha & l_\beta & j_\gamma \\ j_\alpha & j_\beta & l_\gamma \end{Bmatrix}, \tag{8.12}$$

where $(\alpha, \beta, \gamma)$ is any permutation of the numbers $(1, 2, 3)$;

$$\sum_{j_3} (2j_3+1)(2l_3+1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \delta_{l_3 l_3'}. \tag{8.12'}$$

If one of the six indices is taken equal to zero, then the Racah coefficient has the following simple form:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 0 & l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+j_2+j_3}\delta_{l_2 j_3}\delta_{l_3 j_2}(2j_2+1)^{-1/2}(2j_3+1)^{-1/2}.$$

In later sections expressions are given for the Racah coefficients for the cases when one of the subscripts is equal to $\frac{1}{2}$ or 1.

## 8.3. *Spherical Harmonics*

The most common example of an $l$-vector is a spherical harmonic. The covariant components of the unit direction vector $n_m$ $(m = 0, \pm 1)$ coincide up to a normalizing constant with the first order spherical harmonics $Y_{1m}$. By utilizing the multiplication rule (8.3) it is possible to construct from products of $n_m$ successive $l$ -vectors with $l = 2, 3, \ldots,$ etc. They are identical (up to a normalizing constant) with the spherical harmonics $Y_{lm}$. Formula (8.3) expresses the recurrence properties of spherical harmonics.

With the aid of formula (8.5) it is possible to obtain the expansion of the product of spherical harmonics $Y_{lm}(\boldsymbol{n})Y_{l'm'}(\boldsymbol{n})$ in terms of the spherical harmonics $Y_{LM}(\boldsymbol{n})$. This expansion must obviously have the form

$$Y_{lm}(\boldsymbol{n})Y_{l'm'}(\boldsymbol{n}) = \sum_L \varrho_L^{ll'} C_{lml'm'}^{LM} Y_{LM}(\boldsymbol{n}),$$

where the coefficients $\varrho_L^{ll'}$ do not depend on $m$, $m'$, $M$. In order to determine these coefficients we multiply both sides of the preceding equation by $C_{lml'm'}^{L'M'}$. On utilizing the first of formulas (8.4) we obtain

$$\varrho_L^{ll'} Y_{LM}(\boldsymbol{n}) = C_{lml'm'}^{LM} Y_{lm}(\boldsymbol{n}) Y_{l'm'}(\boldsymbol{n}).$$

Further, by choosing $\boldsymbol{n}$ along the $z$-axis $(\vartheta = 0)$, we obtain in virtue of the fact that $Y_{lm}(0) = \delta_{m0}\sqrt{\dfrac{2l+1}{4\pi}}$

$$\varrho_L^{ll'} = \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}}\, C_{l0l'0}^{L0}.$$

Since $Y_{lm}^*(\boldsymbol{n}) = (-1)^m Y_{l,-m}(\boldsymbol{n})$, we have

$$Y_{lm}^*(\boldsymbol{n}) Y_{l'M'}(\boldsymbol{n}) = (-1)^m \sum_L \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}}\, C_{l0l'0}^{L0} C_{l-ml'm'}^{LM} Y_{LM}(\boldsymbol{n}). \tag{8.13}$$

We also give the expansion of the product of two vector spherical harmonics,

$$
\begin{aligned}
Y^*_{jlM}(\boldsymbol{n})\,Y_{j'l'M'}(\boldsymbol{n}) &= (-1)^{M+1}[(2l+1)(2l'+1)(2j+1)(2j'+1)]^{1/2} \\
&\quad \times \frac{1}{(4\pi)^{1/2}} \sum_L (2L+1)^{-1/2} \begin{Bmatrix} j'jL \\ ll'1 \end{Bmatrix} C^{L0}_{l0l'0}\, C^{L\lambda}_{j'M'j-M}\, Y_{L\lambda}(\boldsymbol{n}), \\
|\,Y^{(\lambda)}_{jM}(\boldsymbol{n})\,|^2 &= \frac{(2j+1)^2}{4\pi}(-1)^{M+1} \\
&\quad \times \sum_{n=0}^{j} C^{2n0}_{j0j0}\, C^{2n0}_{jMj-M}\, W(jj,jj;\,2n,\,1)\, P_{2n}(\cos\vartheta).
\end{aligned}
\tag{8.13'}
$$

On the basis of the properties of spherical harmonics regarded as *l*-vectors we can easily determine the character of the expansion of particle wave functions in terms of them.

The wave function $\psi$ of a particle of spin 0 is a scalar. Therefore the expansion of the wave function of a state of definite angular momentum $l$ in terms of spherical harmonics has the following form:

$$\psi = \sum_m a^m Y_{lm}(\boldsymbol{n}) \tag{8.14}$$

where the coefficients $a^m$ form a contravariant *l*-vector which determines the orientation of the angular momentum.

It may be seen from (8.14) that the spherical harmonics $Y_{lm}(\boldsymbol{n})$ can be interpreted as "basis vectors" in *l*-vector space.

If a particle has spin $s$, then its wave function $\psi$ can be represented in the form

$$\psi = \sum_{\mu=-s}^{s} \psi^\mu \chi_\mu(\alpha)$$

where $\chi_\mu(\alpha)$ are the eigenfunctions of the spin operator ($\alpha$ are the spin variables). Thus $\psi$ is characterized by the contravariant *s*-vector $\psi^\mu$. On the other hand, for states of total angular momentum $L\psi$ must have the form

$$\psi = \sum_M a^M Y_{LM}(\boldsymbol{n}, \alpha)$$

where $Y_{LM}(\boldsymbol{n}, \alpha)$ are the eigenfunctions of the total angular momentum operator. On the basis of (8.3) they can be obtained in the form

$$Y_{LM}(\boldsymbol{n}, \alpha) = C^{LM}_{lms\mu}\, Y_{lm}(\boldsymbol{n})\, \chi_\mu(\alpha)$$

## TABLE 5

### *Racah Coefficients*

$W\left(J_1 J_2 l_1 l_2; L\frac{1}{2}\right)$

| | $l_1 = J_1+\frac{1}{2}$ | $l_1 = J_1-\frac{1}{2}$ |
|---|---|---|
| $l_2 = J_2+\frac{1}{2}$ | $(-1)^{J_1+J_2-L}\left[\frac{(J_1+J_2+L+2)(J_1+J_2-L+1)}{(2J_1+1)(2J_1+2)(2J_2+1)(2J_2+2)}\right]^{1/2}$ | $(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2+1)(L+J_1-J_2)}{(2J_1)(2J_1+1)(2J_2+1)(2J_2+2)}\right]^{1/2}$ |
| $l_2 = J_2-\frac{1}{2}$ | $(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2+1)(L-J_1+J_2)}{(2J_1+1)(2J_1+2)(2J_2)(2J_2+1)}\right]^{1/2}$ | $(-1)^{J_1+J_2-L-1}\left[\frac{(J_1+J_2+L+1)(J_1+J_2-L)}{2J_1(2J_1+1)(2J_2)(2J_2+1)}\right]^{1/2}$ |

$W(J_1 J_2 l_1 l_2; L1)$

| | $l_2 = J_2+1$ |
|---|---|
| $l_1 = J_1+1$ | $(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{4(2J_1+3)(J_1+1)(2J_1+1)(2J_2+3)(J_2+1)(2J_2+1)}\right]^{1/2}$ |
| $l_1 = J_1$ | $(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+1)(L+J_1-J_2)}{4J_1(2J_1+3)(J_1+1)(2J_2+1)(J_2+1)(2J_2+3)}\right]^{1/2}$ |
| $l_1 = J_1-1$ | $(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2)(L+J_1-J_2-1)(L-J_1+J_2+2)(L-J_1+J_2+1)}{4(2J_1+1)(2J_1-1)(J_1)(J_2+1)(2J_2+1)(2J_2+3)}\right]^{1/2}$ |

| | $l_2 = J_2$ |
|---|---|
| $l_1 = J_1+1$ | $(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+2)(L+J_1-J_2+1)(J_1+J_2-L+1)(L-J_1+J_2)}{4(2J_1+1)(J_1+1)(2J_1+3)(J_2)(J_2+1)(2J_2+1)}\right]^{1/2}$ |
| $l_1 = J_1$ | $(-1)^{J_1+J_2-L-1}\left[\frac{J_1(J_1+1)+J_2(J_2+1)-L(L+1)}{[4J_1(J_1+1)(2J_1+1)(J_2)(J_2+1)(2J_2+1)]^{1/2}}\right]$ |
| $l_1 = J_1-1$ | $(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(-L+J_1+J_2)(L+J_1-J_2)(L-J_1+J_2+1)}{4(2J_1+1)(J_1)(2J_1-1)(J_2)(2J_2+1)(J_2+1)}\right]^{1/2}$ |
| | $l_2 = J_2-1$ |
| $l_1 = J_1+1$ | $(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{4(2J_1+1)(J_1+1)(2J_1+3)(2J_2-1)(J_2)(2J_2+1)}\right]^{1/2}$ |
| $l_1 = J_1$ | $(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(L+J_1-J_2+1)(L+J_2-J_1)(J_1+J_2-L)}{4J_1(2J_1+1)(J_1+1)(J_2)(2J_2+1)(2J_2-1)}\right]^{1/2}$ |
| $l_1 = J_1-1$ | $(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)}{4(2J_1+1)(J_1)(2J_1-1)(2J_2+1)(J_2)(2J_2-1)}\right]^{1/2}$ |

Thus

$$\psi^{\mu} = \sum_{M} a^{M} Y_{M}^{\mu}(\boldsymbol{n})$$

where

$$Y_{M}^{\mu}(\boldsymbol{n}) = C_{lms\mu}^{LM} Y_{lm}(\boldsymbol{n}).$$

The functions $Y_{M}^{\mu}(\boldsymbol{n})$ (basis vectors in $L$-vector space) can be called generalized spherical harmonics.[1] In particular, for $s = 1$ they coincide with spherical vectors (cf. § 4), and for $s = 1/2$ they coincide with spherical spinors (cf. § 11).

Equation (8.15) defines the transformation to a dual tensor. Just as in vector algebra an antisymmetric tensor of the second rank $A_{ij}$ can be introduced which is dual to the vector $A_k$,

$$A_{ij} = e_{ijk} A_k,$$

so in the algebra of $L$-vectors the tensor $G_{M}^{\mu}$ $\big((L\text{-}s)\text{-vector}\big)$ can be defined which is dual to the $l$-vector $G_m$,

$$G_{M}^{\mu} = C_{m\mu}^{M} G_m.$$

The tensor $G_{M}^{\mu}$ can be represented by a rectangular matrix of $2L+1$ rows and $2s+1$ columns. It depends on all three indices $L$, $l$, and $s$.

---

[1] These wave functions can also be obtained directly by solving the equations for the eigenvalues of the infinitesimal rotation operator. In the general case these wave functions can be expressed in terms of Jacobi polynomials; cf. (75).

# Relativistic Quantum Mechanics of the Electron

## § 9. The Dirac Equation

### 9.1. *Spinors. Pauli Matrices*

In nonrelativistic quantum mechanics (cf., for example, (119) § 54) an electron state is described by a two-component wave function—the *spinor* $\varphi = \begin{pmatrix} \varphi^{1/2} \\ \varphi^{-1/2} \end{pmatrix}$, where $\varphi^\lambda (\lambda = \pm 1/2)$ are the components of the spinor.

When the coordinate system is rotated through the infinitesimal angle $\boldsymbol{\vartheta} = \boldsymbol{\nu}\vartheta$, where $\boldsymbol{\nu}$ is the unit vector along the axis of rotation, the spinor components transform in the following manner:

$$\varphi^\lambda = \left(\delta_{\lambda\mu} + \frac{i}{2}\vartheta\sigma^{(\nu)}_{\lambda\mu}\right)\varphi'^\mu. \tag{9.1}$$

Here $\varphi'^\mu$ are the spinor components in the new coordinate system (a summation is carried out over the index $\mu$), $\sigma^{(\nu)}_{\lambda\mu}$ are the elements of the matrix

$$\sigma^{(\nu)} = \sigma_x \nu_x + \sigma_y \nu_y + \sigma_z \nu_z$$

and $\boldsymbol{\sigma}$ is the set of the three *Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In accordance with the general relation between the infinitesimal rotation operator and angular momentum the matrix $\frac{1}{2}\boldsymbol{\sigma}$ is the operator for the intrinsic angular momentum of the electron—its spin—while the transformation (9.1) is an unambiguous consequence of the basic physical fact that the electron spin is equal to $\frac{1}{2}$.

The Pauli matrices possess the commutation properties of the components of the angular momentum operator

$$\sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z, \quad \sigma_y\sigma_z - \sigma_z\sigma_y = 2i\sigma_x, \quad \sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y.$$

Moreover, they also satisfy the following relations:

$$\sigma_i\sigma_k+\sigma_k\sigma_i = 2\delta_{ik}.$$

From these properties it also follows that

$$\sigma_i\sigma_k = \delta_{ik}+ie_{ikl}\sigma_l, \tag{9.2}$$

where $e_{ikl}$ is the unit antisymmetric tensor of the third rank.

On integrating (9.1) over the angle $\vartheta$ we obtain the transformation law for spinors corresponding to a rotation through a finite angle $\theta$ about the $\boldsymbol{\nu}$ axis:

$$\varphi = \exp\left(\frac{i}{2}\theta\sigma^{(\nu)}\right)\varphi'.$$

Since $(\sigma^{(\nu)})^{2n} = 1$, $(\sigma^{(\nu)})^{2n+1} = \sigma^{(\nu)}$ ($n$ is an integer), then on expanding $\exp\left(\frac{i}{2}\theta\sigma^{(\nu)}\right)$ in a series, we obtain

$$\varphi = \left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\sigma^{(\nu)}\right)\varphi'.$$

From this formula it follows that when $\theta = 2\pi$, $\varphi = -\varphi'$. But rotation through an angle $2\pi$ is physically equivalent to the identity transformation. We thus see that the definition of a spinor contains an ambiguity, since the sign of the spinor remains undetermined. Therefore, physical quantities cannot involve spinors linearly.

Spinors, like tensors, may differ in the way in which they transform under inversion. The eigenvalues of the inversion operator I are usually determined by the fact that a double inversion is equivalent to the identity transformation; i.e., $\mathrm{I}^2 = 1$. Therefore, two cases are possible when the operator I acts on the spinor $\varphi$:

$$\varphi' = \mathrm{I}\varphi = \pm\varphi. \tag{9.3}$$

However, in view of the double sign in the definition of spinors another definition of the operator for the inversion of a spinor is possible. Indeed, a double inversion may be defined not as the identity transformation, but as the transformation corresponding to a rotation through the angle $2\pi$; i.e., instead of the operator I we can introduce the operator $\mathrm{I}_1$, which has the property $\mathrm{I}_1^2\varphi = -\varphi$, and, consequently, the law of transformation of spinors under inversion can be formulated in the following manner:

$$\varphi' = \mathrm{I}_1\varphi = \pm i\varphi. \tag{9.4}$$

(If in (9.3) or (9.4) the upper sign holds, then we adopt the convention of calling $\varphi$ a polar spinor, while in the case of the lower sign we call it a pseudospinor.)

The alternative definitions (9.3) and (9.4) of the inversion operators are equally valid.

### 9.2. *Dirac Equations. Dirac Matrices*

The relativistically invariant equations which in the case of the electron play the same role that Maxwell's equations do in the case of the photon, were obtained by Dirac (47). They consist of the following system of homogeneous first-order differential equations in the two spinor functions $\varphi$ and $\chi$:

$$\begin{aligned} i\frac{\partial\varphi}{\partial t} &= m\varphi+\boldsymbol{\sigma}\mathbf{p}\chi, \\ i\frac{\partial\chi}{\partial t} &= -m\chi+\boldsymbol{\sigma}\mathbf{p}\varphi, \end{aligned} \tag{9.5}$$

where $\mathbf{p} = -i\nabla$ is the momentum operator and $m$ is the electron mass.

We note that Maxwell's equations can also be written in a form analogous to (9.5) if we introduce the following formal substitution: $\varphi \to \boldsymbol{E}$, $\chi \to i\boldsymbol{H}$, $\boldsymbol{\sigma} \to \mathbf{s}$, where $\mathbf{s}$ is the photon spin operator defined by formula (3.3), and if we set $m = 0$.

Later, in Chapter III, we shall obtain the Dirac equations starting from a general variational principle. But here we shall investigate specific properties of the system of equations (9.5).

The set of two spinors $\varphi$ and $\chi$ can be represented as a single four-component quantity:

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi^{1/2} \\ \varphi^{-1/2} \\ \chi^{1/2} \\ \chi^{-1/2} \end{pmatrix} = \begin{Bmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{Bmatrix},$$

which is called a *bispinor.*

In order to be able to write equations (9.5) in the form of a single equation in $\psi$, we introduce the four-dimensional matrices (the *Dirac matrices*)

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9.6}$$

where each element of the matrices $\boldsymbol{\alpha}$ and $\beta$ denotes the corresponding two-dimensional matrix. With the aid of the matrices $\boldsymbol{\alpha}$ and $\beta$ it is possible to write (9.5) in the following form:

$$i\frac{\partial\psi}{\partial t} = (\boldsymbol{\alpha}\mathbf{p}+\beta m)\psi. \tag{9.7}$$

Equation (9.7) has the form of the Schrödinger equation,

$$i\frac{\partial\psi}{\partial t} = H\psi,$$

in which the Hamiltonian operator is given by the following expression:

$$\mathbf{H} = \boldsymbol{\alpha}\mathbf{p}+\beta m. \tag{9.7$'$}$$

From the definition of the matrices $\boldsymbol{\alpha}$ and $\beta$ it follows that

$$\alpha_i\alpha_k+\alpha_k\alpha_i = 2\delta_{ik}, \quad \beta^2 = 1, \quad \boldsymbol{\alpha}\beta+\beta\boldsymbol{\alpha} = 0. \tag{9.8}$$

We also introduce the following auxiliary four-dimensional matrices:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}; \quad \varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{9.9}$$

The matrices $\boldsymbol{\Sigma}$ have the same properties as the matrices $\boldsymbol{\sigma}$. Moreover, as may be easily seen, the following relations hold:

$$\varrho\boldsymbol{\Sigma}-\boldsymbol{\Sigma}\varrho = 0, \quad \beta\boldsymbol{\Sigma}-\boldsymbol{\Sigma}\beta = 0, \quad \beta\varrho+\varrho\beta = 0,$$
$$\varrho^2 = 1, \quad \boldsymbol{\alpha} = \varrho\boldsymbol{\Sigma},$$
$$[\boldsymbol{\alpha}, \boldsymbol{\alpha}] = 2i\boldsymbol{\Sigma}, \quad [\boldsymbol{\alpha}, \boldsymbol{\Sigma}] = 2i\boldsymbol{\alpha},$$
$$\alpha_i\Sigma_k = \Sigma_i\alpha_k = \varrho\delta_{ik}+ie_{ikl}\alpha_l.$$

We note that the unit matrix, the three matrices $\Sigma_j (j = 1, 2, 3)$, the three matrices $\varrho_j$ which commute with them, where $\varrho_1 = \varrho$, $\varrho_2 = i\beta\varrho$, and $\varrho_3 = \beta$, and their nine products $\varrho_j\Sigma_k$ (a total of sixteen matrices) form a complete system of linearly independent four-dimensional matrices.

### 9.3. *Unitary Transformations of Bispinors*

The components of the bispinor wave function $\psi$ can be subjected to the transformation $\psi' = U\psi$, where $U$ is a certain unitary four-rowed matrix. This transformation leaves the form of the Dirac equation (9.5) unaltered if, in accordance with the general rules, the Dirac matrices are also subjected to the following transformation:

$$\alpha_i' = U\alpha_i U^+; \quad \beta' = U\beta U^+. \tag{9.6$'$}$$

The commutation relations and the other properties of these matrices are not altered as a result.

We give examples of such transformations.

If

$$U = \frac{1}{\sqrt{2}}(\beta + \varrho),$$

then the Dirac matrices transform as follows:

$$\beta' = \varrho, \quad \varrho' = \beta, \quad \boldsymbol{\Sigma}' = \boldsymbol{\Sigma},$$
$$\boldsymbol{\alpha}' = \varrho' \boldsymbol{\Sigma}' = \beta \boldsymbol{\Sigma},$$

while the transformed wave function $\psi' = U\psi$ has the following form:

$$\psi' = \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

where

$$\xi = \frac{1}{\sqrt{2}}(\varphi + \chi), \qquad \eta = \frac{1}{\sqrt{2}}(\varphi - \chi).$$

The components of the spinor $\xi$ are usually denoted by $\xi^\lambda (\lambda = 1, 2)$, while the components of the spinor $\eta$ are denoted by $\eta_{\dot\lambda} (\dot\lambda = \dot 1, \dot 2)$.

We note that under Lorentz transformations (cf. subsection 9.6) the spinors $\xi$ and $\eta$ transform independently of each other, while under the transformation of inversion (9.3) they go over into each other: $\mathrm{I}\xi = \eta$, $\mathrm{I}\eta = \xi$.

As a second example we take $U$ in the following form (130):

$$U^{(\mathrm{M})} = \frac{1}{\sqrt{2}}(\alpha_y + \beta).$$

If we designate the quantities transformed by means of this matrix by the index M, we obtain

$$\alpha_x^{(\mathrm{M})} = -\alpha_x; \quad \alpha_z^{(\mathrm{M})} = -\alpha_z; \quad \Sigma_x^{(\mathrm{M})} = \varrho_2 \Sigma_z;$$
$$\alpha_y^{(\mathrm{M})} = \beta; \quad \beta^{(\mathrm{M})} = \alpha_y; \quad \Sigma_y^{(\mathrm{M})} = -\Sigma_y; \quad \Sigma_z^{(\mathrm{M})} = -\varrho_2 \Sigma_x.$$

The matrices $\alpha_i^{(\mathrm{M})}$ are real, while $\beta^M$ and $\Sigma_i^{(\mathrm{M})}$ are purely imaginary, and the Dirac equation contains only real coefficients. It has the following form:

$$\left(\frac{\partial}{\partial t} - \alpha_x \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} - \alpha_z \frac{\partial}{\partial z} + i\alpha_y m\right)\psi^{(\mathrm{M})} = 0.$$

The choice of the matrix $U$ in the form $U^{(\mathrm{M})}=\frac{1}{\sqrt{2}}(\beta+\alpha_y)$ is referred to as the Majorana representation.

### 9.4. *The Necessity for Four-Component Electron Wave Functions*

We now investigate the reason for the doubling of the number of components of the electron wave function in relativistic quantum mechanics in comparison with nonrelativistic quantum mechanics.

In order to do this it is convenient to go over with the aid of the unitary transformation $U=\frac{1}{\sqrt{2}}(\beta+\varrho)$ discussed in subsection 9.3, from the matrices $\boldsymbol{\alpha}$, $\beta$ to the matrices $\boldsymbol{\alpha}'$, $\beta'$ defined by formula (9.6′).

The spinors $\xi$ and $\eta$ forming the bispinor $\psi$ now satisfy the following equations:

$$\begin{aligned} i\frac{\partial\xi}{\partial t} &= \boldsymbol{\sigma}\mathbf{p}\xi+m\eta, \\ i\frac{\partial\eta}{\partial t} &= -\boldsymbol{\sigma}\mathbf{p}\eta+m\xi. \end{aligned} \tag{9.5'}$$

We see that due to the presence in the Dirac equation of the matrix $\beta$ the first equation (9.5′) for the spinor $\xi$ now contains the components of the spinor $\eta$, while the second equation for $\eta$ now contains the components of the spinor $\xi$. If the mass of the particle is equal to zero, then there is no such mixing of components of the two spinors, and equations (9.5′) assume the form

$$i\frac{\partial\xi}{\partial t}=\pm\boldsymbol{\sigma}\mathbf{p}\xi. \tag{9.10}$$

These equations, like the original Dirac equations, are relativistically invariant.

Thus, for a particle of zero rest mass we can obtain a relativistically invariant equation containing a two-component, rather than a four-component, wave function (209). However, it can be easily seen that this equation will not be invariant under the transformation of space inversion.

Indeed, we investigate the transition from one coordinate system to another coordinate system moving with respect to the former with an infinitesimal velocity $\boldsymbol{v}$. Since the four matrices 1, $\sigma_x$, $\sigma_y$, $\sigma_z$ form a complete system of linearly independent two-dimensional matrices, the transformation which contains $\boldsymbol{v}$ linearly and reduces to the identity transformation for $\boldsymbol{v} = 0$ must necessarily be of the following form:

$$\xi = \left(1 + \frac{\lambda}{2} \boldsymbol{v\sigma}\right) \xi'$$

where $\lambda$ is a constant. But an expression of this type is not invariant under an inversion, since $\boldsymbol{v}\xi\xi^*$ is a polar and $\xi\boldsymbol{\sigma}\xi^*$ is an axial vector.

This may also be seen if we compare the foregoing expression with formula (9.1.) Both transformations are of the same form, but in (9.1) $\boldsymbol{\vartheta}$ is a pseudovector, while $\boldsymbol{v}$ is a polar vector.

Thus, the two-component wave function satisfying equation (9.10) may be used to describe a particle of zero mass only if we drop the requirement of the invariance of the equations under space inversion.

Such a situation exists in the case of the neutrino which is described by equations (9.10); here the $-$ sign corresponds to the neutrino, while the $+$ sign corresponds to the antineutrino (114), (123), (166a).

In the case of a particle of finite rest mass the requirement of relativistic invariance leads to the conclusion that the wave function must have four-components. Moreover, as will be shown later, the equations determining this function, i.e., the Dirac equations, will also be invariant under the transformation of space inversion. This invariance is due to the fact that the bispinor $\psi$ comprises the spinor $\varphi$ and the pseudospinor $\chi$, which have different transformation properties under inversion.

## 9.5. *Symmetric Form of the Dirac Equation. Equation of Continuity*

The Dirac equation may be put into a more symmetric form if we introduce the matrices $\gamma_\mu$ ($\mu = 1, 2, 3, 4$), which are defined as follows:

$$\begin{aligned} &\gamma_j = -i\beta\alpha_j \quad (j = 1, 2, 3), \\ &\gamma_4 = \beta. \end{aligned} \tag{9.11}$$

The matrices $\gamma_\mu$ satisfy the following commutation relations:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \quad (\mu, \nu = 1, 2, 3, 4) \tag{9.11'}$$

and are Hermitian, like all the matrices introduced earlier:

$$\gamma_\mu^+ = \gamma_\mu, \quad \tilde{\gamma}_\mu = \gamma_\mu^*$$

($\gamma_\mu^+$ denotes the Hermitian conjugate matrix, while $\tilde{\gamma}_\mu \equiv \gamma_\mu^{\mathrm{T}}$ denotes the transposed matrix).

We multiply both sides of the Dirac equation (9.7) on the left by $i\beta$. On utilizing the definitions (9.10) we obtain

$$(\gamma_4 \mathrm{p}_4 + \boldsymbol{\gamma}\mathbf{p} - im)\psi = 0,$$

where $\mathrm{p}_4 = -i\dfrac{\partial}{\partial x_4} = -\dfrac{\partial}{\partial t}$ and $\boldsymbol{\gamma}$ is the set of the matrices $\gamma_1, \gamma_2, \gamma_3$. If we introduce the abbreviated notation for the "scalar product" $\gamma_\nu \mathrm{p}_\nu$

$$\hat{\mathrm{p}} \equiv \sum_\nu \mathrm{p}_\nu \gamma_\nu = \boldsymbol{\gamma}\mathbf{p} + \gamma_4 \mathrm{p}_4,$$

then the Dirac equation assumes the following form:

$$(i\hat{\mathrm{p}} + m)\psi = 0. \tag{9.12}$$

The operator $i\hat{\mathrm{p}}$ which appears in (9.12) is not self-conjugate. Therefore, the complex-conjugate wave function $\psi^*$ satisfies an equation which does not coincide with (9.12). In place of $\psi^*$ it is convenient to introduce the function

$$\bar{\psi} = \psi^* \beta; \qquad \psi^* = \bar{\psi}\beta,$$

which satisfies the equation

$$\bar{\psi}(-i\hat{\mathrm{p}} + m) = 0 \tag{9.13}$$

(it is understood here that the differential operators $\mathrm{p}_a$ operate on the function $\bar{\psi}$ which appears on their left).

We now prove the validity of formula (9.13). Since $\mathbf{p}^* \equiv i\nabla = -\mathbf{p}$ and $\mathrm{p}_4^* = \mathrm{p}_4$, it follows from (9.12) that

$$(-i\gamma_4^* \mathrm{p}_4 + i\boldsymbol{\gamma}^* \mathbf{p} + m)\psi^* = 0.$$

Further, on noting that $\gamma_a^* \psi^* = \psi^* \gamma_a$, and on multiplying the last equation on the right by $\beta = \gamma_4$ we obtain equation (9.13) after making use of (9.8).

We now multiply equation (9.13) on the right by $-\psi$, multiply equation (9.12) on the left by $\bar{\psi}$, and add them. This yields

$$\bar{\psi}\gamma_\mu(\mathrm{p}_\mu \psi) + (\mathrm{p}_\mu \bar{\psi})\gamma_\mu \psi = 0$$

or

$$\frac{\partial s_\mu}{\partial x_\mu} = 0, \tag{9.14}$$

where the following notation has been introduced:

$$s_\mu = i\bar{\psi}\gamma_\mu \psi. \tag{9.15}$$

Relation (9.14) may be interpreted as an equation of continuity, and $s_\mu$ may be interpreted as a four-dimensional flux density vector.

In three-dimensional form equation (9.14) has the form

$$\frac{\partial s_0}{\partial t} + \operatorname{div} \boldsymbol{s} = 0,$$

where

$$s_0 = \bar{\psi}\gamma_4\psi = \psi^*\psi,$$
$$\boldsymbol{s} = i\bar{\psi}\boldsymbol{\gamma}\psi = \psi^*\boldsymbol{\alpha}\psi.$$

Just as in nonrelativistic quantum mechanics, the quantity $s_0 = \psi^*\psi$ represents the probability density of the electron being localized at a given point in space. The equation of continuity allows us to normalize $\psi$ so that $\int \psi^*\psi d\boldsymbol{r} = 1$.

## 9.6. *Invariance of the Dirac Equation*

We shall now prove the relativistic invariance of the Dirac equations, i.e., their invariance under spatial rotations, proper Lorentz transformations, and the transformation of inversion.

Let us begin with the transformation of space inversion and define this transformation in the case of the bispinor $\psi(\boldsymbol{r}, t)$, in accordance with formulas (9.3) and (9.4) in the following manner:

$$\begin{aligned} \psi(\boldsymbol{r}, t) &\to \psi'(\boldsymbol{r}, t) = \mathrm{I}\psi(\boldsymbol{r}, t) = \eta_p\gamma_4\psi(-\boldsymbol{r}, t), \\ \bar{\psi}(\boldsymbol{r}, t) &\to \bar{\psi}'(\boldsymbol{r}, t) = \bar{\mathrm{I}}\bar{\psi}(\boldsymbol{r}, t) = \eta_p^*\bar{\psi}(-\boldsymbol{r}, t)\gamma_4, \end{aligned} \tag{9.16}$$

where $\eta_p$ has one of the four values, $+i$, $-i$, $+1$, $-1$.

It can be easily shown that this transformation leaves the Dirac equation unaltered. Indeed, in equation

$$\left(-\gamma_4\frac{\partial}{\partial t} - i\boldsymbol{\gamma}\frac{\partial}{\partial \boldsymbol{r}} - im\right)\psi(\boldsymbol{r}, t) = 0$$

we replace $\boldsymbol{r}$ by $-\boldsymbol{r}$:

$$\left(-\gamma_4\frac{\partial}{\partial t} + i\boldsymbol{\gamma}\frac{\partial}{\partial \boldsymbol{r}} - im\right)\psi(-\boldsymbol{r}, t) = 0.$$

and substitute into this expression in place of $\psi(-\boldsymbol{r}, t)$ the function $\eta^{-1}\gamma_4\psi'(\boldsymbol{r}, t)$ in accordance with (9.16):

$$\left(-\gamma_4\frac{\partial}{\partial t} + i\boldsymbol{\gamma}\frac{\partial}{\partial \boldsymbol{r}} - im\right)\gamma_4\psi'(\boldsymbol{r}, t) = 0.$$

By utilizing (9.11′) we can move the matrix $\gamma_4$ immediately preceding $\psi'$ to the left. Then, on multiplying the resulting equation on the left by $\gamma_4$ we obtain the original Dirac equation which now contains $\psi'$ in place of $\psi$.

Corresponding to each of the four values of $\eta_p$ we have four types of bispinors which are usually denoted by $A(\eta_p = i)$, $B(\eta_p = -i)$, $C(\eta_p = 1)$ and $D(\eta_p = -1)$. The bispinors $A$ and $B$ (and also $C$ and $D$) can be used to describe two kinds of particles obeying the same Dirac equation but differing in the way in which they transform under inversion; it is commonly said that these particles differ by the sign of their "intrinsic parity".

We now investigate the transformations under rotation. It follows from (9.1) that under a spatial rotation through an infinitesimal angle $\boldsymbol{\vartheta}$ the transformation of the bispinor $\psi$ has the form

$$\psi = \left(1 + \frac{i}{2}\boldsymbol{\vartheta}\boldsymbol{\Sigma}\right)\psi'.$$

By utilizing the general relation between the infinitesimal rotation operator and the angular momentum operator we can conclude that the matrix $\boldsymbol{\Sigma}$ represents the operator for the intrinsic angular momentum (spin) of the electron.

The following transformation corresponds to a finite rotation through the angle $\theta$ about the $\boldsymbol{\nu}$ axis

$$\psi = e^{\frac{i}{2}\theta\boldsymbol{\nu}\boldsymbol{\Sigma}}\psi' = \left(\cos\frac{\theta}{2} + i\boldsymbol{\nu}\boldsymbol{\Sigma}\sin\frac{\theta}{2}\right)\psi'.$$

In accordance with the arguments presented in subsection 9.4 we assume that the following transformation of the wave function corresponds to the infinitesimal Lorentz transformation

$$\psi = (1 + \tfrac{1}{2}\delta\boldsymbol{u}\boldsymbol{\alpha})\psi',$$

where $\delta\boldsymbol{u}$ is the infinitesimal relative velocity of the two coordinate systems. In this case the following transformation corresponds to a finite Lorentz transformation defined by the finite velocity $\boldsymbol{u}$:

$$\psi = e^{\frac{1}{2}\varphi\boldsymbol{n}\boldsymbol{\alpha}}\psi' = \left(\cosh\frac{\varphi}{2} + \boldsymbol{n}\boldsymbol{\alpha}\sinh\frac{\varphi}{2}\right)\psi',$$

where

$$\varphi = \operatorname{arc\,tanh} u, \qquad \boldsymbol{n} = \frac{\boldsymbol{u}}{u}.$$

Both formulas for the infinitesimal transformations may be amalgamated into one,

$$\psi = (1+\tfrac{1}{2}\vartheta_{\alpha\beta}\gamma_\alpha\gamma_\beta)\psi', \tag{9.17}$$

where $\vartheta_{\alpha\beta}$ is the four-dimensional antisymmetric tensor for the infinitesimal rotation

$$\vartheta_{ij} = \tfrac{1}{2}e_{ijk}\vartheta_k; \qquad \vartheta_{4j} = \frac{i}{2}v_j \qquad (i, j, k = 1, 2, 3).$$

We shall now demonstrate the invariance of the Dirac equation under the transformation (9.17).

On substituting (9.17) into equation (9.12) and on multiplying it on the left by $(1+\frac{1}{2}\vartheta_{\mu\nu}\gamma_\nu\gamma_\mu)$ we obtain

$$(1+\tfrac{1}{2}\vartheta_{\mu\nu}\gamma_\nu\gamma_\mu)(\gamma_\lambda \mathrm{p}'_\lambda+2\vartheta_{\alpha\beta}\gamma_\alpha \mathrm{p}'_\beta-im)(1+\tfrac{1}{2}\vartheta_{\sigma\varrho}\gamma_\sigma\gamma_\varrho)\psi' = 0,$$

where $\mathrm{p}'_\alpha$ are the momentum components in the new coordinate system

$$\mathrm{p}_\alpha = \mathrm{p}'_\alpha+2\vartheta_{\alpha\beta}\mathrm{p}'_\beta.$$

On neglecting terms quadratic in $\vartheta_{\alpha\beta}$ we can rewrite this equation in the following form:

$$\left\{\gamma_\alpha \mathrm{p}'_\alpha-im-\frac{im}{2}\vartheta_{\alpha\beta}(\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha)+\tfrac{1}{2}\vartheta_{\alpha\beta}(\gamma_\beta\gamma_\alpha\hat{\mathrm{p}}'+\hat{\mathrm{p}}'\gamma_\alpha\gamma_\beta)+2\vartheta_{\alpha\beta}\gamma_\alpha \mathrm{p}'_\beta\right\}\psi' = 0.$$

On noting that $\vartheta_{\alpha\beta} = -\vartheta_{\beta\alpha}$ and

$$\gamma_\beta\gamma_\alpha\hat{\mathrm{p}}' = \hat{\mathrm{p}}'\gamma_\beta\gamma_\alpha+2\gamma_\beta \mathrm{p}'_\alpha-2\gamma_\alpha \mathrm{p}'_\beta,$$

we can easily verify that terms linear in $\vartheta_{\alpha\beta}$ vanish, and the transformed equation becomes identical with (9.12).

We can also obtain the transformation law for $\bar{\psi}$. We take the expression complex conjugate to (9.17),

$$\psi^* = \psi^{*\prime}(1+\tfrac{1}{2}\vartheta^*_{\alpha\beta}\gamma_\beta\gamma_\alpha),$$

and multiply it on the right by $\gamma_4$. Since $\vartheta^*_{ij} = -\vartheta_{ij}\,(i, j = 1, 2, 3)$ and $\vartheta^*_{4j} = -\vartheta_{4j}$, we obtain

$$\bar{\psi} = \bar{\psi}'(1-\tfrac{1}{2}\vartheta_{\alpha\beta}\gamma_\alpha\gamma_\beta). \tag{9.18}$$

### 9.7. *Bilinear Combinations of the Components of the Wave Function*

In relativistic quantum mechanics, just as in nonrelativistic quantum mechanics, considerable significance is attached to bilinear combinations of $\psi$ and $\psi^*$. Since the wave function $\psi$ has four-components, sixteen such combinations can be constructed. From them quantities may be formed which behave under Lorentz transformations as a scalar $S$ (one-component), a vector $V$ (four-components), an antisymmetric tensor of the second rank $T$ (six-components), an axial vector $A$ (four-components), and a pseudoscalar $P$ (one-component). We shall show that these quantities have the following form:

$$\begin{aligned} S &= \bar{\psi}\psi, \\ V_\mu &= \bar{\psi}\gamma_\mu\psi, \\ T_{\mu\nu} &= \tfrac{1}{2}\bar{\psi}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\psi, \\ A_\mu &= \bar{\psi}\gamma_\mu\gamma_5\psi, \\ P &= \bar{\psi}\gamma_5\psi, \end{aligned} \tag{9.19}$$

where

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \equiv -\varrho. \tag{9.20}$$

In order to do this we utilize the transformation formulas for $\psi$ and $\bar{\psi}$. First of all we consider $S$. It follows from (9.17) and (9.18) that

$$\bar{\psi}\psi = \bar{\psi}'(1-\tfrac{1}{2}\vartheta_{\alpha\beta}\gamma_\alpha\gamma_\beta)(1+\tfrac{1}{2}\vartheta_{\alpha\beta}\gamma_\alpha\gamma_\beta)\psi'.$$

This equation contains no terms linear in $\vartheta_{\alpha\beta}$, i.e., $\bar{\psi}\psi = \bar{\psi}'\psi'$, and, consequently, $S$ is an invariant. Under inversions $S$ also remains unaltered.

In a similar manner it can be easily shown that

$$\bar{\psi}\gamma_\mu\psi = \bar{\psi}'\gamma_\mu\psi' + \tfrac{1}{2}\vartheta_{\alpha\beta}\bar{\psi}'(\gamma_\mu\gamma_\alpha\gamma_\beta-\gamma_\alpha\gamma_\beta\gamma_\mu)\psi',$$

or by utilizing the properties of the matrices $\gamma_\mu$,

$$V_\mu = V'_\mu + 2\vartheta_{\mu\nu}V'_\nu.$$

Under inversion $\bar{\psi}\gamma_\mu\psi = \bar{\psi}'\gamma_4\gamma_\mu\gamma_4\psi'$ or $V_j = -V'_j$, $V_4 = V'_4$, i.e., $V_\mu$ transforms like a vector. From this it can be seen that the flux density vector introduced in (9.15) is indeed a four-vector.

In a similar manner it can be easily shown that $T_{\mu\nu}$ transforms like a tensor.

In order to establish the transformation properties of the quantities $A_\mu$ and $P$ it is sufficient to utilize the property of the matrix $\gamma_5$ which follows directly from its definition,

$$\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5.$$

From this it follows that the matrix $\gamma_5$ commutes with the matrix $\gamma_\mu\gamma_\nu$ which appears in the transformation under a rotation (9.17), (9.18), and anticommutes with the matrix $\gamma_4$, which defines the transformation of the inversion operator in accordance with (9.16). Therefore, the extra factor $\gamma_5$ in $P$ and $A$ in comparison with $S$ and $V$ does not change the character of the transformation of these quantities under rotations, but changes the sign of the transformation under inversion. Thus, $P$ is a pseudoscalar, and $A$ is an axial vector.

Since the components of $S$, $V$, $T$, $A$, $P$ exhaust all the sixteen bilinear combinations which may be formed from the components of $\bar{\psi}$ and $\psi$, it is clear that there exist no other independent quantities. In particular, the quantity $\bar{\psi}\gamma_{\alpha_1}\gamma_{\alpha_2}\gamma_{\alpha_3} \dots \gamma_{\alpha_n}\psi$, which transforms like a tensor of the $n$th rank may be reduced to quantities of the types already investigated. For example, the tensor $T'_{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi$ is dual with respect to the tensor $T_{\mu\nu}$, $T'_{\mu\nu} = -\frac{1}{2}e_{\alpha\beta\mu\nu}T_{\alpha\beta}$.

We also give expressions for the components of $V$, $A$, and $T$ in three-dimensional form:

$$V = (\mathbf{V}, V_0) = -i\psi^*(\boldsymbol{\alpha}, 1)\psi,$$

$$A = (\mathbf{A}, A_0) = \psi^*(\boldsymbol{\Sigma}, \varrho)\psi,$$

$$T = (\mathbf{H}, \mathbf{E}) = i\psi^*(\beta\boldsymbol{\Sigma}, \beta\boldsymbol{\alpha})\psi.$$

Here $\mathbf{V}$ and $\mathbf{A}$ are the spatial, and $V_0$ and $A_0$ are the time-like components of four-dimensional vectors, $\mathbf{E}$ is a three-dimensional polar vector and $\mathbf{H}$ is an axial one, which together are equivalent to the tensor $T_{\mu\nu}$ ($T_{12} = H_3$, $T_{j4} = -iE_j$).

It follows from (9.19) that any quantity $R_{\mu\nu}$ which transforms like $\psi_\mu\bar{\psi}_\nu$ (we shall call it a spinor of the second rank) can be expressed in terms of the matrices $\gamma_\alpha$ in the following manner:

$$R_{\mu\nu} = (S\cdot 1 + P\gamma_5 + V_\alpha\gamma_\alpha + A_\alpha\gamma_5\gamma_\alpha + T_{\alpha\beta}\gamma_\alpha\gamma_\beta)_{\mu\nu}, \tag{9.21}$$

where $S$ is a scalar, $P$ is pseudoscalar, $V$ is a vector, $A$ is an axial vector, and $T$ is an antisymmetric tensor of the second rank.

## § 10. Electron and Positron States. States of Definite Momentum and Polarization

### 10.1. *Solutions with Positive and Negative Frequencies*

The general solution of the Dirac equation can be represented in the form of a Fourier integral: $\psi = \int \psi_k e^{ikr}\, dk$. The Fourier component $\psi_k$ is the electron wave function in momentum space. In accordance with (9.7) it satisfies the equations

$$i\frac{\partial \psi_k}{\partial t} = (\boldsymbol{\alpha} \boldsymbol{k} + \beta m)\psi_k .$$

If we write $\psi_k$ in the form

$$\psi_k = \begin{pmatrix} \varphi_k \\ \chi_k \end{pmatrix},$$

we have

$$i\frac{\partial \varphi_k}{\partial t} = m\varphi_k + \boldsymbol{\sigma}\boldsymbol{k}\chi_k ,$$

$$i\frac{\partial \chi_k}{\partial t} = -m\chi_k + \boldsymbol{\sigma}\boldsymbol{k}\varphi_k .$$

We seek the solution of this system of equations in the form

$$\psi_k = \psi_0(k)\, e^{-i\omega t}$$

or

$$\begin{pmatrix} \varphi_k \\ \chi_k \end{pmatrix} = \begin{pmatrix} \varphi_0(k) \\ \chi_0(k) \end{pmatrix} e^{-i\omega t} .$$

Then the equation for $\psi_k$ is

$$(\boldsymbol{\alpha}\boldsymbol{k} + \beta m - \omega)\psi_k = 0,$$

or

$$(\omega - m)\varphi_k - \boldsymbol{\sigma}\boldsymbol{k}\chi_k = 0,$$

$$-\boldsymbol{\sigma}\boldsymbol{k}\varphi_k + (\omega + m)\chi_k = 0 .$$

From the condition that this system of equations should have nontrivial solutions

$$\begin{vmatrix} \omega - m & -\boldsymbol{\sigma}\boldsymbol{k} \\ -\boldsymbol{\sigma}\boldsymbol{k} & \omega + m \end{vmatrix} = 0$$

we obtain the frequency $\omega = \pm\varepsilon$, where $\varepsilon = \sqrt{k^2 + m^2}$. The last formula expresses the well-known relativistic relation between energy and momentum.

We see that there exist two kinds of solutions of the Dirac equation: solutions with positive and negative frequencies. The general solution of the Dirac equation has the following form:

$$\psi = \psi^{(+)} + \psi^{(-)},$$

where

$$\psi^{(+)} = \int \psi_0^{(+)}(k) e^{(ikr-\varepsilon t)} dk,$$

$$\psi^{(-)} = \int \psi_0^{(-)}(-k) e^{-i(kr-\varepsilon t)} dk.$$

Such decomposition into positive and negative frequencies is relativistically invariant, since the sign of the frequency cannot change under Lorentz transformations. This can be seen, from the fact that the lowest positive frequency is equal to $m$, while the highest negative frequency is equal to $-m$; i.e., the domains of positive and negative frequencies are separated by the finite interval $2m$, while Lorentz transformations contain a continuous parameter.

Of the two spinors $\varphi_k$ and $\chi_k$ forming a bispinor, one is an arbitrary function of $k$, while the other can be expressed in terms of the first. For example, by specifying $\varphi_k^{(+)}$ and $\chi_k^{(-)}$ we can obtain $\varphi_k^{(-)}$ and $\chi_k^{(+)}$:

$$\begin{aligned} \chi_k^{(+)} &= \frac{\sigma k}{\varepsilon + m} \varphi_k^{(+)}, \\ \varphi_k^{(-)} &= -\frac{\sigma k}{\varepsilon + m} \chi_k^{(-)}. \end{aligned} \tag{10.1}$$

Solutions with positive and negative frequencies belong to different eigenvalues of the self-conjugate operator $H = \boldsymbol{\alpha}\mathbf{p} + \beta m$. Therefore

$$\psi_k^{(+)*} \psi_k^{(-)} = \varphi_k^{(+)*} \varphi_k^{(-)} + \chi_k^{(+)*} \chi_k^{(-)} = 0$$

or

$$\int \psi^{(+)*} \psi^{(-)} dr = 0.$$

These relations can be easily verified with the aid of (10.1).

The existence of two types of solutions is of fundamental significance for the theory of the electron. We examine the special solution of the Dirac equation in the form of a monochromatic wave

$$\psi = \psi_0(r) e^{-i\omega t}.$$

The function $\psi_0(r)$ is an eigenfunction of the Hamiltonian operator

$$H = \boldsymbol{\alpha}\mathbf{p} + \beta m, \qquad H\psi_0 = \omega\psi_0 = \pm\varepsilon\psi_0.$$

In quantum mechanics the eigenvalues of the Hamiltonian operator are interpreted as values of the particle energy. However, in relativistic quantum mechanics of the electron it is impossible to retain this interpretation. Indeed, the existence of wave functions with negative frequencies would imply the existence of electron states of negative energy, and the absence of a lowest energy state. This means that in interacting with other particles an electron could give up unlimited amounts of energy by going to ever lower energy states. The physical absurdity of such a conclusion requires a change in the fundamental assumption of quantum mechanics which relates the possible values of physical quantities to solutions of corresponding wave equations.

The altered assumptions must ascribe positive energies to states with negative frequencies. However, two different states will then correspond to a given value of the energy states with different signs of the frequency. Therefore, the altered assumptions must give a physical interpretation to this twofold degeneracy. We assume that the Dirac equation applies to electrons of both signs of charge (the electron and the positron). In this way a possibility arises of interpreting the sign of the frequency as characterizing the charge state of the electron. Corresponding to a positive frequency the electron must, for example, have the charge $e$, while corresponding to a negative frequency it must have the charge $-e$. We shall see later that such an assumption provides the possibility of constructing a theory of electrons and positrons and of their interaction with the electromagnetic field.

### 10.2. *The Charge Conjugation Transformation*

To facilitate the formulation of changes introduced into quantum mechanics as a result of the foregoing requirements, *charge conjugation*, a transformation of great significance, is introduced. We saw in § 9 that the wave function $\psi$ and the function $\bar{\psi} = \psi^*\beta$ obey different transformation laws in the transition from one coordinate system to another one and satisfy different equations. We shall show that it is possible to construct linear combinations of the components of $\bar{\psi}$ which will satisfy the same equations as the components of $\psi$ and will transform in the same way as the components of $\psi$. We denote these combinations by $\psi^c$ and set

$$\psi^c = C\bar{\psi},$$

where C is a four-rowed matrix.

The function $\psi^c$ is said to be *charge-conjugate* with respect to $\psi$.

We now investigate the conditions that must be satisfied by the matrix C in order that the function $\psi^c$ should satisfy the same equation as $\psi$; i.e.,

$$(i\hat{p}+m)\psi^c = 0, \quad p_a = -i\frac{\partial}{\partial x_a}.$$

In order to do this we use the Dirac equation for $\bar{\psi}$

$$(-ip_a\tilde{\gamma}_a+m)\bar{\psi} = 0.$$

On substituting into this expression in place of $\bar{\psi}$, $\bar{\psi} = C^{-1}\psi^c$, we obtain

$$(-ip_a\tilde{\gamma}_a+m)C^{-1}\psi^c = 0.$$

If the matrix C satisfies the conditions $-\tilde{\gamma}_a C^{-1} = C^{-1}\gamma_a$ then this equation may be rewritten in the form $C^{-1}(i\hat{p}+m)\psi^c = 0$, and after being multiplied on the left by C it may be reduced to the Dirac equation.

Thus, the matrix C, which is called the charge conjugation matrix, must satisfy the following conditions:

$$C\tilde{\gamma}_a = -\gamma_a C \tag{10.2}$$

in order that $\psi$ and $\psi^c$ should satisfy the same equations.

It may be easily shown that if these conditions are satisfied, the function $\psi^c$ will transform under Lorentz transformations in the same way as the function $\psi$. Indeed, in accordance with (9.18), under infinitesimal four-dimensional rotations the function $\psi^c$ transforms in the following manner:

$$\psi^c = C\bar{\psi}'(1-\tfrac{1}{2}\vartheta_{a\beta}\gamma_a\gamma_\beta) = C\big(1-\tfrac{1}{2}\vartheta_{a\beta}(\gamma_a\gamma_\beta)\big)^T\bar{\psi}',$$

where the superscript T denotes transposed matrices.

We require that the function $\psi^c$ should transform in the same way as $\psi$, i.e., that the following relation should hold:

$$\psi^c = (1+\tfrac{1}{2}\vartheta_{a\beta}\gamma_a\gamma_\beta)\psi^{c'},$$

where

$$\psi^{c'} = C\bar{\psi}'.$$

In order to achieve this the matrix C must satisfy the following conditions:

$$\mathrm{C}(\gamma_\alpha \gamma_\beta)^{\mathrm{T}} = -\gamma_\alpha \gamma_\beta \mathrm{C}, \qquad \alpha \neq \beta$$

or

$$\mathrm{C}\tilde{\boldsymbol{\alpha}} = -\boldsymbol{\alpha}\mathrm{C}.$$

But these conditions follow directly from (10.2).

We now investigate how $\psi^{\mathrm{c}}$ transforms under space inversion.

We assume that the transformed functions $\psi^{\mathrm{c}\prime}$ and $\bar{\psi}'$ are related by the same expression which connects the functions $\psi^{\mathrm{c}}$ and $\bar{\psi}$; i.e.,

$$\psi^{\mathrm{c}\prime} = \mathrm{C}\bar{\psi}'.$$

The functions $\psi$ and $\bar{\psi}$ transform in accordance with (9.16) in the following manner:

$$\psi' = \eta_p \gamma_4 \psi,$$
$$\bar{\psi}' = \eta_p^* \bar{\psi} \gamma_4$$

where $\eta_p = \pm i, \pm 1$. We set

$$\psi^{\mathrm{c}\prime} = \eta_p^{\mathrm{c}} \gamma_4 \psi^{\mathrm{c}}$$

where $\eta_p^{\mathrm{c}}$, like $\eta_p^{\wedge}$, may have any one of the four values $\pm i, \pm 1$. On substituting this expression into the equation relating $\psi^{\mathrm{c}\prime}$ to $\bar{\psi}'$, we obtain

$$\psi^{\mathrm{c}} = \frac{\eta_p^*}{-\eta_p^{\mathrm{c}}} \gamma_4 \mathrm{C} \bar{\psi} \gamma_4.$$

Further, on utilizing (10.2) we obtain

$$\psi^{\mathrm{c}} = -\frac{\eta_p^*}{\eta_p^{\mathrm{c}}} \mathrm{C}\bar{\psi},$$

and, consequently, $\eta_p^{\mathrm{c}} = -\eta_p^*$. Thus, if $\eta_p = \pm 1$, then $\eta_p^{\mathrm{c}} = \mp 1$, while if $\eta_p = \pm \mathrm{i}$, then also $\eta_p^{\mathrm{c}} = \pm \mathrm{i}$.

In other words, if under inversion the spinor transforms in accordance with (9.4), then the charge conjugate spinor transforms in the same way; but if the spinor transforms in accordance with (9.3), then the transformation of the charge-conjugate spinor differs in sign from the transformation of the original spinor.

We shall now show that a matrix C with such properties actually does exist. In order to do this we utilize the Majorana representation (cf. subsection 9.3). Since the matrices $\alpha_j^{(M)}$ are real, while the matrix $\beta^{(M)}$ is imaginary, the conditions (10.2) will be satisfied if we set

$$C^{(M)} = -\beta^{(M)} = -\alpha_y. \tag{10.3}$$

It may be easily seen that in the Majorana representation the charge-conjugate function coincides with the complex conjugate function $\psi^{(M)c} = \psi^{(M)*}$.

We obtain the transformation law for the matrix C corresponding to a unitary transformation of $\psi$. Let $\psi' = U\psi, \psi^{c\prime} = U\psi^c$. On introducing the relations $\psi^{c\prime} = C'\overline{\psi}'$ and $\psi^c = C\overline{\psi}$ and on comparing these formulas we obtain

$$\begin{aligned} C' &= UC\tilde{U}, \\ C &= U^+C'U^*. \end{aligned} \tag{10.4}$$

Since in the Majorana representation the matrix C has the following properties, as may be easily seen from formula (10.3):

$$\begin{aligned} C &= -\tilde{C}, \\ C^+C &= 1, \\ C^*C &= -1, \end{aligned} \tag{10.5}$$

then by utilizing (10.4) it may be easily shown that these properties do not depend on the choice of the representation.

On going over from the Majorana representation to the usual one, i.e., on setting

$$C' = C^{(M)}, \quad U = \frac{1}{\sqrt{2}}(\alpha_y + \beta),$$

we obtain in accordance with (10.4)

$$C = \alpha_y. \tag{10.6}$$

We exhibit the different relations between $\psi$ and $\psi^c$:

$$\begin{aligned} \psi^c &= C\overline{\psi}, & \overline{\psi}^c &= C^+\psi, \\ \psi &= C\overline{\psi}^c, & \overline{\psi} &= C^+\psi^c. \end{aligned} \tag{10.7}$$

We also give the expressions for the bilinear combinations (9.19) in terms of the charge-conjugate functions

$$
\begin{aligned}
S &= -\psi^{c}\bar{\psi}^{c}, \\
V_\mu &= \psi^{c}\tilde{\gamma}_\mu\bar{\psi}^{c}, \\
T_{\mu\nu} &= \frac{1}{2}\psi^{c}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\nu)^{T}\bar{\psi}^{c}, \\
A_\mu &= -\psi^{c}(\gamma_\mu\gamma_5)^{T}\bar{\psi}^{c}, \\
P &= -\psi^{c}\tilde{\gamma}_5\bar{\psi}^{c}.
\end{aligned}
\tag{10.8}
$$

### 10.3. *The Positron Wave Function*

At this point let us consider the electron wave function in momentum space. Let

$$\psi = \int \psi(k)e^{ikx}\,d\boldsymbol{k},$$

$$\bar{\psi} = \int \bar{\psi}(k)\,e^{-ikx}\,d\boldsymbol{k},$$

where

$$\boldsymbol{k}x = \boldsymbol{kr}-\omega t, \qquad \omega = \pm\sqrt{\boldsymbol{k}^2+m^2}, \qquad \psi(k) \equiv \psi(\boldsymbol{k}, \omega).$$

On substituting this expansion into (9.12) and (9.13), we obtain:

$$(i\hat{k}+m)\psi(k) = 0, \qquad \bar{\psi}(k)\,(i\hat{k}+m) = 0.$$

On multiplying the last equation on the left by C and on moving $C\bar{\psi}(k)$ to the right, as was done earlier, we obtain

$$(-i\hat{k}+m)\psi^{c}(k) = 0.$$

We see that $\psi^{c}(k)$ satisfies the same equation as $\psi(-k)$. Since the replacement of $k$ by $-k$ includes the replacement of $\omega$ by $-\omega$, we see that $\psi^{c}(k)$ is a solution with positive frequency if $\psi(k)$ is a solution with negative frequency.

We introduce the following notation:

$$\psi^{(e)}(\boldsymbol{k}, \varepsilon) \equiv \psi(\boldsymbol{k}, \varepsilon),$$

$$\psi^{(p)}(\boldsymbol{k}, \varepsilon) \equiv \psi^{c}(-\boldsymbol{k}, -\varepsilon), \qquad (\varepsilon = |\omega|)$$

and correspondingly

$$\psi^{(e)} = \int \psi(\boldsymbol{k}, \varepsilon) e^{i(\boldsymbol{k}\boldsymbol{r}-\varepsilon t)} d\boldsymbol{k},$$

$$\psi^{(p)} = C\overline{\psi}^{(-)}; \qquad \psi^{(-)} = \int \psi(\boldsymbol{k}, -\varepsilon) e^{i(\boldsymbol{k}\boldsymbol{r}+\varepsilon t)} d\boldsymbol{k}.$$

If we take for the particle wave function corresponding to negative frequencies not $\psi^{(-)}$, but $\psi^{(p)}$, then its energy will be always positive. We shall call $\psi^{(p)}$ the positron wave function.

It should be emphasized that in defining the positron wave function by means of these relations we go outside the framework of transformations admissible in quantum mechanics, since we subject a part of the solutions of the Dirac equation to a nonlinear transformation (the transition to a complex-conjugate quantity is a nonlinear operation). As a result of this, the general solution of the Dirac equation is now $\psi = \psi^{(e)} + C\overline{\psi}^{(p)}$ and does not have the meaning of a wave function. On the other hand, the superposition of states of opposite sign of the charge $\psi^{(e)} + \psi^{(p)}$ cannot give a general solution of the Dirac equation since solutions with positive frequencies do not form a complete system of functions. This circumstance is, of course, immaterial as long as we are investigating states of a free electron with a definite sign of the charge. In this case the interpretation of the states with negative frequencies as states of the opposite sign of charge is of a purely formal nature, as long as the interaction with the electromagnetic field is not taken into account. Later we shall see that the use of the transformation $\psi^{(p)} = C\overline{\psi}^{(-)}$ enables us to construct a general theory for a system of electrons, taking into account their interaction with the electromagnetic field, with the states $\psi^{(e)}$ and $\psi^{(p)}$ actually corresponding to the charges $e$ and $-e$.

In concluding this subsection we also investigate the question of the relative intrinsic parity of the electron and the positron. In order to do this we must determine the transformation properties under space inversion of the wave functions of the electron and the positron at rest.

On setting $\boldsymbol{p} = 0$ in the Dirac equation we can easily show that these functions satisfy the following equations $\gamma_4\psi^{(e)} = \psi^{(e)}$ and $\gamma_4\psi^{(p)} = \psi^{(p)}$. As has been shown earlier, the functions $\psi^{(e)}$ and $\psi^{(p)}$ transform under inversion into $\psi^{(e)\prime} = \eta_p\gamma_4\psi^{(e)}$ and $\psi^{(p)\prime} = \eta_p^c\gamma_4\psi^{(p)}$. On utilizing the equations for $\psi^{(e)}$ and $\psi^{(p)}$, we obtain $\psi^{(e)\prime}\psi^{(p)\prime} = \eta_p\eta_p^c\psi^{(e)}\psi^{(p)}$. But in accordance with (10.2″) we have $\eta_p^c = -\eta_p^*$, and therefore $\psi^{(e)'}\psi^{(p)'} = -\psi^{(e)}\psi^{(p)}$.

This relation means that the electron and the positron have intrinsic parities of opposite sign (11).

### 10.4. *Plane Waves*

The electron states of definite frequency may be described by $\mathrm{H}\psi = \omega\psi$, where H is the Hamiltonian operator defined by formula (9.7′).

States corresponding to a definite value of the frequency are degenerate and in consequence we require that the wave functions of these states simultaneously be eigenfunctions of other operators which commute with each other and with the Hamiltonian operator.

We begin by examining states of definite momentum and denote the wave functions of such states by $\psi_p$. They satisfy the equation $\mathrm{p}\psi_p = p\psi_p$. For the sake of definiteness we take the frequency to be positive so that $\omega = \varepsilon = +\sqrt{p^2+m^2}$.

Then we have

$$\psi_p = \frac{u_p}{\sqrt{\Omega}} e^{ipr-i\varepsilon t}, \tag{10.9}$$

where $u_p$ is a constant bispinor (we shall also call it the spinor amplitude), which satisfies the equation $(\boldsymbol{\alpha}p+\beta m-\varepsilon)u_p = 0$, and $\Omega$ is the normalization volume. We assume that $u_p$ satisfies the normalization condition $u_p^* u_p = 1$. We also have: $\int_\Omega |\psi_p|^2 \, dr = 1$.

We write $u_p$ in the form of a column consisting of two spinors $w$ and $w'$:

$$u_p = \begin{pmatrix} w \\ w' \end{pmatrix} N, \tag{10.9'}$$

where $N$ is a normalization constant. The spinors $w$ and $w'$ are related to each other by equations (10.1). The spinor $w$ may be taken to be arbitrary, with

$$w' = \frac{\sigma p}{\varepsilon+m} w. \tag{10.10}$$

We assume that $w$ is a unit spinor, i.e., that it satisfies the condition $w^* w = 1$. Then the constant $N$ has the value $N = \sqrt{(\varepsilon+m)/2\varepsilon}$.

### 10.5. *Polarization of a Plane Wave*

For a given value of the momentum two different electron states are possible corresponding to two linearly independent spinors $w$. We refer to these two states as states of different polarization.

The two linearly independent spinors $w$ may be chosen, for example, in such a way that they are eigenfunctions of the operator $\frac{1}{2}\sigma_z$. We denote these eigenfunctions by $\chi_\mu$:

$$\frac{\sigma_z}{2}\chi_\mu(\lambda) = \mu\chi_\mu(\lambda),$$

where $\lambda$ is the spin variable which assumes the two values $\lambda = \pm\frac{1}{2}$. From the form of the matrix $\sigma_z$ it follows that $\mu = \pm\frac{1}{2}$ and $\chi_\mu(\lambda) = \delta_{\mu\lambda}$.

We note that if $w = \chi_\mu$ then $w'$ is not an eigenfunction of the operator $\sigma_z$. Indeed, in accordance with (10.10) we have

$$w'(\lambda) = \frac{1}{\varepsilon+m}\sum_{\lambda'}(\boldsymbol{\sigma}\boldsymbol{p})_{\lambda\lambda'}\chi_\mu(\lambda') = \frac{1}{\varepsilon+m}(\boldsymbol{\sigma}\boldsymbol{p})_{\lambda\mu}.$$

However, if we choose the $z$-axis in such a way that it is directed along the vector $\boldsymbol{p}$, then $\boldsymbol{\sigma}\boldsymbol{p} = p\sigma_z$ and $w'$ will also be an eigenfunction of the operator $\sigma_z$:

$$\tfrac{1}{2}\boldsymbol{\sigma}_z w'(\lambda) = \mu w'(\lambda) = \frac{\mu p}{\varepsilon+m}\chi_\mu(\lambda).$$

These results are a consequence of the fact that the electron spin operator $\frac{1}{2}\boldsymbol{\Sigma}$ does not commute with the Hamiltonian operator $\mathrm{H} = \boldsymbol{\alpha}\mathbf{p}+\beta m$, while the operator $\mathbf{p}\boldsymbol{\Sigma}$ does. Therefore, the eigenfunctions of the operator H, in general, are not eigenfunctions of the operator $\Sigma_z$, but can be eigenfunctions of the operator $\mathbf{p}\boldsymbol{\Sigma}$:

$$\frac{1}{2}\frac{\boldsymbol{\Sigma}\mathbf{p}}{|\boldsymbol{p}|}u_{p\mu} = \mu u_{p\mu}; \qquad \mu = \pm\tfrac{1}{2}. \tag{10.11}$$

These amplitudes describe states with a definite value of the component of the spin along the direction of motion. We denote the plane waves corresponding to them by

$$\psi_{p\mu} = \frac{u_{p\mu}}{\Omega^{1/2}}e^{ipx}.$$

We shall refer to states for which the bispinor $u_p$ is defined by formula (10.11) as states of definite polarization. The bispinors $u_{p\mu}$ corresponding to different values of $\mu$ are mutually orthogonal $u^*_{p\mu}u_{p\mu'} = \delta_{\mu\mu'}$.

In a similar manner we can discuss states of definite momentum $\boldsymbol{p}$ and of negative frequency $\omega = -\varepsilon$. We shall write the wave functions of such states in the form $\psi_p^{(-)} = (v_p/\sqrt{\Omega})e^{i\boldsymbol{pr}+i\varepsilon t}$, where $v_p$ is a unit bispinor satisfying $(\boldsymbol{\alpha p}+\beta m+\varepsilon)v_p = 0$.

For a given $\boldsymbol{p}$ there exist two linearly independent amplitudes $v_p$ which may be chosen to be the eigenfunctions of the operator $\boldsymbol{\Sigma p}$:

$$\tfrac{1}{2}\boldsymbol{\Sigma p}v_{p\mu} = |\boldsymbol{p}|\mu v_{p\mu}, \quad \mu = \pm\tfrac{1}{2}.$$

These functions are mutually orthogonal, i.e., $v^*_{p\mu}v_{p\mu'} = \delta_{\mu\mu'}$.

We compare the spinor amplitudes $u_p$ and $v_{-p}$.

The first of these amplitudes satisfies

$$(i\hat{p}+m)u_{p\mu} = 0 \tag{10.12}$$

and the second satisfies

$$(i\hat{p}-m)v_{-p\mu} = 0, \tag{10.12'}$$

where $p = (\boldsymbol{p}, i\varepsilon)$. Thus, $u_p$ and $v_{-p}$ may be regarded as eigenfunctions of the matrix $i\hat{p}$, corresponding to the eigenvalues $\mp m$. Since $i\hat{p}$ is a four-rowed matrix, the four linearly independent functions $u_{p\mu}$ and $v_{-p\mu}$ form a complete system of its eigenfunctions. (We note that $v_{-p} = \beta v_p$.).

On introducing $u_{p\mu} \equiv u_{\mu m}$ and $v_{-p\mu} \equiv u_{\mu,-m}$, we rewrite the equations for $u_{p\mu}$ and $v_{p\mu}$ in the following form:

$$i\hat{p}u_{\mu\Lambda} = -\Lambda u_{\mu\Lambda} \quad \Lambda = \mp m, \quad \mu = \pm\tfrac{1}{2}. \tag{10.13}$$

We now prove that

$$\bar{u}_{\mu\Lambda}u_{\mu'\Lambda'} \equiv \sum_{\alpha} \bar{u}_{\mu\Lambda}(\alpha)u_{\mu'\Lambda'}(\alpha) = \frac{\Lambda}{\varepsilon}\delta_{\mu\mu'}\delta_{\Lambda\Lambda'}. \tag{10.14}$$

The functions $u_{\mu\Lambda}$ which are eigenfunctions of the non-Hermitian operator $i\hat{p}$ are not orthogonal in the usual sense of this word, i.e., $u^*_{\mu\Lambda}u_{\mu'\Lambda'} \neq 0$. However, it follows from (10.13) that $\bar{u}_{\mu\Lambda}u_{\mu'\Lambda'} = 0$ for $\Lambda \neq \Lambda'$.

Similarly, it follows from (10.11) that $\bar{u}_{\mu\Lambda}u_{\mu'\Lambda'} = 0$, $\mu \neq \mu'$. We now obtain $\bar{u}_{\mu\Lambda}u_{\mu\Lambda}$. By replacing $m$ in (10.10) by $\Lambda$, we obtain $\bar{u}u = N^2(w^*w - w'^*w') = \dfrac{\Lambda}{\varepsilon}$. Thus, the proof of (10.14) is complete.

In view of the completeness of the system of functions $u_{\mu\Lambda}$ the following relation also holds

$$\sum_{\mu\Lambda}\frac{\varepsilon}{\Lambda}\bar{u}_{\mu\Lambda}(\alpha)u_{\mu\Lambda}(\alpha') = \delta_{\alpha\alpha'}. \tag{10.15}$$

### 10.6. *Polarization Density Matrix for the Electron*

The spinor amplitude of the plane wave $u_p$ may in the general case be expanded in terms of the amplitudes $u_{p\mu}$ corresponding to definite components of the spin along the direction of the momentum:

$$u_p = \sum_{\mu=-1/2}^{+1/2} w^{\mu}u_{p\mu}. \tag{10.16}$$

The expansion coefficients $w^{\mu}$ in this expression are the components of the spinor amplitude of the electron wave function in the rest system. Indeed, in the rest system in accordance with formulas (10.9)–(10.12) the amplitudes have the following form:

$$u_{p\mu} = \begin{pmatrix}\chi_\mu\\ 0\end{pmatrix}, \quad \chi_\mu = \begin{pmatrix}\delta_{\mu 1/2}\\ \delta_{\mu,-1/2}\end{pmatrix}, \quad \boldsymbol{p} = 0, \tag{10.17}$$

and, consequently,

$$u_p = \begin{pmatrix}w\\ 0\end{pmatrix}, \quad w = \begin{pmatrix}w^{1/2}\\ w^{-1/2}\end{pmatrix}.$$

Just as in the case of the photon (cf. subsection 2.4), we can describe the states of an electron of a given momentum by means of the polarization density matrix

$$\varrho_{\mu\nu} = \overline{w^{\mu}w^{\nu *}}, \tag{10.18}$$

where the bar over the right hand side denotes averaging over the parameters which characterize the system of which the electron is a part. The diagonal elements of the matrix $\varrho$ are equal to the probabilities that the component of electron spin along the $z$-axis (the direction of $\boldsymbol{p}$) is equal to $+\frac{1}{2}$ or $-\frac{1}{2}$. The normalization condition for $\varrho$ has the form $\mathrm{Sp}\,\varrho = 1$.

Just as in the case of the photon, the matrix $\varrho$ may be written in the form

$$\varrho = \tfrac{1}{2}(1+\boldsymbol{\zeta}\boldsymbol{\sigma}), \tag{10.19}$$

where $\zeta < 1$. The parameter $\zeta_z = \boldsymbol{\zeta} \boldsymbol{p}/|\boldsymbol{p}|$ determines the probability of polarization along the $z$-axis and $\varrho_{11} = \frac{1}{2}(1+\zeta_z)$. The parameters $\zeta_x$ and $\zeta_y$ appear in an analogous manner in the expressions for $\varrho_{11}$ in the system of coordinates which has been rotated by a certain angle with respect to the original one. Indeed, when the coordinate system has been rotated about the $y$-axis by the angle $\pi/2$ the spinor $w$ is multiplied by the unitary matrix $\exp(\frac{1}{4} i\pi\sigma_y) = (1+i\sigma_y)/\sqrt{2}$ and in accordance with (10.18) the matrix $\varrho$ goes over into $\varrho' = \frac{1}{2}(1+i\sigma_y)\varrho(1-i\sigma_y)$, from which we obtain $\varrho'_{11} = \frac{1}{2}(1+\zeta_x)$.

Similarly, a rotation about the $x$-axis through the angle $-\pi/2$ corresponds to the transformation of the density matrix $\varrho'' = \frac{1}{2}(1-i\sigma_x)\varrho(1+i\sigma_x)$ from which it follows that $\varrho''_{11} = \frac{1}{2}(1-\zeta_y)$. We note that the expression for $\varrho'_{11}$ and $\varrho''_{11}$ cannot be interpreted as the probability that the component of the electron spin along the $x$- or the $y$-axis has the value $+\frac{1}{2}$. However, we can say that $\varrho'_{11}$ is the probability that the component of the electron spin along the $x$-axis is equal to $\frac{1}{2}$ in the rest system of the electron (and similarly in the case of $\varrho''_{11}$).

When $\zeta = 0$ we have a completely unpolarized state, and when $\zeta = 1$ we have a state of complete polarization. If we introduce the notation $\boldsymbol{\zeta} = P\boldsymbol{\zeta}_0$, where $|\boldsymbol{\zeta}_0| = 1$, then the matrix $\varrho$ may be written as a linear combination of two matrices corresponding to the unpolarized and the completely polarized states

$$\varrho = \frac{1-P}{2} + \frac{P}{2}(1+\boldsymbol{\zeta}_0\boldsymbol{\sigma}).$$

The vector $\frac{1}{2}\boldsymbol{\zeta}$ represents the average value of the electron spin in the rest system. Indeed, in the rest system the average value of $\boldsymbol{\Sigma}$ is equal to $\overline{\boldsymbol{\Sigma}} = \overline{\boldsymbol{\sigma}} = w^*\boldsymbol{\sigma}w$. On utilizing (10.18) and (10.19) we obtain

$$\overline{\boldsymbol{\sigma}} = \operatorname{Sp} \varrho\boldsymbol{\sigma} = \boldsymbol{\zeta}. \tag{10.20}$$

The density matrix (10.18) is a matrix in the space of the eigenfunctions $u_{p\mu}$ of the spin operator. We can also investigate the density matrix in the "coordinate" representation, i.e., in the space of the bispinor indices

$$\mathcal{P}_{\lambda\sigma} = \overline{u_p(\lambda)u_p^*(\sigma)} \quad \text{where} \quad \lambda, \sigma = 1,\ 2,\ 3,\ 4. \tag{10.21}$$

In contrast to the two-rowed matrix (10.18) the matrix (10.21) is a four-rowed one. However, its values for different $\lambda$ and $\sigma$ are not independent if we are discussing, as we did earlier, states with a definite sign of the frequency. Therefore the matrix (10.21) may be characterized by the same three parameters as the matrix (10.18).

It is convenient to write the matrix $\mathcal{P}$ in a form in which it is not necessary to take explicitly into account the relationship between the different components $u_p(\lambda)$. In order to do this we write, $u_p = \eta(p)u$, where

$$\eta(p) = \frac{m - i\hat{p}}{2m} \tag{10.22}$$

and $u$ is a certain bispinor, whose components are no longer related by the Dirac equations. The validity of this equation can be easily verified if one notes that $(i\hat{p}+m)\eta(p) = 0$, in virtue of which $u_p$ satisfies the Dirac equation for arbitrary $u$.

Similarly, the amplitude of the state of negative frequency $v_{-p}$ can be written in the form $v_{-p} = \eta(-p)u$.

The matrix $\eta(p)$ is called a projection operator. It can be easily shown that the following relations hold

$$\big(\eta(p)\big)^2 = \eta(p), \qquad \bar{\eta} \equiv \gamma_4 \eta^+ \gamma_4 = \eta, \tag{10.23}$$

$$\bar{u}_p = \bar{u}\eta(p), \qquad \eta(p)u_{\mu\Lambda} = u_{p\mu}\delta_{m\Lambda}. \tag{10.24}$$

On substituting (10.22) into (10.21) we obtain the following expression for $\mathcal{P}$:

$$\mathcal{P}_{\lambda\sigma} = \eta_{\lambda a}(p)\overline{u(a)\bar{u}(\beta)}\big(\eta(p)\gamma_4\big)_{\beta\sigma}. \tag{10.25}$$

We first consider the unpolarized state. We show that in this case

$$\mathcal{P} \doteq \frac{m}{2\varepsilon}\eta\gamma_4 = \frac{m - i\hat{p}}{4\varepsilon}\gamma_4. \tag{10.26}$$

In order to do this we substitute the expansion (10.16) into (10.21), and this leads to the following relation between $\mathcal{P}$ and $\varrho$:

$$\mathcal{P}_{\lambda\sigma} = \sum_{\mu,\nu=-1/2}^{+1/2} \varrho_{\mu\nu} u_{p\mu}(\lambda) u^*_{p\nu}(\sigma). \tag{10.27}$$

Since in the unpolarized state $\varrho_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu}$, therefore

$$\mathcal{P}_{\lambda\sigma} = \tfrac{1}{2}\sum_{\mu} u_{p\mu}(\lambda) u^*_{p\mu}(\sigma).$$

By utilizing the projection operator we can, in accordance with (10.22) and (10.24), rewrite this expression in the following form:

$$\sum_{\mu} u_{\mathbf{p}\mu}(\lambda) u^*_{\mathbf{p}\mu}(\sigma) = \eta_{\lambda\alpha}(\eta\gamma_4)_{\beta\sigma} \sum_{\mu\Lambda} \frac{m}{\Lambda} u_{\mu\Lambda}(\alpha)\bar{u}_{\Lambda\mu}(\beta),$$

where $u_{\mu\Lambda}$ are defined by formula (10.13). In virtue of the orthogonality property (10.15) we obtain

$$\begin{aligned} \sum_{\mu} u_{\mathbf{p}\mu}(\lambda)\bar{u}_{\mathbf{p}\mu}(\sigma) &= \left(\frac{m-i\hat{p}}{2\varepsilon}\right)_{\lambda\sigma}, \\ \sum_{\mu} u_{\mathbf{p}\mu}(\lambda)u^*_{\mathbf{p}\mu}(\sigma) &= \left(\frac{m-i\hat{p}}{2\varepsilon}\gamma_4\right)_{\lambda\sigma}, \end{aligned} \tag{10.28}$$

which leads to the expression (10.26) for $\mathcal{P}$.

On comparing (10.26) and (10.25) we see that in the case of an unpolarized state, $\overline{u(\alpha)\bar{u}(\beta)} = m\delta_{\alpha\beta}/2\varepsilon$ holds.

In the general case of partial polarization the quantity $\overline{u(\alpha)\bar{u}(\beta)}$, which in accordance with (10.25) defines the density matrix, must be of the following form:

$$\overline{u(\alpha)\bar{u}(\beta)} = \frac{m}{2\varepsilon}(\delta_{\alpha\beta}+R_{\alpha\beta}), \tag{10.29}$$

where $R$ is a certain four-rowed matrix (a spinor of the second rank).

For the construction of the matrix $R$ we start from the fact that the density matrix $\mathcal{P}$, and consequently also $R$, depends on three parameters. In the electron rest system these three parameters form the axial vector $\boldsymbol{\zeta}$.

Therefore, the state of polarization may be described in a relativistically invariant manner by two equivalent methods: either with the aid of the antisymmetric tensor $t_{\mu\nu}$, or with the aid of the axial vector $a_\mu$, which is related to $t_{\mu\nu}$ by $t_{\mu\nu} = \frac{i}{m} e_{\mu\nu\lambda\varrho} p_\lambda a_\varrho$, where $e_{\mu\nu\lambda\varrho}$ is the unit antisymmetric tensor of the fourth rank and $p_\lambda$ is the momentum four-vector.

The tensor $t_{\mu\nu}$ must satisfy $t_{\mu\nu}p_\nu = 0$, from which it follows that in the rest system only three of its components differ from zero. The tensor $t_{\mu\nu}$ is equivalent to the two three-dimensional vectors $\boldsymbol{s}$ and $\boldsymbol{d}$

$(t_{ij}=e_{ijk}s_k;\ t_{j4}=-id_j,\ i,j,k=1,2,3)$, which may be expressed in terms of the vector $\boldsymbol{\zeta}$ in the following manner:

$$\boldsymbol{d}=\left[\boldsymbol{s},\frac{\boldsymbol{p}}{\varepsilon}\right];\qquad s^{(\parallel)}=\zeta^{(\parallel)};\qquad s^{(\perp)}=\frac{\varepsilon}{m}\zeta^{(\perp)},\tag{10.30}$$

where the symbols $(\parallel)$ and $(\perp)$ denote longitudinal and transverse components of vectors with respect to $\boldsymbol{p}$.

Similarly, the vector $a_\mu$ must satisfy $a_\mu p_\mu=0$. Its components $a_j$, $j=1,2,3$ and $a_0=-ia_4$ are related to the vector $\boldsymbol{\zeta}$ by the following equations:

$$a^{(\parallel)}=\frac{\varepsilon}{m}\zeta^{(\parallel)};\qquad a^{(\perp)}=\zeta^{(\perp)};\qquad a_0=\frac{\boldsymbol{p}\boldsymbol{a}}{\varepsilon}.\tag{10.30'}$$

In accordance with the general expression (9.21) the matrix $R$ may be represented by either $R=-\frac{1}{2}it_{jk}\gamma_j\gamma_k$ or $R=-i\hat{a}\gamma_5$. On substituting this expression into (10.29) and (10.25), and on noting that $\eta(p)R=R\eta(p)$, we obtain two identical expressions for $\mathcal{P}$:

$$\mathcal{P}=\frac{1}{4\varepsilon}\left(1-\frac{i}{2}t_{\mu\nu}\gamma_\mu\gamma_\nu\right)(m-i\hat{p})\gamma_4\tag{10.31}$$

or (135)

$$\mathcal{P}=\frac{1}{4\varepsilon}(1-i\hat{a}\gamma_5)(m-i\hat{p})\gamma_4.\tag{10.31'}$$

From these expressions it follows that

$$\begin{aligned}t_{\mu\nu}&=\frac{\varepsilon}{m}\operatorname{Sp}\mathcal{P}\gamma_4\frac{\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu}{2i},\\ a_\mu&=\frac{\varepsilon}{im}\operatorname{Sp}\mathcal{P}\gamma_4\gamma_\mu\gamma_5.\end{aligned}\tag{10.32}$$

Thus, if the density matrix $\mathcal{P}$ is known, the state of polarization may be obtained by determining $t_{\mu\nu}$ or $a_\mu$ from formulas (10.32), and then obtaining the vector $\boldsymbol{\zeta}$ with the aid of formulas (10.30), (10.30').

It is easy to obtain results similar to (10.28) and (10.31) also for states with negative frequency and of momentum $-\boldsymbol{p}$:

$$\sum_\mu v_{-p\mu}(\lambda)\bar{v}_{-p\mu}(\sigma)=-\frac{(m+i\hat{p})_{\lambda\sigma}}{2\varepsilon},\tag{10.33}$$

$$\begin{aligned}\mathcal{P}^{(-)} &= -\frac{1}{4\varepsilon}\left(1-\frac{1}{2}it_{\mu\nu}\gamma_\mu\gamma_\nu\right)(m+i\hat{p})\gamma_4\\ &= -\frac{1}{4\varepsilon}(1-i\hat{a}\gamma_5)\,(m+i\hat{p})\gamma_4,\end{aligned} \tag{10.34}$$

where

$$\mathcal{P}^{(-)}_{\lambda\sigma} = \overline{v_{-p}(\lambda)v^*_{-p}(\sigma)}.$$

We note that the density matrix for the positron $\mathcal{P}^{(p)}_{\mu\nu} = u^{(p)}_{p\mu}u^{(p)*}_{p\nu}$ is related to $\mathcal{P}^{(-)}$ by $\mathcal{P}^{(p)}\gamma_4 = -(\mathrm{C}^+\mathcal{P}^{(-)}\gamma_4\mathrm{C})^{\mathrm{T}}$. If $\mathcal{P}^{(-)}$ is of the form (10.34), then $\mathcal{P}^{(p)}$ coincides with expression (10.31).

## 10.7 *Averaging over Polarization States*

In future we shall often encounter the following problem. We are given the expression $M = \bar{a}u_{p\mu} = \sum_\lambda \bar{a}(\lambda)u_{p\mu}(\lambda)$, where $u_{p\mu}$ is the spinor amplitude of a state of definite momentum and polarization (and of positive frequency), while $a$ is a particular bispinor. We are required to find the sum of the squares of the absolute values of $|M|^2$ over the different polarization states of the electron:

$$\sum_\mu |M|^2 = \sum_{\lambda\sigma}\bar{a}(\lambda)\bar{a}^*(\sigma)\sum_\mu u_{p\mu}(\lambda)u^*_{p\mu}(\sigma).$$

This summation can be easily carried out with the aid of formula (10.28). On utilizing it, and also the definitions $\bar{a} = a^*\gamma_4$, $(\bar{a})^* = \gamma_4 a$, we obtain

$$\sum_\mu |\bar{a}u_{p\mu}|^2 = \frac{1}{2\varepsilon}\bar{a}(m-i\hat{p})a. \tag{10.35}$$

Let $a$ be of the form $a = Ru_{p'\nu}$, where $R$ is a four-rowed matrix, while $u_{p'\nu}$ is the amplitude of the state of momentum $p'$ and polarization $\nu$, and consider the problem of evaluating $\sum_{\mu\nu}|\bar{u}_{p'\nu}Ru_{p\mu}|^2$. On utilizing once more the formula (10.28), and also the equality $\bar{a} = \bar{u}_{p'\nu}\bar{R}$, where

$$\bar{R} = \gamma_4 R^+\gamma_4, \tag{10.36}$$

we obtain

$$\sum_{\mu\nu}|\bar{u}_{p'\nu}Ru_{p\mu}|^2 = \frac{1}{4\varepsilon\varepsilon'}\,\mathrm{Sp}\,(m-i\hat{p})R(m-i\hat{p}')\bar{R}. \tag{10.37}$$

Similar results may also be easily obtained for states of negative frequency:

$$\sum_{\mu} |\bar{a}v_{-p\mu}|^2 = -\frac{1}{2\varepsilon}\bar{a}(m+i\hat{p})a, \tag{10.38}$$

$$\sum_{\mu\nu} |\bar{v}_{-p'\nu}Rv_{-p\mu}|^2 = \frac{1}{4\varepsilon\varepsilon'}\,\mathrm{Sp}\,(m+i\hat{p})\,R(m+i\hat{p}')\,\bar{R}, \tag{10.39}$$

$$\sum_{\mu\nu} |\bar{u}_{p'\nu}Rv_{-p\nu}|^2 = -\frac{1}{4\varepsilon\varepsilon'}\,\mathrm{Sp}\,(m+i\hat{p})\,R(m-i\hat{p}')\bar{R}. \tag{10.40}$$

The matrix appearing after the trace symbol Sp in formulas (10.37)–(10.40) represents a sum of terms each of which is a product of several matrices $\gamma_\mu$. For the evaluation of the trace we can utilize the fact that the quantities $\mathrm{Sp}\,\gamma_{\nu_1}\gamma_{\nu_2}\dots\gamma_{\nu_n}$ form a four-dimensional tensor of the $n$th rank. The last assertion follows from the fact that in accordance with the proof given in § 9 the expression $\bar{\psi}\gamma_{\nu_1}\gamma_{\nu_2}\dots\gamma_{\nu_n}\psi$ is a tensor of the $n$th rank. Therefore the quantities $\sum_{A\mu} \bar{u}_{A\mu}\gamma_{\nu_1}\gamma_{\nu_2}\dots\gamma_{\nu_n}u_{A\mu} = \mathrm{Sp}\,\gamma_{\nu_1}\dots\gamma_{\nu_n}$ will also be tensors if the amplitudes are normalized in an invariant manner, i.e., $\bar{u}u = 1$.

Since the matrices $\gamma_\nu$ have the same form in an arbitrary coordinate system, the form of the tensor $\mathrm{Sp}\gamma_{\nu_1}\dots\gamma_{\nu_n}$ also does not depend on the choice of the coordinate system. The only tensor which has this property is $\delta_{\alpha\beta}$. Therefore the desired tensor can be constructed from $\delta_{\alpha\beta}$.

From this it follows immediately that if $n$ is an odd number, then we have

$$\mathrm{Sp}(\gamma_{\nu_1}\gamma_{\nu_2}\dots\gamma_{\nu_n}) = 0 \quad (n = 2k+1).$$

If $n$ is an even number, then we have

$$\mathrm{Sp}(\gamma_{\nu_1}\dots\gamma_{\nu_n}) = \sum_{P} a_P\,\delta_{ik}\delta_{lm}\dots,$$

where $i, k, l, m$, is some combination of indices $\nu_1, \nu_2, \nu_3, \dots$ and $a_p$ are numerical coefficients. The sum is taken over all possible combinations of pairs of numbers $ik$, $lm$, ..., with the number of terms in the sum, i.e., the number of such combinations, evidently being equal to

$$\frac{n!}{2^{n/2}(n/2)!} = 1\cdot 3\cdot 5\dots(\mathrm{n}-1).$$

For the determination of the coefficients $a_p$ it is sufficient to set $i = k$, $l = m$, etc. On utilizing (9.11) and on noting that Sp $1 = 4$, we obtain

$$\tfrac{1}{4}\,\mathrm{Sp}(\gamma_{\nu_1} \cdots \gamma_{\nu_n}) = \sum_P \pm\delta_{ik}\delta_{lm} \cdots \qquad (n = 2k). \qquad (10.41)$$

The signs of the individual terms in the sum may be found in the following manner (215), (153), cf. also (37). We let each matrix $\gamma_\nu$ correspond to a point on the circumference of a circle, and we arrange these points in the same order in which the matrices $\gamma_\nu$ occur. We then join these points in pairs by straight lines. Then to each straight line joining the points $i$ and $k$ there corresponds a factor $\delta_{ik}$, while to each way of joining points (or, in other words, to each decomposition of the subscripts $i, k, l, m, \ldots$ into pairs $i$—$k$, $l$—$m, \ldots$) there corresponds a term in the expansion of the trace $(-1)^P\delta_{ik}\delta_{lm}\cdots$, where $P$ is the number of points of intersection of the straight lines. Thus, we have $a_P = (-1)^P$.

We give the values of $\mathrm{Sp}\,\gamma_1 \ldots \gamma_{\nu_n}$ for $n = 2, 4, 6$:

$$\begin{aligned}
&\tfrac{1}{4}\,\mathrm{Sp}\,\gamma_i\gamma_k = \delta_{ik},\\
&\tfrac{1}{4}\,\mathrm{Sp}\,\gamma_i\gamma_k\gamma_l\gamma_m = \delta_{ik}\delta_{lm}+\delta_{im}\delta_{kl}-\delta_{il}\delta_{km},\\
&\tfrac{1}{4}\,\mathrm{Sp}\,\gamma_i\gamma_k\gamma_l\gamma_m\gamma_r\gamma_s = \delta_{ik}\delta_{lm}\delta_{rs}+\delta_{is}\delta_{kl}\delta_{mr}+\delta_{ik}\delta_{ls}\delta_{mr}+\delta_{im}\delta_{kl}\delta_{rs}\\
&\qquad +\delta_{is}\delta_{kr}\delta_{lm}+\delta_{il}\delta_{kr}\delta_{ms}+\delta_{im}\delta_{ks}\delta_{lr}+\delta_{ir}\delta_{km}\delta_{ls}-\delta_{is}\delta_{\lambda m}\delta_{ls}\\
&\qquad -\delta_{im}\delta_{kr}\delta_{ls}-\delta_{ik}\delta_{lr}\delta_{ms}-\delta_{il}\delta_{km}\delta_{rs}-\delta_{il}\delta_{ks}\delta_{mr}-\delta_{ir}\delta_{kl}\delta_{ms}\\
&\qquad -\delta_{ir}\delta_{ks}\delta_{lm}\,.
\end{aligned} \qquad (10.42)$$

We note that the expression for the matrix $\gamma_{\nu_1} \ldots \gamma_{\nu_n}$ can be simplified if among the subscripts there are two identical subscripts $\nu_i = \nu_k = a$ over which summation is carried out. By utilizing equation (9.11) we can rearrange the product in such a way that the two matrices with identical subscripts turn out to be side by side, and after this we can carry out the summation: $\gamma_a\gamma_a = 4$. For example, if two matrices are separated by one, two, or three factors, the following relations hold (it is understood that summation is carried out over the subscript $a$ from 1 to 4):

$$\begin{aligned}
\gamma_a\gamma_\nu\gamma_a &= -2\gamma_\nu,\\
\gamma_a\gamma_\mu\gamma_\nu\gamma_a &= 4\delta_{\mu\nu},\\
\gamma_a\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_a &= -2\gamma_\nu\gamma_\mu\gamma_\lambda.
\end{aligned} \qquad (10.43)$$

## § 11. Electron States of Definite Angular Momentum and Parity

### 11.1. *Orbital and Spin Functions. Spherical Spinors*

We now investigate electron states characterized by definite values of energy and of angular momentum.

The angular momentum operator $\mathbf{M}$ is a sum of the orbital angular momentum operator $\mathbf{L} = [\mathbf{r}, \mathbf{p}]$ and the spin angular momentum operator $\frac{1}{2}\mathbf{\Sigma}$; $\mathbf{M} = \mathbf{L}+\frac{1}{2}\mathbf{\Sigma}$. We shall denote the eigenfunctions of the operators of the square of the angular momentum $\mathbf{M}^2$ and of its component $\mathrm{M}_z$ corresponding to the eigenvalues $M^2 = j(j+1)$ and $M_z = M$ by

$$\psi_{jM} = \begin{pmatrix} \varphi_{jM} \\ \chi_{jM} \end{pmatrix}.$$

They satisfy the following equations:

$$\mathbf{M}^2\psi_{jM} = j(j+1)\psi_{jM},$$

$$\mathbf{M}_z\psi_{jM} = M\psi_{jM}.$$

Each of the spinors $\varphi_{jM}$ and $\chi_{jM}$ satisfies these same equations if we replace in them the four-rowed matrix $\mathbf{\Sigma}$ by the two-rowed matrix $\boldsymbol{\sigma}$, i.e., if we set $\mathbf{M} = \mathbf{L}+\frac{1}{2}\boldsymbol{\sigma}$.

Just as in Chapter I, we do not solve these equations directly, but in order to obtain the wave functions we use the quantum mechanical rules for the addition of angular momenta, by regarding the orbital and the spin degrees of freedom of the electron as two subsystems.

We already know the eigenfunctions of the operators for the orbital and the spin angular momenta. The eigenfunctions of the orbital angular momentum are the spherical harmonics $Y_{lm}(r/r)$, where $m$ and $l(l+1)$ are the eigenvalues of the operators $\mathrm{L}_z$ and $L^2$, while the eigenfunctions of the spin angular momentum are the functions $\chi_\mu(\lambda) = \delta_{\mu\lambda}$ (cf. subsection 10.5).

In accordance with the rules for the addition of angular momenta, a given value of the total angular momentum $j$ can be obtained for two values of angular momentum of the orbital subsystem $l = j\pm\frac{1}{2}$. We use the notation $\varphi_{jlM}$ to denote the spinor function $\varphi_{jM}$ which corresponds to the same values of $j$ and $M$, but to different values of $l$. We can represent it as a linear combination of products of orbital functions $Y_{lm}$ and spin functions $\chi_\mu$:

$$\varphi_{jlM} = a(r)\sum_{\mu m} C^{jM}_{lm1/2\mu} Y_{lm}(\mathbf{n})\chi_\mu, \qquad \mathbf{n} = \frac{\mathbf{r}}{r},$$

where $a(r)$ is a radial function which we shall define later.

This equation can be regarded as an expansion of the spinor $\varphi$ in terms of the orthonormal spinors $\chi_\mu$, $\varphi(\lambda) = \sum_\mu \varphi^\mu \chi_\mu(\lambda)$, which are the basic unit spinors analogous to the basic unit vectors $\chi_\mu$ in § 3.

We call $\varphi^\mu$ the contravariant spinor components. Since $\chi_\mu(\lambda) = \delta_{\mu\lambda}$, we have $\varphi(\lambda) = \varphi^\lambda$, i.e., the contravariant components coincide with the components defined in subsection 9.1.

In addition to the contravariant components we can also utilize covariant spinor components defined as $\varphi_\mu = (-1)^{\mu-1/2}\varphi^{-\mu}$. In virtue of the formulas for the transformation of a spinor under rotation (9.1) this definition guarantees the invariance of the scalar product $\varphi^\mu_{\mathrm{I}}\varphi_{\mathrm{II}\mu}$ of the two spinors $\varphi_{\mathrm{I}}$ and $\varphi_{\mathrm{II}}$ under three-dimensional rotations.

The expression for $\varphi_{jlM}$ can also be written in the following form:

$$\varphi_{jlM} = a(r)\Omega_{jlM}(\mathbf{n}), \tag{11.1}$$

where $\Omega_{jlM}(\mathbf{n})$ is a spinor whose contravariant components are given by

$$\left(\Omega_{jlM}(\mathbf{n})\right)^\mu = C^{jM}_{l,M-\mu,1/2,\mu} Y_{l,M-\mu}(\mathbf{n}). \tag{11.2}$$

We refer to the quantity $\Omega_{jlM}$ as a spinor spherical harmonic or a spherical spinor.

The coefficients $C^{jM}_{l,M-\mu,s\mu}$, appearing in (11.2) differ from the corresponding coefficients in Chapter I by the value $s = \frac{1}{2}$. They are normalized in accordance with (8.4) and have the values in Table 6.

TABLE 6

| | $j$ | |
|---|---|---|
| $\mu$ | $l+\frac{1}{2}$ | $l-\frac{1}{2}$ |
| $\frac{1}{2}$ | $\sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}$ | $-\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}$ |
| $-\frac{1}{2}$ | $\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}$ | $\sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}$ |

The spherical spinors form an orthonormal system of functions. For different values of $j$, $M$, or $l$ the functions are mutually orthogonal, since they are eigenfunctions of self-conjugate operators belonging to different eigenvalues. In virtue of (8.4) they are also normalized: $\int \Omega^*_{jlM}\Omega_{j'l'M'}\,do = \delta_{jj'}\delta_{ll'}\delta_{MM'}$, where $do$ is an element of solid angle in the direction $\boldsymbol{n}$.

### 11.2. *Wave Function of a State of Definite Angular Momentum*

In order to determine the radial dependence of $\varphi_{jlM}$, i.e., the function $a(r)$, we make use of the Fourier transformation:

$$\varphi(\boldsymbol{r}) = \int \varphi(\boldsymbol{k}) e^{i\boldsymbol{kr}}\,d\boldsymbol{k}.$$

States of definite angular momentum correspond to wave functions $\varphi(\boldsymbol{k})$ in momentum space which are eigenfunctions of the operators for the square of the angular momentum and for one of its components. The angular momentum operator has the same structure in the momentum representation as in the coordinate representation, i.e., $\mathbf{M} = \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ has the same meaning as before, while $\mathbf{L} = -i[\boldsymbol{k}\nabla_k]$. Therefore, we can directly write $\varphi_{jlM}(\boldsymbol{k}) = a(k)\Omega_{jlM}(\boldsymbol{\nu})$, where $\boldsymbol{\nu} = \boldsymbol{k}/k$. Since the electron energy has a definite value, this fixes the absolute value of its momentum $p = (\varepsilon^2 - m^2)^{1/2}$. We, therefore, assume that $a(k)$ differs from zero only when $k = p$. On substituting $\varphi_{jlM}(\boldsymbol{k})$ into the Fourier integral we obtain $\varphi_{jlM}(\boldsymbol{r}) = C\int e^{ip\boldsymbol{\nu r}}\Omega_{jlM}(\boldsymbol{\nu})\,do_\nu$, where $C$ is a constant.

We then make use of the expansion of $\exp(ip\boldsymbol{\nu r})$ in terms of spherical harmonics (cf. (4.24)). This yields

$$\varphi_{jlM} = C\sum_{l'm'} g_{l'}(pr)\, Y_{l'm'}(\boldsymbol{n}) \int Y^*_{l'm'}(\boldsymbol{\nu})\Omega_{jlM}(\boldsymbol{\nu})\,do_\nu.$$

On substituting into this equation the expression for the spherical spinors (11.2) we obtain

$$\sum_{m'} Y_{l'm'}(\boldsymbol{n}) \int Y^*_{l'm'}(\boldsymbol{\nu})\Omega_{jlM}(\boldsymbol{\nu})\,do_\nu = \delta_{ll'}\Omega_{jl'M}(\boldsymbol{n})$$

and, consequently, $\varphi_{jlM} = Cg_l(pr)\Omega_{jlM}(\boldsymbol{n})$, where $g_l$ is the function defined by formula (4.25). On comparing this expression with (11.1) we see that $a(r) = Cg_l(pr)$.

We now obtain the second spinor $\chi_{jlM}$ which appears in the electron wave function $\psi_{jlM}$. On making use of the fact that in momentum space the spinors $\chi(\boldsymbol{k})$ and $\varphi(\boldsymbol{k})$ are related by the simple expression (10.1) we obtain

$$\chi_{jlM}(\boldsymbol{k}) = a(k)\frac{k}{\varepsilon+m}(\boldsymbol{\sigma}\boldsymbol{\nu})\Omega_{jlM}(\boldsymbol{\nu}), \quad \boldsymbol{\nu} = \frac{\boldsymbol{k}}{k}\cdot$$

On the other hand, $\chi_{jlM}$, like $\varphi_{jlM}$, is an eigenfunction of the operators $\mathbf{M}^2$ and $\mathrm{M}_z$ belonging to the same values of $j$ and $M$. This means that its angular dependence must be determined by a spinor spherical harmonic. From this we can conclude that the product $(\boldsymbol{\sigma}\boldsymbol{\nu})\Omega_{jlM}(\boldsymbol{\nu})$ is a spherical spinor. Since in accordance with (11.2) $\Omega_{jlM}$ contains the spherical harmonic $Y_{lm}$, then (cf. (4.9)) the product $\boldsymbol{\nu}Y_{lm}(\boldsymbol{\nu})$ contains the spherical harmonics $Y_{l'm}$, where $l' = l\pm 1$. Of these values only the one value of $l'$, viz.,

$$l' = 2j-l = \begin{cases} l+1, & l = j-\frac{1}{2}, \\ l-1, & l = j+\frac{1}{2}, \end{cases}$$

is compatible with the given value of $j$ in accordance with the rules for the addition of angular momenta.

Therefore $\boldsymbol{\sigma}\boldsymbol{\nu}\Omega_{jlM}(\boldsymbol{\nu}) = c\Omega_{jl'M}(\boldsymbol{\nu})$, where $c$ is a constant. In order to find this constant we choose the direction along the $z$-axis and make use of the definition (11.2) and the explicit expression for the coefficient $C^{j\mu}_{l0\,1/2\mu}$ and for the spherical harmonic $Y_{l0}\,(0)$. This yields

$$(\boldsymbol{\sigma}\boldsymbol{\nu})\Omega_{jlM}\,(\boldsymbol{\nu}) = -\Omega_{jl'M}(\boldsymbol{\nu}), \quad \text{where} \quad l' = 2j-l. \tag{11.3}$$

From this we obtain in the coordinate representation, in a manner analogous to the transition from $\varphi_{jlM}(\boldsymbol{k})$ to $\varphi_{jlM}(\boldsymbol{r})$, the following expression for $\chi_{jlM}(\boldsymbol{r})$:

$$\chi_{jlM} = -C\frac{p}{\varepsilon+m}g_{l'}(pr)\Omega_{jl'M}\,(\boldsymbol{n}) \quad \left(l' = 2j-l; \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r}\right)\cdot$$

The coefficient $C$ is determined from the normalization condition $\int|\psi_{jlM}|^2 d\boldsymbol{r} = 1$. The integration is here taken over the normalization volume which may be chosen in the form of a sphere of radius $R$ (cf. subsection 1.4). Since $R$ may be taken to be arbitrarily large, the integration can be easily carried out if we make use of the asymptotic expression for the functions $g_l$ (4.26).

We give the expressions for the normalized wave functions:

$$\begin{aligned}\varphi_{jlM} &= \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{(\varepsilon+m)p^2\delta}{2\varepsilon}}\,g_l(pr)\,\Omega_{jlM}\left(\frac{\mathbf{r}}{r}\right),\\ \chi_{jlM} &= -\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{(\varepsilon-m)p^2\delta}{2\varepsilon}}\,g_{l'}(pr)\,\Omega_{jl'M}\left(\frac{\mathbf{r}}{r}\right),\end{aligned} \tag{11.4}$$

where $\delta = \pi/R$, $l' = 2j-l$.

We note that the wave functions normalized per unit energy interval, i.e., in accordance with the condition $\int \psi^*_{\varepsilon jlM}\psi_{\varepsilon' jlM}d\mathbf{r} = \delta(\varepsilon-\varepsilon')$, differ from the functions (11.4) only by the replacement of $\delta$ by $\varepsilon/p$.

In momentum space the normalized wave functions have the following form:

$$\begin{aligned}\varphi_{jlM}(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\varepsilon(\varepsilon-m)\delta}}\,\Omega_{jlM}\left(\frac{\mathbf{k}}{k}\right),\\ \chi_{jlM}(\mathbf{k}) &= -\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\varepsilon(\varepsilon+m)\delta}}\,\Omega_{jl'M}\left(\frac{\mathbf{k}}{k}\right).\end{aligned} \tag{11.5}$$

### 11.3. *Parity of a State*

We note that $\varphi_{jlM}$ and $\chi_{jlM}$ belong to different eigenvalues of the operator $\mathbf{L}^2$: $l(l+1)$ in the former case and $l'(l'+1)$ in the latter case. This means that the electron wave function $\psi_{jlM}$ is not an eigenfunction of the orbital angular momentum operator. It can also be easily verified directly that the operator $\mathbf{L}$ does not commute with the Hamiltonian operator $H$. We therefore conclude that the decomposition of the total angular momentum into orbital and spin parts has a restricted physical meaning in the case of the electron, just as in the case of the photon. However, this decomposition becomes fully valid in the nonrelativistic approximation. As may be seen from (10.1) the ratio of $\chi$ to $\varphi$ tends to zero as $k \to 0$. This means that for low values of the energy we can use a wave function consisting of only two components (the remaining two components are small). In such a case $l$ acquires the meaning of orbital angular momentum. However, in the general case the index $l$ on the wave function serves only to designate two different states of the electron with the same values of $j$ and $M$.

We can uniquely associate the quantum number $l$ with the parity of the electron state. In accordance with the law for the inversion of spinors the inversion operator has the following form:

$$\mathrm{I} = \eta_p \beta \mathrm{I}_r \tag{11.6}$$

where $\mathrm{I}_r$ is the operator for the reversal of sign of the space coordinates $\mathrm{I}_r \psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$.

We apply the operator (11.6) to the wave function $\psi_{jlM}$

$$\mathrm{I}\psi_{jlM} = \eta_p \mathrm{I}_r \beta \begin{pmatrix} \varphi_{jlM} \\ \chi_{jlM} \end{pmatrix} = \eta_p \mathrm{I}_r \begin{pmatrix} \varphi_{jlM} \\ -\chi_{jlM} \end{pmatrix}.$$

Since $\varphi_{jlM}$ contains the spherical harmonic $Y_{lm}$, while $\chi_{jlM}$ contains the spherical harmonic $Y_{l'm}$, and $\mathrm{I}_r Y_{lm} = (-1)^l Y_{lm}$, $\mathrm{I}_r Y_{l'm} = (-1)^{l'} Y_{l'm} = (-1)^{l+1} Y_{l'm}$, we have

$$\mathrm{I}\psi_{jlM} = I\psi_{jlM}, \quad I = (-1)^l. \tag{11.7}$$

Thus, $\psi_{jlM}$ is an eigenfunction of the inversion operator corresponding to the eigenvalue $I = (-1)^l$. We shall refer to the quantity $I$ as the parity of the state. Thus the quantum number $l$ which for a given value of $j$ is uniquely related to $I$ defines parity, rather than orbital angular momentum, which does not have a rigorous meaning. Two states characterized by different values of $l$ for a given $j$ differ in parity.

### 11.4 *Expansion in Spherical Waves*

We have constructed two different complete systems of electron wave functions $\psi_{p\mu}$ and $\psi_{jlM}$. An arbitrary solution of the Dirac equation may be expanded in terms of either one of these systems: $\psi = \sum\limits_n a_n \psi_n$, where $n$ denotes the set of quantum numbers $p\mu$ or $jlM$ (and also $\varepsilon$). If $\psi$ and $\psi_n$ are normalized, then we have $a_n = \int \psi\psi_n^* d\boldsymbol{r}$. In particular, the wave function of a state of definite momentum and polarization can be expanded in terms of wave functions of states of definite angular momentum and parity:

$$\psi_p = \sum_{jlM} a_{jlM}^{(p)} \psi_{jlM}; \qquad \psi_{p\mu} = \sum_{jlM} a_{jlM}^{(p\mu)} \psi_{jlM},$$

or the converse expansion can be performed

$$\psi_{jlM} = \sum_{p\mu} a_{p\mu}^{(jlM)} \psi_{p\mu}.$$

It is evident that

$$a_{jlM}^{(p\mu)} = (a_{p\mu}^{(jlM)})^*.$$

These coefficients can be conveniently calculated with the aid of wave functions in momentum space. On utilizing expressions (11.5) and (10.9) and on going over from Fourier series to Fourier integrals (cf. subsection 1.4) we obtain

$$a_{jlM}^{(\mathbf{p})} = \Omega_{jlM}^*\left(\frac{\mathbf{p}}{p}\right) w \sqrt{\delta_0}\,, \tag{11.8}$$

where $w$ is the spinor amplitude of the plane wave defined in accordance with (10.9′), while $\delta_0$ is an element of solid angle. In the case of states of definite component of spin along the direction of the momentum, i.e., when $w = \chi_\mu$ we denote $a_{jlM}^{(\mathbf{p})}$ by $a_{jlM}^{(p\mu)}$.

The quantity $|a_{p\mu}^{(jlM)}|^2 \dfrac{do}{\delta_0}$ defines the angular distribution of the electrons in a state of definite angular momentum. If we sum this quantity over the polarizations, we obtain

$$\frac{do}{\delta_0} \sum_\mu |a_{p\mu}^{(jlm)}|^2 = |\Omega_{jlM}(\mathbf{v})|^2 do.$$

For a given value of $j$ this quantity does not depend on $l$, since

$$|\Omega_{jlM}|^2 = \Omega_{jl'M}^*(\boldsymbol{\sigma}\mathbf{v})(\boldsymbol{\sigma}\mathbf{v})\Omega_{jl'M} = |\Omega_{jl'M}|^2.$$

The quantity $|\Omega_{jlM}|^2$ can be evaluated by the method presented in § 8; this yields

$$|\Omega_{jlM}|^2 = \frac{1}{4\pi} \sum_{n=0}^{j-\frac{1}{2}} a_{2n} P_{2n}(\cos\vartheta),$$

where

$$a_{2n} = (-1)^{2n-1-M} \frac{(2l+1)(2j+1)}{4\pi} C_{l0\,l0}^{2n0} C_{j,-M,\,jM}^{2n0} W\left(lljj;\, 2n\frac{1}{2}\right), \tag{11.9}$$

$W$ are the Racah coefficients. Explicit expressions for the coefficients $a_{2n}$ are given below in Table 7.

By utilizing (11.8) we can easily obtain the expansion of a plane wave in terms of spherical waves. If the amplitude of the plane wave

$$u = N\begin{pmatrix} w \\ w' \end{pmatrix}$$

is normalized ($uu^* = 1$), while the polarization is determined by the normalized spinor $w$ ($w^*w = 1$), then we have

$$ue^{i\mathbf{pr}} = \frac{1}{\sqrt{2}} \sum_{jlM} \left(\Omega^*_{jlM}(\mathbf{v})w\right) \begin{pmatrix} \sqrt{1+\frac{m}{\varepsilon}}\, g_l(pr)\Omega_{jlM}(\mathbf{n}) \\ -\sqrt{1-\frac{m}{\varepsilon}}\, g_{l'}(pr)\Omega_{jl'M}(\mathbf{n}) \end{pmatrix}, \tag{11.10}$$

where

$$\mathbf{n} = \frac{\mathbf{r}}{r}, \qquad \mathbf{v} = \frac{\mathbf{p}}{p}, \qquad l' = 2j-l.$$

TABLE 7

| | $j$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\frac{3}{2}$ | | $\frac{5}{2}$ | | | $\frac{7}{2}$ | | | |
| $a_{2n}$ $\lvert M\rvert$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{7}{2}$ |
| $a_0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a_2$ | 1 | $-1$ | 8/7 | 2/7 | $-10/7$ | 25/21 | 195/28 | $-5/3$ | $-5/21$ |
| $a_4$ | | | 6/7 | $-9/7$ | 3/7 | 81/77 | 288/77 | 63/77 | $-117/77$ |
| $a_6$ | | | | | | 25/33 | $-5/11$ | $-5/33$ | 25/33 |

For $pr \gg 1$ we can utilize the asymptotic expression for the function $g_l$ (4.26). Then we have

$$ue^{i\mathbf{pr}} \approx 4\pi \frac{1}{pr} \sum_{jlM} \left(\Omega^*_{jlM}(\mathbf{v})\,w\right) \begin{pmatrix} i^l \sqrt{\frac{m+\varepsilon}{2\varepsilon}}\, \Omega_{jlM}(\mathbf{n}) \sin\left(pr - \frac{l\pi}{2}\right) \\ -i^{l'} \sqrt{\frac{\varepsilon-m}{2\varepsilon}}\, \Omega_{jl'M}(\mathbf{n}) \sin\left(pr - \frac{l'\pi}{2}\right) \end{pmatrix}. \tag{11.11}$$

## 12. Electron in an External Field

### 12.1. *The Dirac Equation with an External Field*

In considering the one-body theory exclusively, only a limited number of problems can be formulated. These are problems in which the number of particles does not change, and the interaction can be introduced by means of the concept of an external field.

The equation of motion for an electron in a given external field may be obtained in the same way as in nonrelativistic quantum mechanics. Let $A$ be the four-potential of the external electromagnetic field ($A$ is the vector and $A_0 = -iA_4$ is the scalar potential). We shall obtain the desired equation if in the Dirac equation we replace the four-momentum operator p by p$-eA$, where $e$ is the electron charge. Evidently an equation of this type is relativistically invariant, since the transformation properties of the quantities p and $A$ are the same.

Thus, the Dirac equation, taking the external field into account, has the form

$$(i\hat{\mathrm{p}} - ie\hat{A} + m)\psi = 0, \tag{12.1}$$

where

$$\hat{A} = \gamma_\mu A_\mu .$$

We confine ourselves to the case when the external field is independent of the time. Then stationary solutions such as

$$\psi(\boldsymbol{r}, t) = \psi_0(\boldsymbol{r})e^{-i\omega t}$$

exist, where $\psi_0(r)$ is an eigenfunction of the Hamiltonian operator, i.e.,

$$\mathrm{H}\psi_0 = \omega\psi_0, \qquad \mathrm{H} = \boldsymbol{\alpha}\mathbf{p} + \beta m + eA_0 - e\boldsymbol{\alpha}A. \tag{12.2}$$

The general solution of (12.1) can be represented as a superposition of wave functions of different frequencies $\omega$.

From the point of view of the values of the frequencies a significant difference from the free electron case now arises. As we have seen in § 10, in the absence of an external field the frequencies form a continuous spectrum with a gap in the range from $-m$ to $m$. In the presence of an external field, in addition to the continuous spectrum, a discrete spectrum can also occur (as we shall see later), i.e., bound states can occur in the interval $-m < \omega < m$.

We have interpreted the solutions with values of $\omega > m$ as wave functions of electron states, and solutions with $\omega < -m$, transformed in accordance with (10.7), as wave functions of positron states. Now the problem arises of the interpretation of the discrete spectrum.

We shall confine ourselves here to the practically important case when the discrete spectrum is situated close to one of the boundaries of the continuous spectrum. In this case we can, as before, separate the frequencies into positive and negative ones, and interpret them in accordance with the rules given in § 10. If the wave function $\psi = \psi^{(-)}$

belongs to negative frequencies, it is necessary to go over to the charge-conjugate wave function $\psi^{(p)} = C\bar{\psi}^{(-)}$. We derive the equation satisfied by $\psi^{(p)}$. On carrying out the same operations as in the case of the transition from equation (9.12) to (9.13) and on noting that $A^* = \mathrm{A}$, $A_4^* = -A_4$, we obtain $\bar{\psi}(-i\hat{p}+ie\hat{A}+m) = 0$. On multiplying this equation on the left by $C$ and on utilizing (10.5), we finally obtain

$$(i\hat{\mathrm{p}}+ie\hat{A}+m)\psi^{(p)} = 0. \tag{12.3}$$

We see that equation (12.3) differs from (12.1) by a reversal of the sign of the charge. Thus, instead of solutions of positive and negative frequencies we can discuss solutions of only positive frequencies corresponding to the two signs of the charge. This is in accord with the basic idea for the interpretation of negative frequencies presented in § 10.

### 12.2. *Separation of Variables in a Central Field*

We consider the problem of the motion of an electron in an electrostatic central field

$$A = 0, \qquad A_0 = A_0(r).$$

The Hamiltonian operator (12.2) has in this case the form

$$\mathrm{H} = \boldsymbol{\alpha}\mathbf{p}+\beta m+V(r),$$

where $V(r) = eA_0(r)$.

In view of the spherical symmetry of the field the angular momentum operator and the inversion operator both commute with the Hamiltonian operator. Therefore, there exist states of definite energy, angular momentum, and parity. As in § 11 we denote the corresponding wave functions by

$$\psi_{jlM} = \begin{pmatrix} \varphi_{jlM} \\ \chi_{jlM} \end{pmatrix}.$$

The eigenfunctions of the angular momentum and the inversion operators were obtained in § 11. Therefore, we know the angular dependence of the wave function, and we can write $\varphi_{jlM}$ and $\chi_{jlM}$ in the following form:

$$\begin{aligned} \varphi_{jlm} &= ig(r)\,\Omega_{jl'M}(\boldsymbol{n}), \\ \chi_{jlM} &= -f(r)\,\Omega_{jl'M}(\boldsymbol{n}), \end{aligned} \tag{12.4}$$

where $l' = 2j-l$, $\boldsymbol{n} = \boldsymbol{r}/r$ and $\Omega_{jlM}$ is the spherical spinor defined by formula (11.2). Our problem thus reduces to the determination of the radial functions $g(r)$ and $f(r)$. In order to do this we substitute (12.4) into the equation for the eigenfunctions of the energy operator $H\psi = \varepsilon\psi$, i.e., into the equations

$$(\varepsilon-m-V)\varphi = \boldsymbol{\sigma}\mathbf{p}\chi, \quad (\varepsilon+m-V)\chi = \boldsymbol{\sigma}\mathbf{p}\varphi. \tag{12.5}$$

We rearrange the right-hand sides of equations (12.5). On noting that $\boldsymbol{\sigma}\mathbf{p}\varphi = dg/dr(\boldsymbol{\sigma}\boldsymbol{n})\,\Omega_{jlM} + ig\boldsymbol{\sigma}\mathbf{p}\Omega_{jlM}$, and on making use of the fact that $(\boldsymbol{\sigma}\boldsymbol{n})\Omega_{jlM} = -\Omega_{jl'M}$ we obtain

$$-(\boldsymbol{\sigma}\mathbf{p})\Omega_{jlM} = (\boldsymbol{\sigma}\mathbf{p})\,(\boldsymbol{\sigma}\boldsymbol{n})\Omega_{jl'M} = (\mathbf{p}\boldsymbol{n} + i[\mathbf{p}\boldsymbol{n}]\boldsymbol{\sigma})\Omega_{jl'M} = -i\left(\frac{2}{r}+\frac{1}{r}\mathbf{L}\boldsymbol{\sigma}\right)\Omega_{jl'M},$$

where $\mathbf{L} = [\boldsymbol{r}\mathbf{p}]$ is the orbital angular momentum operator. The function $\Omega_{jl'M}$ is an eigenfunction of the operators $\mathbf{L}^2$, $\mathbf{M}^2$ and $(\frac{1}{2}\boldsymbol{\sigma})^2$ which corresponds to the eigenvalues $l'(l'+1)$, $j(j+1)$ and $3/4$. Therefore we have

$$\mathbf{L}\boldsymbol{\sigma}\Omega_{jl'M} = \left(\mathbf{M}^2-\mathbf{L}^2-(\tfrac{1}{2}\boldsymbol{\sigma})^2\right)\Omega_{jl'M} = \{j(j+1)-l'(l'+1)-\tfrac{3}{4}\}\,\Omega_{jl'M}.$$

On introducing the following notation:

$$\varkappa = \mp(j+\tfrac{1}{2}) = \begin{cases} -(l+1), & j = l+\tfrac{1}{2}, \\ \quad l, & j = l-\tfrac{1}{2}, \end{cases} \tag{12.6}$$

and on noting that

$$j(j+1)-l'(l'+1)-\tfrac{3}{4} = \begin{cases} -(l+2), & j = l+\tfrac{1}{2}, \\ \quad l-1, & j = l-\tfrac{1}{2}, \end{cases}$$

we obtain

$$(2+\mathbf{L}\boldsymbol{\sigma})\Omega_{jl'M} = (1+\varkappa)\Omega_{jl'M},$$

$$\boldsymbol{\sigma}\mathbf{p}\varphi = -\Omega_{jl'M}\left(\frac{dg}{dr}+\frac{\varkappa+1}{r}g\right).$$

In an analogous manner we obtain

$$\boldsymbol{\sigma}\mathbf{p}\chi = -i\Omega_{jlM}\left(\frac{df}{dr}-\frac{\varkappa-1}{r}f\right).$$

On substituting these expressions into (12.5) and on cancelling the angular functions appearing on both sides of these equations, we obtain the desired equations for the radial functions

$$\frac{dg}{dr}+(1+\varkappa)\frac{g}{r}-(\varepsilon+m-V)f = 0,$$

$$\frac{df}{dr}+(1-\varkappa)\frac{f}{r}+(\varepsilon-m-V)g = 0,$$

or

$$\begin{aligned}
&\frac{d(rg)}{dr}+\frac{\varkappa}{r}(rg)-(\varepsilon+m-V)(rf)=0,\\
&\frac{d(rf)}{dr}-\frac{\varkappa}{r}(rf)+(\varepsilon-m-V)(rg)=0.
\end{aligned}\tag{12.7}$$

The functions $rg$ and $rf$ must in addition satisfy two boundary conditions which guarantee the possibility of normalizing the wave function: they must remain finite as $r \to 0$ and $r \to \infty$.

When $V=0$ the solutions of the system (12.7) are given by the radial functions found in subsection 11.2

$$ig = cg_l(pr),$$

$$f = c\sqrt{\frac{\varepsilon-m}{\varepsilon+m}}\,g_{l'}(pr), \qquad l' = 2j-l.$$

### 12.3. *Asymptotic Behavior of the Radial Functions*

We now obtain the asymptotic behavior of the functions $g$ and $f$. If we assume that the potential tends to zero sufficiently rapidly for large values of $r\,(V \to 0$ as $r \to \infty)$ we obtain from (12.7) for $r \to \infty$

$$\begin{aligned}
&\frac{d(rg)}{dr}-(\varepsilon+m)rf=0,\\
&\frac{d(rf)}{dr}+(\varepsilon-m)rg=0.
\end{aligned}$$

The general solution of these equations is of the form

$$\begin{aligned}
rg &= C(e^{-\lambda r}+qe^{\lambda r}),\\
rf &= C\left(-\sqrt{\frac{m-\varepsilon}{m+\varepsilon}}\,e^{-\lambda r}+q\sqrt{\frac{m-\varepsilon}{m+\varepsilon}}\,e^{\lambda r}\right),
\end{aligned}\tag{12.8}$$

where $\lambda=(m^2-\varepsilon^2)^{1/2}$ and $C$ and $q$ are constants. The constant $C$ is determined from the normalization condition for the wave function, while $q$ is obtained from the condition that the solutions of equations (12.7) remain finite for $r=0$. In particular, this quantity depends on the energy $q=q(\varepsilon)$.

The nature of the behavior of the functions (12.8) is essentially different in the two cases $\varepsilon > m$ and $\varepsilon < m$. If $\varepsilon > m$ the coefficient in the exponential is imaginary, and the condition at infinity is satisfied independently of the value of $\varepsilon$. The asymptotic expression of the radial functions (12.8) can then be given in the following form:

$$
\begin{aligned}
irg &= i^{l}C_1\sqrt{1+\frac{\varepsilon}{m}}\,\sin\left(pr-\frac{l\pi}{2}+\delta\right),\\
rf &= i^{l'}C_1\sqrt{\frac{\varepsilon}{m}-1}\,\sin\left(pr-\frac{l'\pi}{2}+\delta\right),
\end{aligned}
\tag{12.9}
$$

where $p = -i\lambda = \sqrt{\varepsilon^2-m^2}$ while $C_1$ is a constant determined by the normalization condition. The phase $\delta$ depends on the nature of the external field, and also on the energy and the momentum of the electron. In the absence of an external field $\delta = 0$, which may be easily seen on comparing (12.9) with the asymptotic expression for $g_l$ (4.26).

If $\varepsilon < m$, then the quantity $\lambda$ is real, and therefore the second terms in (12.8) increase exponentially. In this case the condition at infinity may be satisfied only for such values of $\varepsilon$ for which $q(\varepsilon) = 0$. The roots of this equation determine the allowed energy values. Thus, as was asserted earlier, we have a continuous spectrum for $\varepsilon > m$, and a discrete spectrum for $\varepsilon < m$. The latter may be completely absent if $q(\varepsilon)$ has no roots.

### 12.4. *Behavior of Energy Levels as Functions of the Potential Well Depth*

We investigate the frequency spectrum in the special case of the spherically symmetric potential well $V = -V_0$, $r < r_0$, $V = 0$, $r > r_0$, where $V_0$ is a constant. On this simple example we shall demonstrate certain general features of the behavior of the discrete spectrum.

On substituting these values of $V$ into the system of equations (12.7), and on eliminating $rf$ from it, we obtain the following second-order equation for the function $W \equiv rg$:

$$
\frac{d^2W}{dr^2}-\frac{\varkappa(\varkappa+1)}{r^2}W+[(\varepsilon+V_0)^2-m^2]W = 0, \qquad r < r_0,
$$

$$
\frac{d^2W}{dr^2}-\frac{\varkappa(\varkappa+1)}{r^2}W+(\varepsilon^2-m^2)W = 0, \qquad r > r_0.
$$

The general solution of these equations has the following form:

$$
\begin{aligned}
W &= \sqrt{r}\,\{A_1J_{|\varkappa+1/2|}(\beta r)+B_1N_{|\varkappa+1/2|}(\beta r)\}, \qquad r < r_0,\\
W &= \sqrt{r}\,\{A_2J_{|\varkappa+1/2|}(\beta' r)+B_2N_{|\varkappa+1/2|}(\beta' r)\}, \qquad r > r_0,
\end{aligned}
$$

where $J$ is a Bessel function, and $N$ is a Neumann function

$$\beta = \sqrt{(\varepsilon+V_0)^2-m^2}, \qquad \beta' = \sqrt{\varepsilon^2-m^2}.$$

The condition that $W$ should remain finite *at* $r = 0$ requires that $B_1 = 0$. Instead of the condition that the function should remain finite as $r \to \infty$ we introduce the condition $W(R) = 0$ where $R \gg r_0$. This corresponds to placing the system in a spherical box with impenetrable walls of radius $R$. Such a change in the boundary condition has no fundamental significance. However, practically it is convenient because it makes the whole spectrum discrete with a very small distance of the order of $1/R$ between the frequencies in the range $|\omega| > m$. Due to the discrete nature of the spectrum we can follow the changes in the position of each level as $V_0$ is varied.

From the boundary conditions it follows that

$$B_2 = -A_2\frac{J_{|\varkappa+1/2|}(\beta' R)}{N_{|\varkappa+1/2|}(\beta' R)}.$$

The requirement of continuity of $W(r)$ at $r = r_0$ finally leads to the following formula:

$$W(r) = A\sqrt{r}\begin{cases} [N_{|\varkappa+1/2|}(\beta' R)J_{|\varkappa+1/2|}(\beta' r_0) \\ \qquad -J_{|\varkappa+1/2|}(\beta' R)N_{|\varkappa+1/2|}(\beta' r_0)]J_{|\varkappa+1/2|}(\beta r), & r < r_0, \\ J_{|\varkappa+1/2|}(\beta r_0)\,[N_{|\varkappa+1/2|}(\beta' R)J_{|\varkappa+1/2|}(\beta' r) \\ \qquad -J_{|\varkappa+1/2|}(\beta' R)N_{|\varkappa+1/2|}(\beta' r)], & r > r_0, \end{cases}$$

where $A$ is a normalization constant.

The energy spectrum is determined from the following condition:

$$\left(\frac{W_1}{W}\right)_{r=r_0-0} = \left(\frac{W_1}{W}\right)_{r=r_0+0},$$

where $W_1 = rf$. On expressing $W_1$ in terms of $W$ with the aid of equations (12.7), and on utilizing asymptotic expressions for the functions containing $R$ In their arguments, we obtain for $\varepsilon < m$

$$a\frac{J'_{|\nu|}(a)}{J_{|\nu|}(a)} = \frac{m\sqrt{1+\left(\frac{a}{mr_0}\right)^2}-\varepsilon}{m+\varepsilon}\left\{a'\frac{H^{(1)\prime}_{|\nu|}(a')}{H^{(1)}_{|\nu|}(a')}\cdot\frac{m\left(1+\sqrt{1+\left(\frac{a}{mr_0}\right)^2}\right)}{-\varepsilon+m\sqrt{1+\left(\frac{a}{mr_0}\right)^2}}+\nu\right\},$$

where $\nu = \varkappa+\frac{1}{2}$, $a = \beta r_0$, $a' = \beta' r_0$ and $H$ is a Hankel function.

The roots of this equation may be found graphically in the general case.

We discuss the case $l = 0$, $\varkappa = -1$, $\nu = -\frac{1}{2}$. Then we have

$$J_{1/2}(a) = \sqrt{\frac{2}{\pi a}}\sin a, \qquad H^{(1)}_{1/2}(a) = -i\sqrt{\frac{2}{\pi a}}\,e^{ia}$$

and the last equation takes on the following form:

$$1 - a\cot a = \frac{mr_0\sqrt{1-\dfrac{\varepsilon^2}{m^2}}+1}{1+\dfrac{\varepsilon}{m}}\left\{1+\sqrt{1+\left(\frac{a}{mr_0}\right)^2}\right\}.$$

This equation can be easily solved in the limiting cases of a wide ($mr_0 \gg 1$) and a narrow ($mr_0 \ll 1$) well. In the former case ($mr_0 \gg 1$)

$$1 - a\cot a = \frac{2}{1+\dfrac{\varepsilon}{m}}\left[1+mr_0\left(1-\frac{\varepsilon^2}{m^2}\right)^{1/2}\right].$$

From this it may be seen that the first bound state $\varepsilon = m$ appears when $a = \pi/2$, i.e., for $V_0 = \pi^2/8mr_0^2$, just as in the corresponding nonrelativistic problem. The energy of this level attains the value $\varepsilon = 0$ for

$$V_0 = m\left(1+\frac{\pi^2}{2m^2r_0^2}\right)$$

and the value $\varepsilon = -m$ for

$$V_0 = 2\left(1+\frac{1}{4}\frac{\pi^2}{m^2r_0^2}\right)m.$$

In the latter case ($mr_0 \ll 1$) we have

$$\varepsilon = m \quad \text{for} \quad V_0 = \frac{\pi}{r_0} - m,$$

$$\varepsilon = 0 \quad \text{for} \quad V_0 = \frac{\pi}{r_0},$$

$$\varepsilon = -m \quad \text{for} \quad V_0 = \frac{\pi}{r_0} + m.$$

We see that in both these limiting cases the width of the range of $V_0$ within which the level shifts from $m$ to $-m$ is equal in order of magnitude to $2m$. This also holds in the general case.

Figure 3 shows schematically the dependence of the frequency spectrum on the potential depth $V_0$ for a given well radius $r_0$ (170). We see that for $V_0 < V_0^{(1)}$ there are no bound states ($|\omega| < m$). The frequency spectrum consists, just as in the case when there is no external field, of the two regions $\omega > m$ and $\omega < -m$ to which we shall refer respectively as the upper and the lower continuum. For $V_0 > V_0^{(1)}$ the lowest level of the upper continuum attains a value less than $m$, i. e., there appears one bound state; for $V_0 = V_0^{(2)}$ a second bound state appears, etc. The value of the frequency for each bound state diminishes continuously as $V_0$ increases, and even becomes negative for $V_0 > V_0'$. Nevertheless, we may include these states among the electron states, since such a state can be returned to the upper continuum by an adiabatic variation of the external field. A difficulty arises when at a value $V_0 = V_{0c}$ the level crosses the boundary $\omega = -m$ and merges with the lower continuum which represents the set of positron states. The problem of the behavior of an electron in a potential whose depth exceeds $V_{0c}$ cannot be solved within the framework of the quantum mechanics of a single particle. We shall return to this question later (cf. Subsection 18.2).

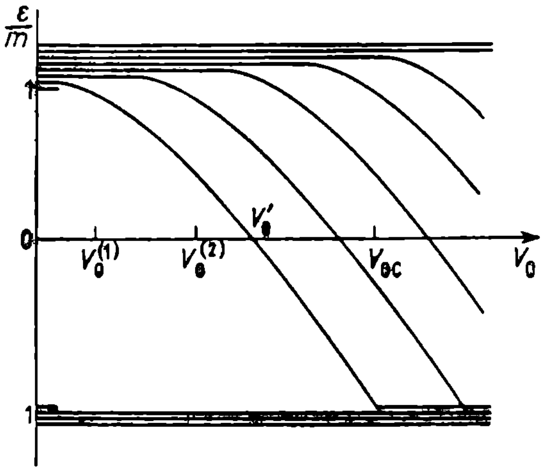


Fig. 3.

### 12.5. *Electron in a Constant Homogeneous Magnetic Field*

We now discuss the motion of an electron in a constant homogeneous magnetic field $H$. If it is directed along the $z$-axis, then $A_0 = A_x = A_z = 0$, $A_y = Hx$. We write the Dirac equation in the form $\{\boldsymbol{\alpha}(\mathbf{p}-eA)+\beta m\}\psi = \varepsilon\psi$ or $(\varepsilon-m)\varphi = \boldsymbol{\sigma}(\mathbf{p}-eA)\chi$ and $(\varepsilon+m)\chi = \boldsymbol{\sigma}(\mathbf{p}-eA)\varphi$. By eliminating $\chi$ from these equations we obtain a second order equation for $\varphi$:

$$\{\mathbf{p}^2+e^2H^2x^2-eH(\sigma_z+2xp_y)\}\varphi = (\varepsilon^2-m^2)\varphi. \tag{12.10}$$

The solution of this equation may be sought in the form $\varphi = e^{i(p_y y + p_z z)} f(x)$ where $f(x)$ satisfies the equation

$$\left\{-\frac{d^2}{dx_2} + (eHx - p_y)^2 - eH\sigma_z\right\} f = (\varepsilon^2 - m^2 - p_z^2) f$$

or

$$(-(d^2/d\xi^2) + \xi^2 - \sigma_z) f = af,$$

where

$$\xi = \sqrt{eH}\left(x - \frac{p_y}{eH}\right), \qquad a = \frac{\varepsilon^2 - m^2 - p_z^2}{eH}.$$

The spinor function $f$ may be so chosen that it is an eigenfunction of the operator $\sigma_z$: $\sigma_z f = \mu f$, where $\mu = \pm 1$. This means that $f = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ for $\mu = 1$ and $f = \begin{pmatrix} 0 \\ f_{-1} \end{pmatrix}$ for $\mu = -1$. For $f_\mu (\mu = \pm 1)$ we obtain the following equation $(d^2 f_\mu / d\xi^2) - \xi^2 f_\mu = -(a + \mu) f_\mu$, the solution of which is given by

$$f_\mu = C e^{-\xi^2/2} H_n(\xi), \tag{12.11}$$

where $H_n(\xi)$ is an Hermite polynomial, $C$ is a normalizing constant and $n = (a + \mu - 1/2) = 0, 1, 2, \ldots$ From this we obtain for the electron energy $\varepsilon^2 = m^2 + p_z^2 + eH(2n - \mu + 1)$.

The function $\chi$ may be expressed in terms of $\varphi$ as follows:

$$\chi = \frac{1}{\varepsilon + m}\left\{\mu p_z + (eH)^{1/2}\left(\xi\sigma_y - i\sigma_x \frac{\partial}{\partial \xi}\right)\right\}\varphi. \tag{12.12}$$

We give the expressions for the normalized wave functions $\psi_\mu$:

$$\left.\begin{aligned}
\psi_{+1} &= \frac{\exp i(p_y y + p_z z)}{(L_y L_z)^{1/2}[2(\varepsilon^2 + m\varepsilon)]^{1/2}} \begin{pmatrix} (\varepsilon + m) \upsilon^{(n)} \\ 0 \\ p_z \upsilon^{(n)} \\ -i(eH2n)^{1/2} \upsilon^{(n-1)} \end{pmatrix}, \\
\psi_{-1} &= \frac{\exp i(p_y y + p_z z)}{(L_y L_z)^{1/2}[2(\varepsilon^2 + m\varepsilon)]^{1/2}} \begin{pmatrix} 0 \\ (\varepsilon + m) \upsilon^{(n)} \\ i[2eH(n+1)]^{1/2} \upsilon^{(n+1)} \\ -p_z \upsilon^{(n)} \end{pmatrix},
\end{aligned}\right\} \tag{12.13}$$

where $L_y$ and $L_z$ are normalization lengths along the $y$-and $z$-axes, and

$$\upsilon^{(n)} = e^{-\xi^2/2} H_n(\xi) \left[\frac{(eH)^{1/4}}{\pi^{1/4} 2^{n/2} (n!)^{1/2}}\right].$$

## § 13. Motion of an Electron in the Field of a Nucleus

### 13.1. *Solution of the Radial Equations for the Coulomb Field*

The most important application of the Dirac equation is the investigation of the states of an electron in the field of a nucleus. The latter is not a strictly central field. The deviation from spherical symmetry is associated with the fact that in the general case nuclei possess electric quadrupole, magnetic dipole and also higher multipole moments. If we neglect these effects, which give rise to the hyperfine structure of electron levels, and assume that the nuclear field is spherically symmetrical, then we can discuss electron states with definite values of energy, angular momentum and parity. In this case the wave function is of the form (12.4) and the problem reduces to the solution of the equations for the radial functions (12.7).

At large distances the field of the nucleus is a Coulomb field, i.e., $V = -Za/r$ where $r > r_0$ and where $-Ze$ is the nuclear charge, $a = e^2/4\pi$, and $r_0$ is the nuclear radius. We shall at first discuss this problem in the approximation when the finite nuclear dimensions may be neglected, and we shall assume that the above expression for $V$ holds over all space.

In accordance with the nature of the asymptotic behavior of the radial wave functions (12.8) we seek the solution of equations (12.7) in the following form:

$$rg = \sqrt{1+\frac{\varepsilon}{m}}\, e^{-\lambda r}(F_1+F_2),$$

$$rf = \sqrt{1-\frac{\varepsilon}{m}}\, e^{-\lambda r}(F_1-F_2).$$

Substitution of these expressions leads to $\lambda = \sqrt{m^2-\varepsilon^2}$ and the following equations for $F_1$ and $F_2$:

$$\begin{aligned} \frac{dF_1}{d\varrho} &= \left(1-\frac{\varepsilon}{\lambda}\frac{Za}{\varrho}\right)F_1-\left(\frac{\varkappa}{\varrho}+\frac{m}{\lambda}\frac{Za}{\varrho}\right)F_2, \\ \frac{dF_2}{d\varrho} &= \left(-\frac{\varkappa}{\varrho}+\frac{m}{\lambda}\frac{Za}{\varrho}\right)F_1+\frac{\varepsilon}{\lambda}\frac{Za}{\varrho}F_2, \end{aligned} \tag{13.1}$$

where $\varrho = 2\lambda r$.

The solution of the system of equations (13.1) for small values of $\varrho$ is of the form $F_1 = a_1\varrho^\gamma$, and $F_2 = a_2\varrho^\gamma$, $a_1$ and $a_2$ are related by the following equations:

$$a_1\left(\gamma+Za\frac{\varepsilon}{\lambda}\right)+a_2\left(\varkappa+\frac{Za}{\lambda}m\right)=0,$$

$$a_1\left(\varkappa-Za\frac{m}{\lambda}\right)+a_2\left(\gamma-Za\frac{\varepsilon}{\lambda}\right)=0,$$

from which it follows that $\gamma=\sqrt{\varkappa^2-Z^2a^2}$. The sign of the square root must be chosen positive in order to satisfy the condition that the wave function should remain finite at $\varrho = 0$.

If we eliminate one of the functions, for example $F_1$, from equations (13.1), we obtain a second order equation for $\varrho^{-\gamma}F_2$ of the following form:

$$\left(\varrho\frac{d^2}{d\varrho^2}+(b-\varrho)\frac{d}{d\varrho}-a\right)\varrho^{-\gamma}F_2=0,$$

where

$$b=2\gamma+1,\qquad a=\gamma-Za\frac{\varepsilon}{\lambda}.$$

As is well known, the solution of the last equation which is finite at $\varrho = 0$ is given by the confluent hypergeometric function

$$F(a,b;\varrho)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{\varrho^n}{n!}$$

($\Gamma$ is the gamma-function). Thus we have

$$F_2 = c\varrho^\gamma F\left(\gamma-Za\frac{\varepsilon}{\lambda},\ 2\gamma+1;\varrho\right), \tag{13.2}$$

where $c$ is a constant. By using the recurrence formula for the hypergeometric functions,

$$\varrho\frac{d}{d\varrho}F(a,b;\varrho)=a[F(a+1,b;\varrho)-F(a,b;\varrho)],$$

we can obtain $F_1$ from (13.1):

$$F_1=c\frac{\gamma-Za\frac{\varepsilon}{\lambda}}{-\varkappa+Za\frac{m}{\lambda}}\varrho^\gamma F\left(\gamma+1-\frac{Za\varepsilon}{\lambda},2\gamma+1;\varrho\right). \tag{13.2'}$$

In the case $\varepsilon < m$ these solutions, as we have seen earlier, have meaning only for discrete values of $\varepsilon$. In order to find them we use the asymptotic expression for the hypergeometric function

$$F(a, b; \varrho) \approx e^{-i\pi a} \frac{\Gamma(b)}{\Gamma(b-a)} \varrho^{-a} + \frac{\Gamma(b)}{\Gamma(a)} \varrho^{a-b} e^{\varrho}, \qquad \varrho \gg 1, \tag{13.3}$$

from which the following conditions for the absence of exponentially increasing terms in $F_1$ and $F_2$ can be obtained:

$$\frac{1}{\Gamma\left(\gamma - Za\frac{\varepsilon}{\lambda}\right)} = 0, \qquad \frac{1}{\left(Za\frac{m}{\lambda} - \varkappa\right)\Gamma\left(\gamma - Za\frac{\varepsilon}{\lambda}\right)} = 0. \tag{13.3$'$}$$

If $(Za/\lambda)m - \varkappa \neq 0$, which always holds for $\varkappa < 0$, then the second condition coincides with the first one. Since the poles of the $\gamma$ function are given by negative integers and zero, we have $\gamma - Za\varepsilon/\lambda = -n_r$, where $n_r$ is a non-negative integer. From this it follows that

$$\varepsilon = \frac{m}{\sqrt{1 + [Z^2a^2/(\gamma + n_r)^2]}}. \tag{13.4}$$

If $\varkappa = Zam/\lambda$, then $\gamma = Za\varepsilon/\lambda$ and $n_r = 0$. At the same time the second condition (13.3′) is not satisfied. Thus,

$$n_r = \begin{cases} 0, 1, 2, \ldots, & \varkappa < 0, \\ 1, 2, \ldots, & \varkappa > 0. \end{cases}$$

Formula (13.4) gives the fine structure of the levels of a hydrogen-like atom. The lowest level corresponds to the values $n_r = 0$, $\varkappa = -1$ and is equal to $\varepsilon = m(1 - Z^2a^2)^{1/2}$. For $Za \ll 1$ the expansion of the expression (13.4) in terms of this parameter to terms of order $(Za)^4$ yields the following expression for $(\varepsilon - m)/m$:

$$\frac{\varepsilon - m}{m} = -(Za)^2 \left\{ \frac{1}{2n^2} + \frac{(Za)^2}{2n^3} \left( \frac{1}{|\varkappa|} - \frac{3}{4n} \right) \right\}, \tag{13.4$'$}$$

where $n = n_r + |\varkappa|$.

The quantum number $n$ coincides with the principal quantum number of the nonrelativistic theory. The first term in (13.4′) yields Balmer's formula. Formula (13.4′) can also be written in the form

$$\varepsilon - m = -\frac{Z^2a}{2a_0 n^2} \left[ 1 + \frac{(Za)^2}{n} \left( \frac{1}{|\varkappa|} - \frac{3}{4n} \right) \right],$$

where $a_0 = 1/ma$ is the Bohr radius.

A characteristic feature of the fine structure formula is the fact that it contains only the absolute value of $\varkappa$. Therefore, all the energy levels, with the exception of the lowest one, are twofold degenerate. In particular, the two levels $2s_{1/2}$ ($n_r = 1$, $\varkappa = -1$, $l = 0$, $\mathrm{j} = \frac{1}{2}$, $n = 2$) and $2p_{1/2}$ ($n_r = 1$, $\varkappa = 1$, $l = 1$, $\mathrm{j} = \frac{1}{2}$, $n = 2$) coincide.

Later (cf. § 53) we shall see that formula (13.4) based on the use of the concept of a given external field is not exact. Corrections to this formula remove the degeneracy with respect to $\varkappa$. The order of magnitude of these corrections for small values of $Z$ is equal to $(Za)^4 a$. From this it may be seen that, for example, the use of formula (13.4) for hydrogen, attempting an accuracy which exceeds the accuracy of the expansion (13.4′), is meaningless.

We now give the final expressions for the radial wave functions of the discrete spectrum:

$$g = -\frac{\sqrt{\Gamma(2\gamma+n_r+1)}}{\Gamma(2\gamma+1)\sqrt{n_r!}}\sqrt{\frac{1+(\varepsilon/m)}{4N(N-\varkappa)}}\left(\frac{2Z}{Na_0}\right)^{3/2}$$

$$\times e^{-\frac{Z}{Na_0}r}\left(\frac{2Zr}{Na_0}\right)^{\gamma-1}\Big\{n_r F\Big(-n_r+1,\ 2\gamma+1;\ \frac{2Zr}{Na_0}\Big)$$

$$-(N-\varkappa)F\Big(-n_r,\ 2\gamma+1;\ \frac{2Zr}{Na_0}\Big)\Big\}, \quad (13.5)$$

$$f = -\frac{\sqrt{\Gamma(2\gamma+n_r+1)}}{\Gamma(2\gamma+1)\sqrt{n_r!}}\sqrt{\frac{1-(\varepsilon/m)}{4N(N-\varkappa)}}\left(\frac{2Z}{Na_0}\right)^{3/2}$$

$$\times e^{-\frac{Z}{Na_0}r}\left(\frac{2Zr}{Na_0}\right)^{\gamma-1}\Big\{n_r F\Big(-n_r+1,\ 2\gamma+1;\ \frac{2Zr}{Na_0}\Big)\Big\}$$

$$+(N-\varkappa)F\Big(-n_r,\ 2\gamma+1;\ \frac{2Zr}{Na_0}\Big)\Big\}, \quad (13.5')$$

where

$$N = \sqrt{n^2-2n_r(|\varkappa|-\gamma)}, \qquad a_0 = \frac{1}{ma}, \qquad n = n_r+|\varkappa|,$$

$$\gamma = \sqrt{(\varkappa^2-Z^2a^2)}, \qquad a = \frac{e^2}{4\pi} = \frac{1}{137}.$$

They are normalized in accordance with the condition $\int (f^2+g^2)r^2dr = 1$.

### 13.2. *Wave Functions for the Continuous Spectrum*

We now proceed to investigate the continuous spectrum ($\varepsilon > m$); in this case $\lambda$ is a purely imaginary quantity, $\lambda = ip = i(\varepsilon^2 - m^2)^{1/2}$.

It is now convenient to write the formulas for $F_1$ and $F_2$ in the following form:

$$\begin{aligned} F_1 &= ce^{i\xi}(\gamma+i\nu)F(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr)(2pr)^{\gamma}, \\ F_2 &= ce^{-i\xi}(\gamma-i\nu)F(\gamma+i\nu,\ 2\gamma+1;\ 2ipr)(2pr)^{\gamma}, \end{aligned} \tag{13.6}$$

where

$$e^{2i\xi} = \frac{\varkappa - iZ\alpha[m/p]}{-\gamma + iZ\alpha[\varepsilon/p]}, \quad \nu = Z\alpha\frac{\varepsilon}{p} = Z\frac{\alpha}{v},$$

$v$ is the electron velocity ($\xi$ is real).

The hypergeometric functions appearing in (13.6) are related by a simple expression which can be obtained by utilizing the integral representation for the hypergeometric function:

$$F(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)2^{b-1}} \int_{-1}^{1} (1+z)^{a-1}(1-z)^{b-a-1} e^{\frac{x}{2}(1+z)} dz.$$

If $x$ is an imaginary quantity and $2\mathrm{Re}\,a+1 = b$, then

$$F(a, b; x) = e^{x}F^{*}(a+1, b; x).$$

Upon utilizing this equation and the expressions for $F_1$ and $F_2$, we obtain the radial functions in the following form:

$$\begin{aligned} irg &= c\sqrt{1+(\varepsilon/m)}\,(2pr)^{\gamma}\{e^{-ipr+i\xi}(\gamma+i\nu)F(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr) \\ &\qquad + e^{ipr-i\xi}(\gamma-i\nu)F^{*}(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr)\}, \\ rf &= c\sqrt{(\varepsilon/m)-1}\,(2pr)^{\gamma}\{e^{-ipr+i\xi}(\gamma+i\nu)F(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr) \\ &\qquad - e^{ipr-i\xi}(\gamma-i\nu)F^{*}(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr)\}. \end{aligned} \tag{13.7}$$

Now we obtain the asymptotic expressions for the functions (13.7). To do this we utilize formula (13.3). Since now $\varrho = 2ipr$ is a purely imaginary quantity, the nature of the falling off of the hypergeometric function is determined not by an exponential function, but by the power-series factor in (13.3). In our case we have $\mathrm{Re}\,a = \gamma+1$, $b$

$= 2\gamma+1$ and the first term falls off faster than the second one, so that we can retain only the second term. Thus we have

$$F(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr) \approx \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1+i\nu)}\,\frac{1}{(2ipr)^{\gamma}}\,(2ipr)^{i\nu}e^{2ipr}$$

$$= \frac{\Gamma(2\gamma+1)}{(\gamma+i\nu)\,\Gamma(\gamma+i\nu)}\,\frac{1}{(2pr)^{\gamma}}\,e^{-\frac{\pi}{2}\nu}\,e^{i\left(2pr-\frac{\pi}{2}\gamma+\nu\ln 2pr\right)}, \qquad pr \gg 1.$$

On substituting this expression into (13.7) we obtain

$$\begin{aligned} irg &= c\sqrt{1+\frac{\varepsilon}{m}}\,\frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+i\nu)|}\,e^{-\frac{\pi}{2}\nu}\sin\left(pr-\frac{l\pi}{2}+\delta+\nu\ln 2pr\right), \\ rf &= c\sqrt{\frac{\varepsilon}{m}-1}\,\frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+i\nu)|}\,e^{-\frac{\pi}{2}\nu}\,i^{l'-l}\sin\left(pr-\frac{l'\pi}{2}+\delta+\nu\ln 2pr\right), \\ &\qquad pr \gg 1, \end{aligned} \tag{13.8}$$

where

$$\delta = \xi - \arg\Gamma(\gamma+i\nu) - \frac{\pi}{2}(\gamma-l-1),$$

$$\nu = \frac{Z\alpha}{v}, \qquad \gamma = \sqrt{(j+\tfrac{1}{2})^2 - Z^2\alpha^2}, \qquad l' = 2j-l.$$

Formula (13.8) agrees with the general asymptotic expression for the radial functions (12.9). In this case the phase turns out not to be constant, but to depend logarithmically on $r$, which is a characteristic feature of the Coulomb field associated with its slow falling off with distance.

For the determination of the constant $c$ we shall make use of the normalizing condition in the form

$$\int_0^R (|g|^2+|f|^2)r^2\,dr = 1.$$

Since the integrand does not fall off as $r \to \infty$, while the radius of the normalizing sphere $R$ is arbitrarily large, we can substitute into this integral the asymptotic expressions for the radial functions (13.8); the integral in this case turns out to be an elementary one, and we obtain

$$c = i^{l}\,\frac{e^{\frac{\pi}{2}\nu}\,|\Gamma(\gamma+i\nu)|}{\Gamma(2\gamma+1)}\sqrt{\frac{m}{\varepsilon R}}\,. \tag{13.8'}$$

The phase factor $i^l$ in this expression has been chosen so that at $Z=0$ the wave functions should coincide with the free electron wave functions in the form (11.4).

If the wave functions are normalized per unit energy interval, then $R$ should be replaced by $\pi p/\varepsilon$ in (13.8′).

### 13.3. *Isotopic Level Shift*

The results obtained so far have been based on the use of Coulomb's law for all values of $r$. In those cases when the value of the electron wave function in the neighborhood of the nucleus is important, it is necessary to take into account the fact that the expression for the potential $V$ is valid only for $r > r_0$ ($r_0$ is the nuclear radius).

An example of an effect due to the finite nuclear size is the isotopic level shift in which a difference becomes noticeable between the electron energy levels in the field of nuclei having the same charge, but different radii (34), (159). We shall show that the isotopic shift may be directly expressed in terms of the wave function at $r = r_0$ (186).

Let $\varepsilon$ denote the value of the electron energy in the Coulomb field, and $\varepsilon+\varepsilon_1$ the shifted value of the energy in the field of a nucleus of finite size. We denote the radial wave functions in the former case by $f_0$ and $g_0$, and in the latter case by $f$ and $g$. We discuss the case $\varkappa=-1$. The functions $f_0$ and $g_0$ in this case satisfy equations (12.7):

$$\frac{d}{dr}(rg_0)-\frac{1}{r}(rg_0)=(\varepsilon+m-V)\,(rf_0),$$

$$\frac{d}{dr}(rf_0)+\frac{1}{r}(rf_0)=(m+V-\varepsilon)\,(rg_0),$$

where $V=-Z\alpha/r$. For $r > r_0$ the functions and $g$ satisfy the equations

$$\frac{d}{dr}(rg)-\frac{1}{r}(rg)=(\varepsilon+m-V)rf+\varepsilon_1 rf,$$

$$\frac{d}{dr}(rf)+\frac{1}{r}(rf)=(m+V-\varepsilon)\,(rg)-\varepsilon_1 rg.$$

From the equations for $f_0$, $g_0$ and $f$, $g$ it is simple to obtain the relation $(d/dr)\,(gf_0r^2-g_0fr^2)=\varepsilon_1 r^2(g_0g+f_0f)$. We integrate it with respect to $r$ between the limits $r_0$ and $\infty$. On assuming that the functions $f_0$, $g_0$ differ from $f$, $g$ only within a small region near the nucleus, and that this region makes a small contribution to the normalization integral, we can approximate that

$$\int_{r_0}^{\infty} r^2 (g_0 g + f_0 f)\, dr \approx \int_0^{\infty} (g_0^2 + f_0^2) r^2 dr = 1.$$

Thus we have

$$\varepsilon_1 = r_0^2 [g_0(r_0) f(r_0) - g(r_0) f_0(r_0)]. \tag{13.9}$$

The isotopic shift is equal to the difference in the values of expression (13.9) for two different values of the nuclear radius $r_0$ corresponding to two isotopes.

### 13.4 *General Investigation of the Effect of Finite Nuclear Size*

The fact that the nucleus is of finite size may also play a significant role for the states of the continuous spectrum in those cases when the effect is due to the behavior of the electron wave function in the neighborhood of the nucleus. Such cases occur, for example, in $\beta$ decay (181).

In order to estimate the distortion of the wave function we return to the equations for the radial functions (12.7) in the general case of an arbitrary central field. The nature of the behavior of the wave functions for small $r$ may be determined in the following manner (166). We construct the formal solution of each of the two equations (12.7) by treating the last terms as inhomogeneous ones. This yields

$$\begin{aligned} rg &= c_1 r^{-\varkappa} + r^{-\varkappa} \int_0^r x^{\varkappa} (\varepsilon + m - V) f(x)\, dx, \\ rf &= c_2 r^{\varkappa} - r^{\varkappa} \int_0^r x^{-\varkappa} (\varepsilon - m - V) g(x)\, dx. \end{aligned} \tag{13.10}$$

From the boundary condition at $r = 0$ we obtain $c_1 = 0$ where $\varkappa > 0$, $c_2 = 0$ where $\varkappa < 0$, and therefore for small $r$,

$$\begin{aligned} &\left.\begin{aligned} rg &= c_1 r^{|\varkappa|}, \\ rf &= -c_1 r^{-|\varkappa|} \int_0^r x^{2|\varkappa|} (\varepsilon - m - V)\, dx \end{aligned}\right\} \quad (\varkappa < 0), \\ &\left.\begin{aligned} rf &= c_2 r^{\varkappa}, \\ rg &= c_2 r^{-\varkappa} \int_0^r x^{2\varkappa} (\varepsilon + m - V)\, dx \end{aligned}\right\} \quad (\varkappa > 0). \end{aligned} \tag{13.11}$$

We see that for $\varkappa < 0$ the dependence of the function $g$ on $r$ is determined by the magnitude of $\varkappa$, i.e., by the electron angular momentum, and in the first approximation is independent of the field. But the function $f$ is essentially determined by the field, and, therefore, it is affected by the difference between the actual field and the Coulomb field. If we can assume that for small $r$ $V \gg m$ (this holds for not very high energies and for nuclei with not too small a value of $Z$), then we have $rf = c_1 r^{-|\varkappa|} \int\limits_0^r x^{2|\varkappa|} V(x) dx$ where $\varkappa < 0$. A similar situation occurs when $\varkappa > 0$ with the functions $f$ and $g$ interchanging their roles.

In order that the expansion (13.7) be meaningful it is necessary that the integrals obtained by substituting (13.11) into (13.10) should exist. This will be true for all $\varkappa$ if $\lim\limits_{r \to 0} rV = 0$. Thus, this expansion is not applicable to the pure Coulomb field. However, the field of a nucleus of finite size remains finite (or vanishes) at $r = 0$, and therefore we can utilize expansion (13.11).

We shall not give in detail the solutions of the equations for the radial functions taking finite nuclear size into account. We shall merely point out the main features of this problem.

First of all, it should be noted that the function $V(r)$ is not exactly defined for $r < r_0$, and the nuclear boundary itself is to a certain extent arbitrary. We shall obtain a qualitatively correct picture if we assume that the charge density is constant within a sphere of radius $r_0$. In such a case we have

$$V = -\frac{Z\alpha}{r_0}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{r_0^2}\right) \equiv V_1 \qquad \text{for} \qquad r < r_0.$$

Another limiting case is the field of a charge distributed over the surface of a sphere of radius $r_0$. In this case

$$V = -Z\alpha/r_0 \equiv V_2 \qquad \text{for} \qquad r < r_0.$$

The fields $V_1$ and $V_2$ lead to results for the distortion of the wave function near $r_0$ for $r > r_0$ which differ little from each other. The essential feature here is not the exact shape of the potential for $r < r_0$, but the fact that it does not increase in accordance with Coulomb's law as $r \to 0$.

The solutions of the equations for the radial functions in the case of the field $V_2$ (constant potential) coincide with the solutions for the free electron if $\varepsilon$ is replaced by $\varepsilon + V_2$. In practice it is sufficient to restrict ourselves to the first terms in the expansions of $f$ and $g$ in terms of $r$. For this purpose we can use formula (13.11) both in the case of the field $V_1$, and the field $V_2$. This determines the solution in the range $r < r_0$ up to a constant factor.

The expression for the potential $V = -Z\alpha/r$ is valid in the region $r > r_0$. Therefore solutions exist in the form (13.1), (13.2). However, since now the point $r = 0$ no longer lies within the region where the solutions of equations (12.7) hold, the corresponding boundary condition no longer applies. Therefore, in formula (13.2) we must take into account the double sign of $\gamma$. We shall, as before, utilize the notation $\gamma = +\sqrt{\varkappa^2 - Z^2\alpha^2}$, then $F_2 = F_2(\gamma) + qF_2(-\gamma)$, where $F_2(\gamma)$ is given by formula (13.2), and $q$ is a constant.

Similarly $F_1 = F_1(\gamma) + qF_1(-\gamma)$, where expression (13.2) holds for $F_1(\gamma)$.

Thus, the functions $f$ and $g$ depend on the two constants $c$ and $q$ with the former appearing as a common factor. Therefore, their ratio contains only the constant $q$ which may be determined from the condition of continuity at $r = r_0$ and may be expressed as $(f/g)_{r=r_0+0} = (f/g)_{r=r_0-0}$.

The finite size of nuclei alters in a radical way the question of the existence of bound states for large values of nuclear charge. According to (13.4) when $Z\alpha = 1$ the value of the lowest electron energy level $\varepsilon_{\min}(\varkappa = -1, n_r = 0)$ becomes equal to zero.

For $Z\alpha > 1$ formula (13.4) leads to imaginary values, and this indicates that corresponding states do not exist. However, this conclusion is valid only for the Coulomb field and is inapplicable in the case of the actual field of the nucleus. Taking finite nuclear dimensions into account leads to the possibility of bound states occurring also for $Z\alpha > 1$ (156). As $Z$ increases, the value of $\varepsilon_{\min}$ diminishes and, just as in the case of the example of the potential well discussed in § 12, for a certain value of $Z = Z_c$ it attains the value $-2m$. The value of $Z$ amounts to $Z_c = 200$ for $r_0 = 1.2 \cdot 10^{-12}$ cm, and $Z_c = 175$ for $r_0 = 8 \cdot 10^{-13}$ cm.

## § 14. Electron Scattering

### 14.1 *Spinor Scattering Amplitude*

The problem of the scattering of particles by an external field may be solved in relativistic quantum mechanics by the same methods as in nonrelativistic quantum mechanics (cf., for example, (119) and (139)).

In order to obtain the differential scattering cross section in the general case of an arbitrary stationary external field, it is necessary to find the asymptotic solution of the Dirac equation which represents a superposition of a plane and an outgoing spherical wave:

$$\psi \approx u e^{ipz} + G(\boldsymbol{n}) \frac{e^{ipr}}{r}\cdot$$

Here $u$ is a constant bispinor which defines the polarization of the incident wave (the $z$-axis coincides with the direction of the primary beam); $G$ is the amplitude of the scattered wave and, like $u$, is a bispinor. It depends on the direction of scattering $\boldsymbol{n} = \boldsymbol{r}/r$. The differential scattering cross section $d\sigma$ is equal to

$$\frac{d\sigma}{do} = \frac{r^2 \boldsymbol{n}\boldsymbol{j}(\boldsymbol{r})}{j_0} = \frac{G^* \boldsymbol{\alpha} \boldsymbol{n} G}{u^* \alpha_z u},$$

where $\boldsymbol{j}(\boldsymbol{r})$ and $j_0$ are the flux densities of the scattered and the incident waves. The amplitude $G(\boldsymbol{n})$ is equivalent to the amplitude of a plane wave propagating in the direction $\boldsymbol{n}$. Therefore $d\sigma/do = G^*G/u^*u$ or if we represent $u$ and $G$ in the form

$$u = \begin{pmatrix} w \\ w' \end{pmatrix}; \qquad G = \begin{pmatrix} F \\ F' \end{pmatrix},$$

then we have $d\sigma/do = |F|^2/|w|^2$. Thus, in the case of the scattering problem it is sufficient to investigate only a part of the wave function $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$, viz., the spinor $\varphi$. If we write the asymptotic expression for $\varphi$ in the form

$$\varphi = w e^{ipz} + F \frac{e^{ipr}}{r} \tag{14.1}$$

and if the spinor $w$ is normalized ($w^*w = 1$), then we have

$$d\sigma = |F|^2 do. \tag{14.2}$$

In order to obtain the solution of the Dirac equation in the form (14.1) we use the asymptotic expressions for the wave functions found in § 12. Special solutions corresponding to definite energy, angular momentum and parity, are determined by formulas (12.4) and (12.9). The general solution corresponding to the definite energy $\varepsilon = (p^2+m^2)^{1/2}$

may evidently be written in the form of a superposition of these solutions:

$$\varphi = \sum_{jlM} a_{jlM}\Omega_{jlM}(\boldsymbol{n})\frac{\sin(pr+\delta_{jl}-l\pi/2)}{r}, \qquad \boldsymbol{n}=\frac{\boldsymbol{r}}{r}. \tag{14.3}$$

In order to bring expression (14.3) into the form (14.1) we compare it with the asymptotic expansion of a plane wave. In accordance with (11.11) we have

$$we^{ipz} \approx \frac{4\pi}{p}\sum_{jlM} i^l\left(\Omega^*_{jlM}(\boldsymbol{n}_0)w\right)\Omega_{jlM}(\boldsymbol{n})\,\frac{\sin(pr-l\pi/2)}{r},$$

where $\boldsymbol{n}_0 = \boldsymbol{p}/p$. It may be easily seen that expressions (14.3) and (14.1) will coincide if $a_{jlM} = (4\pi/p)i^l e^{i\delta_{jl}}\left(\Omega^*_{jlM}(\boldsymbol{n}_0)w\right)$. The amplitude of the scattered wave turns out to be equal to

$$F = \frac{2\pi}{ip}\sum_{jlM}\left(\Omega^*_{jlM}(\boldsymbol{n}_0)w\right)(e^{2i\delta_{jl}}-1)\,\Omega_{jlM}(\boldsymbol{n}). \tag{14.4}$$

We also give expressions, useful in a number of applications, for the exact solutions of the Dirac equations which have the asymptotic form $\psi_{\boldsymbol{p}}^{(\mathrm{out})} \approx ue^{i\boldsymbol{p}\boldsymbol{r}}+G(\boldsymbol{n})e^{ipr}/r$. On replacing in (14.3) the asymptotic value of the radial functions by the exact one it may be easily shown that

$$\psi_{\boldsymbol{p}}^{(\mathrm{out})} = \frac{4\pi}{p}\sum_{jlM} e^{i\delta_{jl}}\left(\Omega^*_{jlM}(\boldsymbol{n}_0)w\right)\begin{pmatrix} ig_{jl}(pr)\Omega_{jlM}(\boldsymbol{n}) \\ -f_{jl}(pr)\Omega_{jl'M}(\boldsymbol{n})\end{pmatrix}, \tag{14.5}$$

where $\boldsymbol{n}_0 = \boldsymbol{p}/p$, $\boldsymbol{n} = \boldsymbol{r}/r$, $l' = 2j-l$, $g_{jl}$ and $f_{jl}$ are radial functions satisfying equations (12.7) and normalized in such a way that in the asymptotic expressions (12.9) the coefficient $c_1$ is equal to unity.

In an analogous manner we can construct the solutions

$$\psi_{\boldsymbol{p}}^{(\mathrm{in})} = \frac{4\pi}{p}\sum_{jlM} e^{-i\delta_{jl}}\left(\Omega^*_{jlM}(\boldsymbol{n}_0)w\right)\begin{pmatrix} ig_{jl}(pr)\Omega_{jlM}(\boldsymbol{n}) \\ -f_{jl}(pr)\Omega_{jl'M}(\boldsymbol{n})\end{pmatrix}, \tag{14.5'}$$

which asymptotically go over into a superposition of a plane and an incoming spherical wave

$$\psi_{\boldsymbol{p}}^{(\mathrm{in})} \approx ue^{i\boldsymbol{p}\boldsymbol{r}}+G(\boldsymbol{n})\frac{e^{-ipr}}{r}.$$

Solutions (14.5) and (14.5′) for different values of $p$ form two complete orthogonal systems of functions:

$$\frac{1}{(2\pi)^3}\int \psi_p^{(\mathrm{out})*}\,\psi_{p'}^{(\mathrm{out})}\,dr = \delta(p-p'),$$

$$\frac{1}{(2\pi)^3}\int \psi_p^{(\mathrm{in})*}\psi_{p'}^{(\mathrm{in})}\,dr = \delta(p-p'),$$

in terms of which the general solution of the Dirac equation may be expanded. In the absence of an external field this expansion goes over into the expansion in terms of plane waves.

### 14.2. *Expression for the Cross Section in Terms of Phases*

The relation between the amplitudes $F$ and $w$ can be conveniently represented in the form

$$F = R(\boldsymbol{n})\,w. \tag{14.6}$$

Here $R(\boldsymbol{n})$ is a two-dimensional matrix which must obviously have the structure $R(\boldsymbol{n}) = a(\boldsymbol{n}) + \mathbf{a}_1(\boldsymbol{n})\boldsymbol{\sigma}$, where $a$ is a scalar, and $\mathbf{a}_1$ is an axial vector. But the only axial vector which can be introduced in a scattering problem is the vector $[\boldsymbol{n}_0\boldsymbol{n}]$. Therefore, $R$ may be written in the form

$$R = a(\vartheta) - ib(\vartheta)\boldsymbol{\tau}\boldsymbol{\sigma} = a(\vartheta) - ib(\vartheta)\,(\sigma_y\cos\varphi - \sigma_x\sin\varphi), \tag{14.6'}$$

where $\boldsymbol{\tau}$ is a unit vector perpendicular to the scattering plane

$$\boldsymbol{\tau} = \frac{[\boldsymbol{n}_0\boldsymbol{n}]}{\sin\vartheta},$$

$\vartheta$ and $\varphi$ are spherical polar coordinates of the vector $\boldsymbol{n}$ in the system in which the vector $\boldsymbol{n}_0$ is taken as the polar axis. The functions $a(\vartheta)$ and $b(\vartheta)$ may be obtained by comparing (14.6) with (14.4). On utilizing the explicit form of the spherical spinors $\Omega_{jlM}$ and the matrices $\sigma_i$ we obtain

$$\begin{aligned} a(\vartheta) &= \frac{1}{2ip}\sum_{\varkappa}(e^{2i\delta_\varkappa}-1)\,|\varkappa|\,P_l(\cos\vartheta),\\ b(\vartheta) &= \frac{1}{2ip}\sum_{\varkappa}(e^{2i\delta_\varkappa}-1)\frac{\varkappa}{|\varkappa|}P_l^{(1)}(\cos\vartheta), \end{aligned} \tag{14.7}$$

where

$$\delta_{\varkappa} \equiv \delta_{jl}, \quad \varkappa = \mp(j+\tfrac{1}{2}), \quad (j = l \pm \tfrac{1}{2}),$$

$P_l$ and $P_l^{(1)}$ are Legendre polynomials:

$$P_l = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0},$$

$$P_l^{(1)} = \sin\vartheta \frac{dP_l}{d\cos\vartheta}.$$

We note that the function $b\ (\vartheta)$ contains the factor $\sin\vartheta$. Therefore, the matrix $R$ does not contain the spin matrices $\boldsymbol{\sigma}$ when $\vartheta = 0$.

We also introduce another form in which the matrix $R$ can be expressed:

$$R = A + B(\boldsymbol{\sigma n})(\boldsymbol{\sigma n}_0). \qquad (14.6'')$$

Comparing with (14.6′) we can easily see that

$$a = A + B\cos\vartheta,$$

$$b = B\sin\vartheta.$$

The functions $A$ and $B$ have the following expansions

$$A = \frac{1}{p}\frac{\mathrm{d}}{d\cos\vartheta}\sum_l \left[f_{l^+}P_{l+1}(\cos\vartheta) + f_{l^-}P_{l-1}(\cos\vartheta)\right],$$

$$B = \frac{1}{p}\frac{\mathrm{d}}{d\cos\vartheta}\sum_l (f_{l^{-1}} - f_{l^+})P_l(\cos\vartheta), \qquad (14.7')$$

where

$$f_{l\pm} = \frac{1}{2i}(e^{2i\delta_{l\pm}} - 1),$$

$$\delta_{l\pm} = \delta_{jl}\left(j = l \pm \frac{1}{2}\right).$$

We now determine the total scattering cross section. According to (14.2) and (14.4) we have

$$\sigma = \int |F|^2 do = \frac{(4\pi)^2}{p^2}\sum_{jl}\sin^2\delta_{jl}\sum_M |\Omega^*_{jlM}(\boldsymbol{n}_0)w|^2.$$

Since $Y_{lM}(\boldsymbol{n}_0) = \sqrt{(2l+1/4\pi)}\,\delta_{M0}$, we have

$$\Omega^*_{jlM}(\boldsymbol{n}_0)w = \sum_{\lambda m} w(\lambda) C^{jM}_{lm\frac{1}{2}\lambda} Y_{lm}(\boldsymbol{n}_0) = w(M) C^{jM}_{l0\frac{1}{2}M} \sqrt{\frac{2l+1}{4\pi}}\,.$$

But in accordance with the table in subsection 11.1 we have

$$C^{jM}_{l0\frac{1}{2}M} = \sqrt{\frac{|\varkappa|}{2l+1}}\,, \qquad |\varkappa| = j+\tfrac{1}{2}.$$

Therefore, $4\pi \sum |\Omega^*_{jlM}(\boldsymbol{n}_0)w|^2 = |\varkappa|$ and the cross section assumes the following form:

$$\sigma = \frac{4\pi}{p^2} \sum_{\varkappa} |\varkappa| \sin^2 \delta_\varkappa. \tag{14.8}$$

If for all values of $l$ the quantities $\delta_\varkappa$ coincide for $\varkappa = -(l+1)$ and $\varkappa = l$, then formula (14.8) goes over into the nonrelativistic formula for the scattering cross section

$$\sigma = \frac{4\pi}{p^2} \sum_{l} (2l+1) \sin^2 \delta_l.$$

### 14.3. *Polarization and Azimuthal Asymmetry*

An essential feature of electron scattering is its azimuthal asymmetry, i.e., the dependence of the differential scattering cross section on the angle $\varphi$. According to (14.6) the angle $\varphi$ is contained in the matrix $\boldsymbol{R}$ which determines the relationship between the amplitudes of the incident and the scattered waves, with the coefficients of cos $\varphi$ and sin $\varphi$ containing electron spin operators. Therefore, the azimuthal scattering asymmetry is closely connected with the nature of the polarization of the incident wave.

The differential scattering cross section can be easily related to the state of polarization of the incident wave if we use the polarization density matrix $\varrho$ defined in accordance with (10.18). It follows from (14.2) and (14.6) that

$$\frac{d\sigma}{do} = \mathrm{Sp}\,\varrho \boldsymbol{R}^+ \boldsymbol{R}. \tag{14.9}$$

On substituting into this equation the general expression (10.19) $\varrho = \frac{1}{2}(1+\boldsymbol{\zeta}\boldsymbol{\sigma})$, where $\frac{1}{2}\boldsymbol{\zeta}$ is the average value of the spin in the rest system of the electron, and expression (14.6′) for $\boldsymbol{R}$, we obtain

$$\frac{d\sigma}{do} = Q(\vartheta)\left(1+\Delta(\vartheta\boldsymbol{\zeta}\boldsymbol{\tau}\right) = Q(\vartheta)\,[1+\Delta(\vartheta)\,(\zeta_y \cos\varphi - \zeta_x \sin\varphi)], \tag{14.10}$$

where

$$Q(\vartheta) = |a|^2+|b|^2, \quad \Delta(\vartheta) = \frac{i(ab^*-a^*b)}{|a|^2+|b|^2}. \tag{14.11}$$

It may be seen from (14.10) that the scattering system is an "analyzer" of the polarization. A measurement of the azimuthal asymmetry enables us to determine the component of $\zeta$ perpendicular to the scattering plane. On choosing for $\varphi$ the values 0 and $\pi$ or $\pm\pi/2$, we obtain from (14.10)

$$\Delta(\vartheta)\zeta_y = \frac{d\sigma(\vartheta, 0)-d\sigma(\vartheta, \pi)}{d\sigma(\vartheta, 0)+d\sigma(\vartheta, \pi)}, \quad \Delta(\vartheta)\zeta_x = \frac{d\sigma\left(\vartheta, -(\pi/2)\right)-d\sigma\left(\vartheta, +(\pi/2)\right)}{d\sigma\left(\vartheta, -(\pi/2)\right)+d\sigma\left(\vartheta, +(\pi/2)\right)}.$$

The longitudinal component of the polarization $\zeta_z$ cannot be determined by single scattering.

If $\Delta(\vartheta) = 0$, then the scattering does not have any azimuthal asymmetry. This occurs, in particular, in the nonrelativistic approximation when scattering by a central field gives rise to the same phases $\delta_{jl}$ for a given value of $l$ and different values of j. In such a case in accordance with (14.7) $b = 0$ and, consequently, $\Delta = 0$.

Moreover, there is no azimuthal asymmetry in the Born approximation. Indeed, in this approximation $\exp(2i\delta_\varkappa)-1 \approx 2i\delta_\varkappa$, $a(\vartheta)$ and $b(\vartheta)$ are real, and according to (14.11) $\Delta = 0$.

We note that $\Delta \to 0$ when $\vartheta \to 0$, since $\Delta(\vartheta)$, like $b(\vartheta)$, contains the factor $\sin\vartheta$.

Because the matrix $R$ depends on $\boldsymbol{\sigma}$, the state of polarization of the electron is altered by scattering. The polarization density matrix $\varrho'$ of the scattered electron can be defined as $\varrho'_{\alpha\beta} = F^\alpha F^{\beta *}/|F|^2$ where $F^\alpha$ are the components of the spinor $F$. We obtain from (14.6)

$$\varrho' = \frac{R\varrho R^+}{\mathrm{Sp}\,\varrho R^+R}. \tag{14.12}$$

Therefore, the polarization vector $\boldsymbol{\zeta}'$ of the scattered electron is given by

$$\boldsymbol{\zeta}' = \frac{\mathrm{Sp}\, \varrho R^{+} \boldsymbol{\sigma} R}{\mathrm{Sp}\, \varrho R^{+} R}. \tag{14.13}$$

If the incident electron is unpolarized, then it follows from this that

$$\boldsymbol{\zeta}' = \Delta(\vartheta)\boldsymbol{\tau}, \quad \boldsymbol{\tau} = \frac{[\boldsymbol{n}_0 \boldsymbol{n}]}{\sin\vartheta}; \tag{14.14}$$

i.e., the scattered electron is polarized in the direction perpendicular to the scattering plane, while its degree of polarization is equal to $\Delta$. Thus, the scattering system acts as a "polarizer".

On utilizing (14.14) and (14.10) we conclude that secondary scattering of unpolarized electrons will exhibit azimuthal asymmetry. The differential cross section for secondary scattering is given by

$$\frac{d\sigma'}{do'} = Q(\vartheta')\left(1 + \Delta(\vartheta)\Delta(\vartheta')\boldsymbol{\tau}\boldsymbol{\tau}'\right), \tag{14.15}$$

where $\cos\vartheta' = (\boldsymbol{n}\boldsymbol{n}')$, $\boldsymbol{\tau}' = [\boldsymbol{n}\boldsymbol{n}']/\sin\vartheta'$, $\boldsymbol{n}'$ is the unit vector in the direction of the secondary scattering.

In the general case, if initial polarization is present we have

$$\boldsymbol{\zeta}' = \frac{(|a|^2 - |b|^2)\boldsymbol{\xi} + i(ab^* - a^*b)\boldsymbol{\tau} + (ab^* + a^*b)[\boldsymbol{\tau}\boldsymbol{\xi}] + 2|b|^2(\boldsymbol{\xi}\boldsymbol{\tau})\boldsymbol{\tau}}{|a|^2 + |b|^2 + i(ab^* - a^*b)\boldsymbol{\xi}\boldsymbol{\tau}}.$$

### 14.4. *Scattering by a Coulomb Field*

The foregoing general theory of scattering by a central field is also applicable to the case of the Coulomb field. However, since (cf. § 13) the phase in the asymptotic expression for the wave function depends logarithmically on $r$, the solution of the Dirac equation which describes scattering should be sought in a form which takes into account the distortion of the plane wave at infinity, viz., in the form (cf. 13.8)

$$\varphi = w e^{ipz - i\nu \ln p(r-z)} + F\frac{e^{ipr - i\nu \ln 2pr}}{r}.$$

Formula (14.4) remains valid and, in accordance with (13.8), we have

$$\delta_{\varkappa} = \xi - \arg\Gamma(\gamma + i\nu) - \frac{\pi}{2}(\gamma - l - 1), \quad \nu = \frac{Z\alpha}{v}, \quad \gamma = \sqrt{\varkappa^2 - Z^2\alpha^2}. \tag{14.16}$$

The series (14.7) for the functions $a(\vartheta)$ and $b(\vartheta)$ in which $\delta_\varkappa$ is defined by formula (14.16) cannot be summed analytically. For low values of $Z$ it is possible to expand these functions in terms of the parameter $Z\alpha$. In this case the scattering problem can be solved by perturbation theory. This solution will be obtained in §§ 26, 51. Here we shall only state the results (138), (133).

In the lowest order approximation (to terms of order $(Z\alpha)^2$) we have

$$\frac{d\sigma}{do} = Q(\vartheta) = \left(\frac{Z\alpha}{2pv}\right)^2 \frac{1}{\sin^4(\vartheta/2)}\left(1-v^2\sin^2\frac{\vartheta}{2}\right) \quad \text{and} \quad \Delta(\vartheta) = 0. \qquad (14.17)$$

This formula differs from Rutherford's formula by the last factor in (14.17). In the next approximation (to terms of order $(Z\alpha)^3$) an azimuthal asymmetry appears, and also a difference between the scattering of electrons and positrons:

$$Q(\vartheta) = \left(\frac{Z\alpha}{2pv}\right)^2 \frac{1}{\sin^4(\vartheta/2)}[1-v^2\sin^2(\vartheta/2)+\pi Z\alpha\sin(\vartheta/2)(1-\sin(\vartheta/2))], \qquad (14.18)$$

$$\Delta(\vartheta) = -2Z\alpha\frac{v(1-v^2)^{1/2}}{1-v^2\sin^2(\vartheta/2)}\cdot\frac{\sin^3(\vartheta/2)}{\cos(\vartheta/2)}\ln\frac{1}{\sin(\vartheta/2)} \qquad (14.19)$$

(in the case of positron scattering $Z\alpha$ should be taken as negative).

In the general case of arbitrary $Z$ the determination of the cross section requires numerical calculation. Figure 4 shows some results obtained in this manner (50). The curves show the ratio of the differential scattering cross section for unpolarized electrons to the value obtained from Rutherford's formula $d\sigma_R$, i.e.,

$$\frac{d\sigma}{d\sigma_R} = \frac{Q(\vartheta)}{\left(Z\alpha/2pv\sin^2(\vartheta/2)\right)^2}$$

as a function of the scattering angle $\vartheta$ for different values of the energy and of $Z$. Figure 5 shows similar results in the case of positrons ($Z < 0$). Figure 6 gives the curves of the dependence of $\Delta$ on the energy for given $\vartheta$ and $Z$ in the case of electrons (179), while Fig. 7 gives the same information for positrons (8). We note that the signs of $\Delta$ shown on the graphs are different for electrons and positrons.

At very high electron energies it is necessary to take finite nuclear size into account. The change in the phases due to it may be taken into account by the method presented in § 12. However, we shall not reproduce the corresponding results here (164), (61), (2).

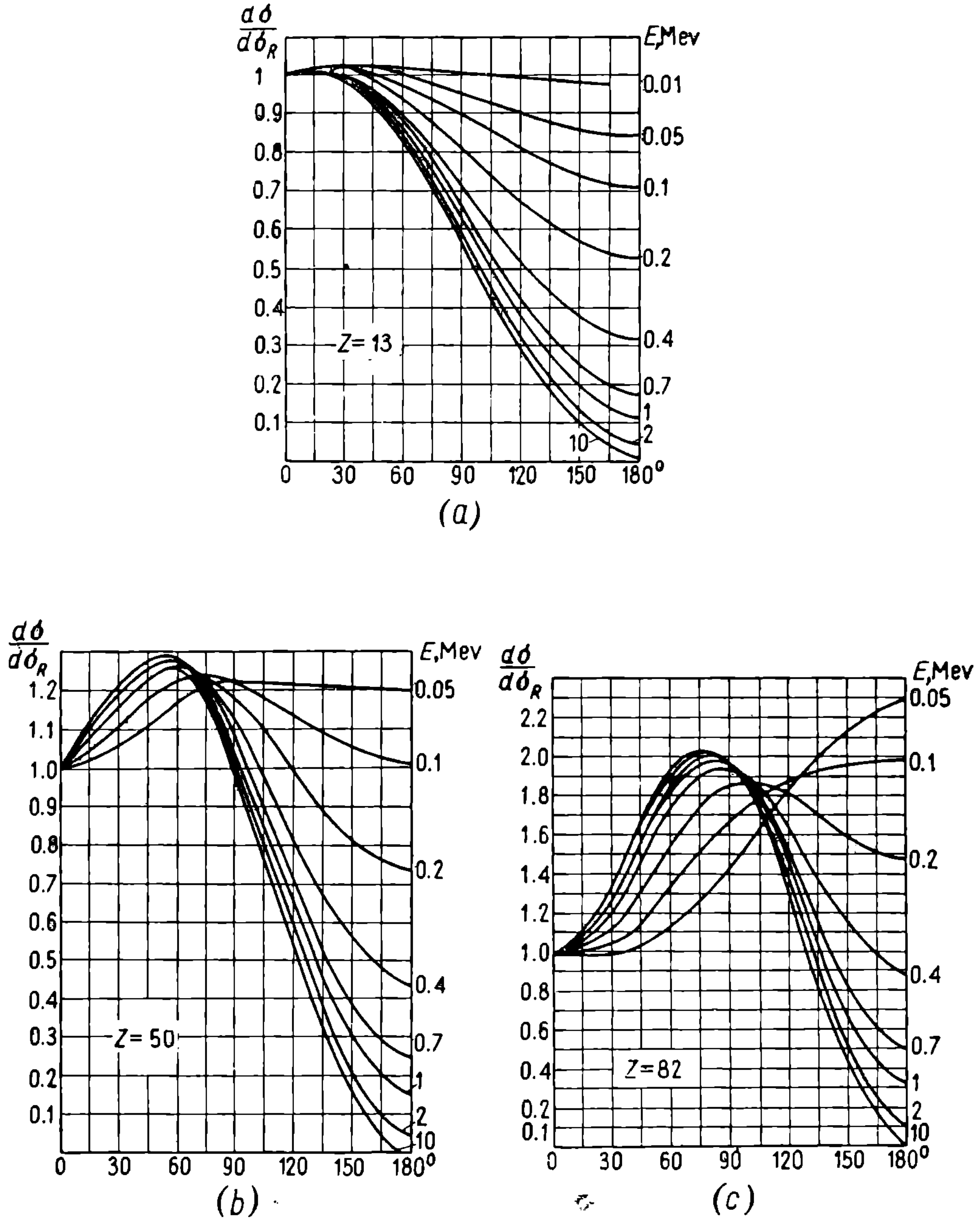
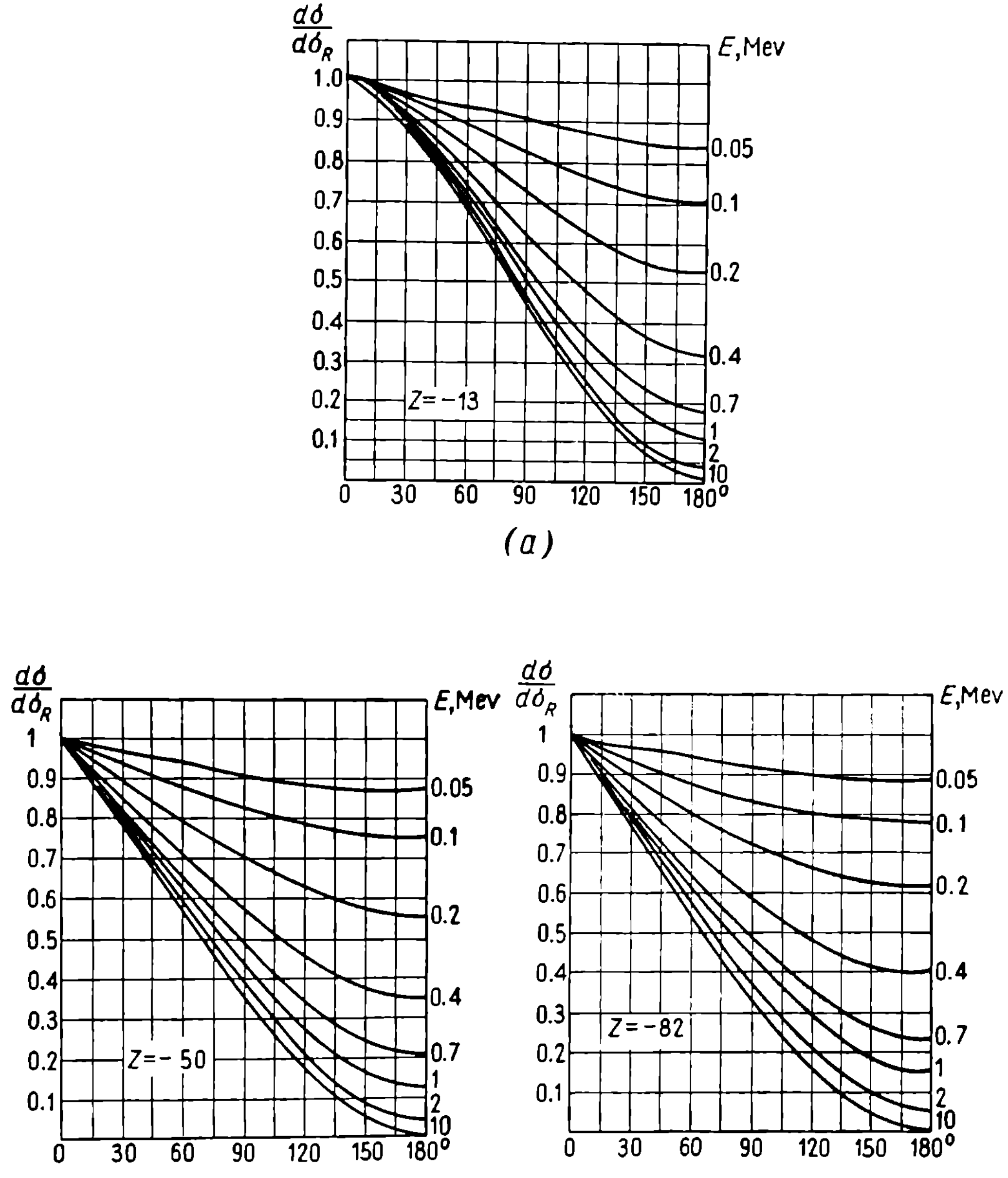

dσ/dσ_R
E, Mev
0.01
0.05
0.1
0.2
0.4
0.7
1
2
10
Z = 13
0 30 60 90 120 150 180°
1 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1
(a)
dσ/dσ_R
E, Mev
0.05
0.1
0.2
0.4
0.7
1
2
10
Z = 50
1.2 1.1 1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1
0 30 60 90 120 150 180°
(b)
dσ/dσ_R
E, Mev
0.05
0.1
0.2
0.4
0.7
1
2
10
Z = 82
2.2 2.0 1.8 1.6 1.4 1.2 1.0 0.8 0.6 0.4 0.2 0.1
0 30 60 90 120 150 180°
(c)

Fig. 4.


$\frac{d\sigma}{d\sigma_R}$
E,Mev
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
30
60
90
120
150
180°
0.05
0.1
0.2
0.4
0.7
1
2
10
Z=−13
Z=− 50
Z=−82
1

(a)

(b)

(c)

Fig. 5.

### 14.5. *Small Angle Scattering*

As has already been noted earlier (cf. (14.5)), the solution of the Dirac equation for an electron in a Coulomb field which has the asymptotic form of a plane and an outgoing spherical wave

$$\psi_p = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} = ue^{ipz} + G(\boldsymbol{n})\frac{e^{ipr}}{r},$$

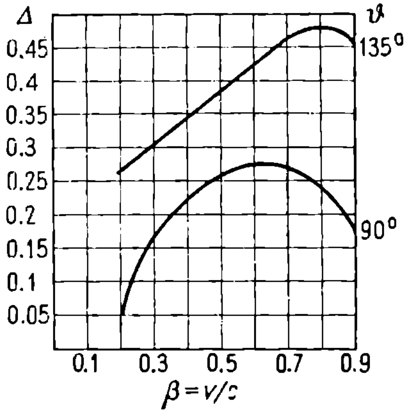


Fig. 6.

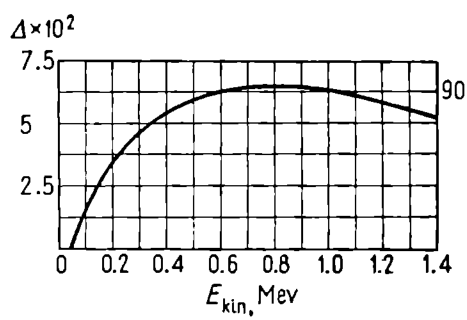


Fig. 7.

can be expressed in the form of the following series:

$$\varphi = \sum 4\pi i^l \left(\Omega^*_{jlM}(0)\, w\right) \frac{1}{p} e^{i\delta_\kappa} \Omega_{jlM}\left(\frac{\boldsymbol{r}}{r}\right) \frac{1}{2} \frac{|\Gamma(\gamma+i\nu)|}{\Gamma(2\gamma+1)} e^{\frac{\pi}{2}\nu}$$

$$\times \{(\gamma+i\nu)e^{-i(pr-\eta)}F(\gamma+1+i\nu,\ 2\gamma+1;\ 2ipr) + \text{compl. conj.}\} \qquad (14.20)$$

and of an analogous series for $\chi$, neither of which can be summed analytically.

However, it is possible to obtain an approximate solution $\psi^{(1)}$ having the following properties: if $\psi^{(1)}$ is expanded in a series in terms of spherical waves $\psi^{(1)} = \sum_l \psi_l^{(1)}$, then the terms of this series will differ significantly from the terms of the exact series $\psi_p = \sum_l \psi_l$ only for low values of $l$. This approximate solution is useful for the investigation of those processes in which the principal role is played by large values of $l$, for example, the case of small angle scattering.

The function $\psi^{(1)}$ can be obtained in the following manner (69), (22). We transform the Dirac equation

$$\left(\varepsilon-\frac{Za}{r}-\beta m+i\boldsymbol{\alpha}\nabla\right)\psi=0$$

into a second-order equation. In order to do this we apply to the equation given above the operator $(\varepsilon+Za/r+\beta m-i\boldsymbol{\alpha}\nabla)$:

$$\left(\Delta+p^2+\frac{2\varepsilon Za}{r}\right)\psi=\left(Zai\boldsymbol{\alpha}\nabla\frac{1}{r}-\frac{(Za)^2}{r^2}\right)\psi,\quad p^2=\varepsilon^2-m^2. \tag{14.21}$$

If in this equation we omit the right-hand side, then we obtain an equation which coincides with the nonrelativistic Schrödinger equation for a Coulomb field. It is well known that its solution having the required asymptotic form is given by $\psi=cue^{ipz}F$, where $F$ is the confluent hypergeometric function $F=F(i\nu,\ 1;\ ipr-ipr)$, $\nu=Za/v$, $v$ is the electron velocity, $u$ is a unit bispinor, $c$ is a constant.

On this basis it is useful to seek the solution of equation (14.21) in the form $\psi=ce^{ipz}(1+\Omega)uF$, where $\Omega$ is a particular operator. In this form it is possible to take partially into account the first term on the right-hand side of (14.21). The second term may be neglected since it contains $1/r^2$, and therefore it may be expected that for large values of $l$ and for small angles small values of $r$ are not important.

It may be shown that the function

$$\psi^{(1)}=ce^{ipz}\left(1-\frac{i}{2\varepsilon}\boldsymbol{\alpha}\nabla\right)uF \tag{14.22}$$

actually possesses the required properties. In order to prove this the following expansion should be used (43):

$$e^{ipz}F=\sum_l(2l+1)i^lP_l(\cos\vartheta)e^{\frac{\pi}{2}\nu}\frac{\Gamma(l+1-i\nu)}{\Gamma(2l+2)}(2pr)^le^{ipr}G_l,$$

$$G_l=F(l+1-i\nu,\ 2l+2;\ -2ipr),$$

the function $\psi^{(1)}$ should be expanded by means of it, and then this expansion should be compared with the series (14.20). It then turns out (22) that for $(Za)^2/l^2\ll 1$ the terms of the two series coincide.

From the asymptotic form $\psi^{(1)} \sim u e^{ipz} + f(\boldsymbol{n}) u \dfrac{e^{ipr}}{r}$ it follows that

$$f(\boldsymbol{n}) = \frac{Z\alpha}{2pv \sin^2(\vartheta/2)} \left(1 + \frac{1}{2v}(\boldsymbol{\alpha n} - \alpha_z)\right).$$

For small angles the second term may be neglected and we have $f = 2Z\alpha/pv\vartheta^2$. In this case the cross section coincides with formula (14.17) for $\vartheta \ll 1$.

## § 15. Nonrelativistic Approximation

### 15.1. *Transition to the Pauli Equation*

If the electron energy differs but little from the rest energy, i.e., $|\varepsilon - m| \ll m$, then, as we have seen earlier, one of the two spinors composing the electron wave function is much smaller than the other: $\chi \ll \varphi$. This enables us to obtain an approximate equation which contains only the spinor $\varphi$ by means of a formal expansion of the wave functions in a series in powers of $1/c$, where $c$ is the velocity of light.

We start with the Dirac equation for an electron in an external field which we write in the form

$$i\frac{\partial \psi}{\partial t} = \left[c\boldsymbol{\alpha}\left(\mathbf{p} - \frac{e}{c}A\right) + \beta mc^2 + eA_0\right]\psi.$$

Since in nonrelativistic theory the energy is taken to mean the difference between the total energy and the rest energy it is convenient to transform the wave function by introducing instead of $\psi$ the function $\psi'$:

$$\psi = \psi' e^{-imc^2 t}.$$

On substituting this expression into the Dirac equation

$$i\frac{\partial \psi'}{\partial t} = \left\{c\boldsymbol{\alpha}\left(\mathbf{p} - \frac{e}{c}A\right) + (\beta - 1)mc^2 + eA_0\right\}\psi'$$

and by representing as usual $\psi'$ in the form $\psi' = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$, we obtain the following system of equations:

$$\begin{aligned} i\frac{\partial \varphi}{\partial t} &= c\boldsymbol{\sigma}\left(\mathbf{p} - \frac{e}{c}A\right)\chi + eA_0\varphi, \\ i\frac{\partial \chi}{\partial t} &= c\boldsymbol{\sigma}\left(\mathbf{p} - \frac{e}{c}A\right)\varphi + eA_0\chi - 2mc^2\chi \end{aligned} \qquad (15.1)$$

(here we have omitted primes on the spinors $\varphi$ and $\chi$; this will not lead to any confusion since in future we shall use only the transformed function $\psi'$).

In order to carry out the expansion in powers of $1/c$, we assume that $\chi$ is of the order of magnitude of $\varphi/c$ (this is justified by the result of subsequent computations). Then in order to obtain the lowest-order approximation it is possible to neglect in the second equation all terms containing $\chi$ with the exception of the last one the coefficient of which contains $c^2$. We thus obtain

$$\chi = \frac{1}{2mc}\boldsymbol{\sigma}\left(\mathbf{p}-\frac{e}{c}A\right)\varphi. \tag{15.2}$$

The substitution of this expression into the first equation yields

$$i\frac{\partial\varphi}{\partial t} = \left\{\frac{1}{2m}\left[\boldsymbol{\sigma}\left(\mathbf{p}-\frac{e}{c}A\right)\right]^2 + eA_0\right\}\varphi.$$

By utilizing the properties of the matrices $\sigma_i$, it can be easily shown that

$$\left[\boldsymbol{\sigma}\left(\mathbf{p}-\frac{e}{c}A\right)\right]^2 = \left(\mathbf{p}-\frac{e}{c}A\right)^2 - \frac{e}{c}\boldsymbol{\sigma}H,$$

where $H = \text{curl}\, A$ is the magnetic field. By utilizing this transformation it is possible to write the equation to first approximation in the form

$$i\frac{\partial\varphi}{\partial t} = \mathrm{H}\varphi = \left[\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}A\right)^2 + eA_0 - \frac{e}{2mc}\boldsymbol{\sigma}H\right]\varphi. \tag{15.3}$$

Equation (15.3) is known as the Pauli equation. It differs from the nonrelativistic Schrödinger equation by the presence of the Hamiltonian H in the last term, which has the form of the potential energy of a magnetic dipole in an external field. Thus, in first approximation the electron behaves as a nonrelativistic particle having in addition to its charge a magnetic moment

$$\boldsymbol{\mu} = \frac{e\hbar}{2mc}\boldsymbol{\sigma}. \tag{15.4}$$

## 15.2. *Second Approximation*

We now obtain the second-order approximation by continuing the expansion to terms of order $1/c^2$. At first we confine ourselves to he case when only an external electric field is present ($A = 0$).

On substituting the first-order expression (15.2) for $\chi$ into the terms of the second of equations (15.1) which were neglected earlier, and on solving the resulting equation with respect to $\chi$ we obtain

$$\chi = \frac{1}{2mc}\left(\sigma\mathrm{p} + \frac{eA_0}{2mc^2}\sigma\mathrm{p} - \frac{i}{2mc^2}\sigma\mathrm{p}\frac{\partial}{\partial t}\right)\varphi. \tag{15.5}$$

On substituting this value of $\chi$ into the first of equations (15.1) we obtain the equation for $\varphi$:

$$i\left(1+\frac{\mathrm{p}^2}{4mc^2}\right)\frac{\partial\varphi}{\partial t} = \left\{\frac{\mathrm{p}^2}{2m} + eA_0 + \frac{e}{4m^2c^2}(\sigma\mathrm{p})A_0(\sigma\mathrm{p})\right\}\varphi.$$

On multiplying the last equation by $(1-\mathrm{p}^2/4m^2c^2)$ and on retaining terms of order not higher than $1/c^2$ we rewrite it in the following form:

$$i\frac{\partial\varphi}{\partial t} = \mathrm{H}'\varphi, \quad \mathrm{H}' = \left(1-\frac{\mathrm{p}^2}{4m^2c^2}\right)\left(\frac{\mathrm{p}^2}{2m}+eA_0\right) + \frac{e}{4m^2c^2}(\sigma\mathrm{p})A_0(\sigma\mathrm{p}). \tag{15.6}$$

Although this equation does have the form of a Schrödinger equation, nevertheless it cannot be directly regarded as a wave equation, since the function does not have the exact meaning of an electron wave function, and the operator H′ cannot be regarded as a Hamiltonian operator. Indeed, in the first place the integral $\int \varphi^*\varphi\, dr$ is not conserved in time, since according to the equation of continuity only the integral $\int (\varphi^*\varphi+\chi^*\chi)dr$ is conserved. Therefore the function $\varphi$ cannot be normalized. Secondly, H′ is not a self-conjugate operator, since it contains the term $(e/4m^2c^2)\mathrm{p}^2A_0$. Both these deficiencies can be eliminated if we carry out a transformation of the wave function by introducing in place of $\varphi$ the wave function $\Phi$:

$$\Phi = \mathrm{O}\varphi, \tag{15.7}$$

and by choosing O in such a way that $\Phi$ can be normalized.

After such a transformation equation (15.6) assumes the form

$$i\frac{\partial\Phi}{\partial t} = \mathrm{H}\Phi, \tag{15.8}$$

where $\mathrm{H} = \mathrm{OH'O^{-1}}$.

We obtain the operator O by requiring that for terms of order $1/c^2$ the equation

$$\int(\varphi^*\varphi+\chi^*\chi)dr = \int \Phi^*\Phi\, dr$$

should hold. To achieve this it is sufficient to substitute into the above equation the first-order expression (15.2) for $\chi$, which yields with the required degree of accuracy

$$\int \varphi^*\left(1+(\mathbf{p}^2/4m^2c^2)\right)\varphi\, dr = \int \Phi^*\Phi\, dr.$$

From this it follows that the operator O may be chosen in the form

$$\mathrm{O} = \left(1+\frac{\mathbf{p}^2}{8m^2c^2}\right). \tag{15.9}$$

Since the operator O differs from unity only by a term of order $1/c^2$, the transformation (15.7) does not affect the first-order equation in which $\varphi$ is a normalized wave function to terms of order $1/c$.

We now find the Hamiltonian operator in the second-order approximation. On substituting expressions (15.9) and (15.6) for the operators O and H′ into (15.8) and retaining terms of order $1/c^2$, we obtain

$$\mathrm{H} = \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3c^2} + eA_0 + \frac{1}{8m^2c^2}(\mathbf{p}^2A_0 - A_0\mathbf{p}^2) - \frac{e}{4m^2c^2}\mathbf{p}^2A_0$$
$$+\frac{e}{4m^2c^2}(\boldsymbol{\sigma}\mathbf{p})A_0(\boldsymbol{\sigma}\mathbf{p}).$$

The last two terms in this expression contain operators which are not self-conjugate. However, it can be easily shown that they cancel. Indeed, $(\boldsymbol{\sigma}\mathbf{p})A_0(\boldsymbol{\sigma}\mathbf{p}) = A_0\mathbf{p}^2 - \boldsymbol{\sigma}(A_0\mathbf{p} - \mathbf{p}A_0)\boldsymbol{\sigma}\mathbf{p} = A_0\mathbf{p}^2 - i(\boldsymbol{\sigma}\nabla A_0)(\boldsymbol{\sigma}\mathbf{p})$ $= A_0\mathbf{p}^2 + iE\mathbf{p} - [E\mathbf{p}]\boldsymbol{\sigma}$, where $E = -\nabla A_0$ is the electric field. On noting that $\mathbf{p}^2A_0 - A_0\mathbf{p}^2 = -\Delta A_0 + 2iE\mathbf{p}$, we finally obtain the following expression for the Hamiltonian operator:

$$\mathrm{H} = \frac{\mathbf{p}^2}{2m} + eA_0 - \frac{\mathbf{p}^4}{8m^3c^2} - \frac{e}{4m^2c^2}\boldsymbol{\sigma}[E\mathbf{p}] + \frac{e}{8m^2c^2}\Delta A_0, \tag{15.10}$$

which is Hermitian.

The last three terms in (15.10) are corrections of order $1/c^2$. The first of them takes into account the dependence of the mass on the velocity, while the second one can be interpreted as the energy of interaction between the moving magnetic dipole and the electric field. Since the equation $\Delta A_0 = -\varrho$ holds for the electrostatic field where $\varrho$ is the charge density giving rise to the external field, the last term differs from zero only at those points where these charges are present. In particular, for the Coulomb field of the charge $Ze$ we have

$$\Delta A_0 = -Ze\delta(\mathrm{r}).$$

By utilizing the Hamiltonian (15.10) it is possible to solve by perturbation theory methods the problem of the fine structure of the levels of a hydrogen-like atom, taking for the perturbation the last three terms in (15.10). This leads to formula (13.4′) obtained earlier by means of expanding the exact solution (13.4) of the Dirac equation.

In the presence of an external magnetic field the Hamiltonian differs from (15.10) by the replacement of $\mathbf{p}$ by $\mathbf{p}-\frac{e}{c}A$ and of $\mathrm{p}^2$ by $\left[\boldsymbol{\sigma}\left(\mathbf{p}-\frac{e}{c}A\right)\right]^2 = \left(\mathbf{p}-\frac{e}{c}A\right)^2 - \frac{e}{c}\boldsymbol{\sigma}H$, and has the form

$$\mathrm{H} = \frac{\left(\mathrm{p}-\frac{e}{c}A\right)^2}{2m} - \frac{\mathrm{p}^4}{8m^3c^2} + eA_0 - \frac{e}{2mc}\boldsymbol{\sigma}H + \frac{e}{8m^2c^2}\Delta A_0 - \frac{e}{4m^2c^2}\boldsymbol{\sigma}\left[E,\left(\mathrm{p}-\frac{e}{c}A\right)\right]. \qquad (15.11)$$

The transition to the approximate equation for the two-component function $\Phi$ can also be accomplished (66) by means of the unitary transformation $\psi' = \mathrm{U}\psi$, $\mathrm{H}' = \mathrm{UHU}^{-1}$, which leads to the system of equations $i\frac{\partial\psi'}{\partial t} = \mathrm{H}'\psi' = \mathrm{H}'\begin{pmatrix}\varphi'\\ \chi'\end{pmatrix}$, which breaks up into two pairs of independent equations in $\varphi'$ and $\chi'$.

There exists no matrix U which has precisely the properties indicated earlier, but it can be obtained approximately in the nonrelativistic domain in the form of an expansion in powers of $1/mc^2$. In order to do this we write the Hamiltonian H in the form

$$\mathrm{H} = \begin{pmatrix}\mathrm{H}_{11} & \mathrm{H}_{12}\\ \mathrm{H}_{21} & \mathrm{H}_{22}\end{pmatrix},$$

where $\mathrm{H}_{ik}$ are two-rowed matrices, and we break up H into the diagonal $\mathrm{H}_+ = \begin{pmatrix}\mathrm{H}_{11} & \mathrm{O}\\ \mathrm{O} & \mathrm{H}_{22}\end{pmatrix}$ and the nondiagonal $\mathrm{H}_- = \begin{pmatrix}\mathrm{O} & \mathrm{H}_{12}\\ \mathrm{H}_{21} & \mathrm{O}\end{pmatrix}$ terms

$$\mathrm{H} = \mathrm{H}_+ + \mathrm{H}_-;$$

$$\mathrm{H}_+ = \beta mc^2 + eA_0; \quad \mathrm{H}_- = ec(\mathbf{p}-(e/c)A)\boldsymbol{\alpha}.$$

The problem consists of making the matrix $H'_-$ equal to zero after the unitary transformation. This may be achieved approximately by choosing $U = \exp(\beta H_-/2mc^2)$. On expanding U into a series it can be easily seen that in the new Hamiltonian H′ the nondiagonal part contains only terms of order $1/mc$ and higher. Further, on carrying out the transformation $H'' = U'H'U'^{-1}$ with the aid of the matrix $U' = \exp\{-(\beta/2mc^2)H'_-\}$ we obtain $H''_-$ of order $1/(mc)^2$, etc.

### 15.3. *Application of the Dirac Equation to Nucleons*

The Dirac equation describes any free particle of spin $\frac{1}{2}$. However, its application to the behavior of a particle in an external electromagnetic field is limited to electrons or to other particles ($\mu$ mesons) for which the electromagnetic interactions are the strongest ones.

In view of the restricted nature of the concept of an external field the Dirac equation (12.1) is not exact even for an electron. Thus, for example, as has already been noted earlier, the formula for the energy levels of the hydrogen atom is correct only to terms of order $\alpha^4$. If the interaction of the electron with its own electromagnetic field is taken into account (radiation corrections), then, as we shall see later (Chapters VII and VIII), this will lead to correction terms in the equation describing the behavior of the electron in an external field. Thus, in equation (15.11) the coefficient of the term containing $\boldsymbol{\sigma H}$ will be altered, and this can be interpreted as a change in the magnetic moment, and also the coefficient of $\Delta A_0$ will be altered. The use of equations (12.1) and (15.11) is justified by the fact that these corrections are small.

The interaction of nucleons with meson fields is of the greatest importance. Therefore, an investigation of the behavior of a proton in an electromagnetic field cannot be carried out in even a partially satisfactory manner. As a result of its strong interaction with the meson field, which is associated with charged particles, the motion of a proton in an electromagnetic field does not coincide with the motion of a particle which is characterized only by its charge. Equation (12.1) is not applicable to the proton; the exact equation must take into account the additional interaction with the electromagnetic field through the meson fields.

In particular, as is shown by experiment, equation (15.3) of the nonrelativistic approximation turns out to be inapplicable to the proton.

For $v/c \ll 1$ the proton has a magnetic moment which is not equal to the nuclear magneton $\mu_0 = e\hbar/2Mc$ ($M$ is the proton mass), as would follow from (15.4), but which has the value $\mu_p = 2.7\mu_0$. The neutron, which has no charge, has a magnetic moment given by $\mu_n = -1.9\,\mu_0$.

Finally we can take into account the additional "anomalous" magnetic moment of the nucleons by introducing into the Dirac equation an additional interaction in the form $-\mu'\boldsymbol{\sigma}\boldsymbol{H}$ where $\mu'$ is the "anomalous" part of the magnetic moment equal to $\mu' = \mu_p - \mu_0$ in the case of the proton and $\mu' = \mu_n$ in the case of the neutron.

The corresponding relativistically invariant equation containing the electromagnetic field tensor $F_{\alpha\beta}$ must be of the form

$$\left(i\left(\hat{\mathrm{p}}-\frac{e}{c}\hat{A}\right)+i\frac{\mu'}{2}\gamma_\alpha\gamma_\beta F_{\alpha\beta}+Mc^2\right)\psi = 0, \qquad (15.12)$$

or

$$i\frac{\partial\psi}{\partial t} = \mathrm{H}\psi = \left\{c\boldsymbol{\alpha}\left(\mathbf{p}-\frac{e}{c}A\right)-\mu'\beta(\boldsymbol{\Sigma}\boldsymbol{H}-i\boldsymbol{\alpha}\boldsymbol{E})+\beta Mc^2\right\}\psi. \qquad (15.12')$$

We investigate with the aid of this equation the behavior of a neutral particle which has a magnetic moment ($e = 0$, $\mu' \neq 0$).

We now go over to the nonrelativistic approximation. On setting, as we have done previously, $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} e^{-iMc^2t}$, we obtain

$$\left(i\frac{\partial}{\partial t}+\mu'\boldsymbol{\sigma}\boldsymbol{H}\right)\varphi = c\boldsymbol{\sigma}\left(\mathbf{p}-\frac{i\mu'}{c}\boldsymbol{E}\right)\chi$$

and

$$\left(i\frac{\partial}{\partial t}-\mu'\boldsymbol{\sigma}\boldsymbol{H}+2Mc^2\right)\chi = c\boldsymbol{\sigma}\left(\mathbf{p}+\frac{i\mu'}{c}\boldsymbol{E}\right)\varphi.$$

From here we obtain in first approximation $\chi = (1/2Mc)\boldsymbol{\sigma}(\mathbf{p}+(i\mu'/c)\boldsymbol{E})\varphi$. The substitution of this expression into the first equation yields

$$i\frac{\partial\varphi}{\partial t} = \mathrm{H}\varphi.$$

$$\mathrm{H} = \frac{\mathbf{p}^2}{2m}-\mu'\boldsymbol{\sigma}\boldsymbol{H}-\frac{\mu'}{2Mc}\operatorname{div}\boldsymbol{E}+\frac{\mu'}{2Mc}([\mathbf{p},\boldsymbol{E}]-[\boldsymbol{E},\mathbf{p}])+\frac{(\mu'\boldsymbol{E})^2}{2Mc^2}. \qquad (15.13)$$

In accordance with (15.13) a neutral particle which has a magnetic moment interacts with a static electric field (65). In particular, if this equation is applicable to the neutron then the latter must interact with the Coulomb field of the electron. On neglecting the quadratic term in (15.13) we obtain for the energy of this interaction

$$V = -\frac{\mu'}{2Mc}\operatorname{div}\boldsymbol{E} = \mu_0\mu_n\delta(\boldsymbol{r}).$$

This interaction is equivalent to a potential well of finite width. For example, for a well of width $r_0 = e^2/4\pi mc^2$, the depth of the well turns out to be approximately equal to $4 \times 10^3$ ev. This value apparently agrees with the available data on the interaction between neutrons and electrons. We note that the Hamiltonian (15.11) also contains the energy of interaction between a magnetic moment and an electric field (the term $-(\mu_0/4Mc)\operatorname{div}\boldsymbol{E}$), but with a coefficient which is only half as big as the one in the analogous expression for the interaction of the anomalous magnetic moment.

However, it should be emphasized that the conclusion about the existence of the interaction (15.14) on the basis of equation (15.12) is illusory. In actual fact there can exist no equation of the type (15.12), nor one similar to it, which does not take into account the interaction of a nucleon with its own meson fields. We have already noted that even for an electron the radiation corrections distort the coefficients of $\boldsymbol{\sigma H}$ and div $\boldsymbol{E}$ in its Hamiltonian. It is *a fortiori* impossible to establish a relation between these coefficients for a nucleon in the presence of strong meson interactions. Such a relation can be obtained only on the basis of a theory of meson-nucleon interactions, which must also give a quantitative expression for the neutron magnetic moment. Since at the present time such a theory does not exist, the behavior of nucleons in external electromagnetic fields may be approximately described by equations of the type (15.11), (15,13) with empirical coefficients. In variable fields these coefficients may depend on the field frequency.

# Quantized Electromagnetic and Electron-Positron Fields

## § 16. Quantization of the Electromagnetic Field

### 16.1. *Four-Dimensional Form of the Field Equations*

In Chapter I, starting from the classical Maxwell's equations we obtained photon wave functions in momentum space, and investigated quantum photon states. However, for the general formulation of the theory of the electromagnetic field and of its interaction with electrons, and also for the investigation of specific phenomena to which this interaction gives rise, it is more convenient to utilize another method, to the presentation of which we now turn. It consists of treating the electromagnetic field as a certain generalized dynamical system, and of regarding the quantities defining the field as generalized coordinates of this system. The quantum nature of the system is taken into account in this treatment by replacing the classical quantities by the corresponding operators. These operators act on the wave function of the system which depends on the "occupation numbers," i.e., on the numbers of particles—photons—present in the different quantum states. This method of describing the electromagnetic field as a quantum mechanical system, utilizing the wave function of the system in occupation number space, is referred to as the *method of second quantization.*[1]

We note that the transition from the wave function in configuration or in momentum space to the wave function in occupation number space is a canonical transformation. Therefore, the two approaches to the quantum theory of electromagnetic processes from the "corpuscular" (photons and their wave functions) and the "field" (field and its quantum mechanical study) points of view are equivalent.

Thus, we shall treat the four-potential ($A$, $iA_0$) and the electromagnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ not as ordinary quantities ($c$-numbers), but as operators ($q$-numbers). In the same way that Newton's mechanical equations which are satisfied by the momentum and coordinate operators are retained in nonrelativistic quantum mechanics, so in quantum electrodynamics we retain Maxwell's equations which are satisfied by the field operators. But in quantum mechanics the equations of motion acquire a concrete meaning only after the establishment of commutation relations between the components of the momenta and the coordinates; therefore in quantum electrodynamics it is also necessary, first of all, to establish the commutation relations between the dynamical quantities characterizing the electromagnetic field. These relations are referred to as the *field quantization rules.*

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[1] This method, as applied to radiation theory, was first developed by Dirac (46).

Since the field quantization rules must be invariant under Lorentz transformations, we shall, before formulating these rules, give the invariant four-dimensional form of the equations of the electromagnetic field (cf., for example, (118)).

As is well known, the electric and the magnetic fields $E$ and $H$ together form the antisymmetric electromagnetic field tensor $F_{\mu\nu}$ whose components are equal to $F_{ik} = e_{ikl} H_l$, $F_{4k} = iE_k$, where $e_{ikl}$ is the unit antisymmetric tensor $(i, k, l = 1, 2, 3)$, or

$$F_{\mu\nu} = \begin{pmatrix} 0, & H_z, & -H_y, & -iE_x \\ -H_z, & 0, & H_x, & -iE_y \\ H_y, & -H_x, & 0, & -iE_z \\ iE_x, & iE_y, & iE_z, & 0 \end{pmatrix}$$

With the aid of the tensor $F_{\mu\nu}$ the first and the second pairs of Maxwell's equations can be written in the following form:

$$\frac{\partial F_{\mu\nu}}{\partial x_\varrho} + \frac{\partial F_{\nu\varrho}}{\partial x_\mu} + \frac{\partial F_{\varrho\mu}}{\partial x_\nu} = 0,$$
$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0. \tag{16.1}$$

Here all the subscripts assume the values 1, 2, 3, 4 and the repeated subscript $\nu$ is summed over from $\nu = 1$ to $\nu = 4$; $x_4 = ix_0 = it$.

The tensor components can be expressed in terms of the four potential of the field $A_\mu(A, iA_0)$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

These relations are equivalent to the first pair of Maxwell's equations (16.1).

The four-potential $A_\mu$ is not uniquely determined by the field $F_{\mu\nu}$, since the addition to $A_\mu$ of the gradient of any scalar does not alter the field. In other words, Maxwell's equations are invariant under the transformation $A_\mu \to A'_\mu = A_\mu + (\partial\eta/\partial x_\mu)$, where $\eta$ is an arbitrary scalar. Such a transformation is referred to as a *gradient* or a *gauge* transformation.

All physical relations containing the electromagnetic field must obviously be invariant under gauge transformations.

Since the potential $A_\mu$ is not uniquely defined, a subsidiary condition (the Lorentz condition)

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{16.2}$$

may be imposed upon it.

By utilizing equations (16.1) and the Lorentz condition it can be easily shown that the components of the potential satisfy the wave equation

$$\Box A_\mu \equiv \frac{\partial^2}{\partial x_\nu^2} A_\mu = 0, \qquad \mu = 1, 2, 3, 4. \tag{16.3}$$

These equations, together with the subsidiary condition (16.2), are equivalent to Maxwell's equation.

Condition (16.2) does not completely determine the gauge of the components of the potential $A_\mu$. It restricts gauge transformations to those functions $\eta$ which satisfy $\Box\eta = 0$.

We note that in order to satisfy condition (16.2) it is sufficient to require that at some initial instant of time $t = 0$ the quantity $\chi \equiv \partial A_\mu/\partial x_\mu$ and its time derivative $\partial\chi/\partial t$ should vanish. Then the function $\chi$ will also vanish at all later times (208).

### 16.2. *Variational Principle. Energy-Momentum Tensor of the Electromagnetic Field*

The second pair of Maxwell's equations (16.1) can be obtained from a variational principle if we assume that a definite Lagrangian $\mathcal{L}$ corresponds to the electromagnetic field regarded as a generalized dynamical system. In such an approach the field potentials at different points

$A_\mu(x)$ play the role of generalized coordinates of the system, while $\partial A_\mu/\partial x_\nu$ are the generalized velocities. The integral of the Lagrangian with respect to the time $I = \int \mathfrak{L}\, dt$ is known as the action. It can be represented in the form of a four-dimensional integral $I = \int L d^4x$, where $L$ is known as the Lagrangian density, and the integration is carried out over the invariant four-volume (both now and later we denote the element of volume $dx_1 dx_2 dx_3 dx_0 = dr\, dt$ by $d^4x$).

The field equations can be obtained with the aid of the variational principle $\delta I = 0$, analogous to the variational principles of classical mechanics. In earrying out this variation the generalized coordinates, i.e., $A_\mu(x)$, are varied, and it is assumed that the variation of $A_\mu(x)$ vanishes over the boundaries of the region of integration.

We derive the field equations by assuming that, in general, $L$ depends both on $A_\mu$, and on $\partial A_\mu/\partial x_\nu$. The variation of the action is obviously equal to

$$\delta I = \int \left( \frac{\partial L}{\partial A_\mu} \delta A_\mu + \frac{\partial L}{\partial(\partial A_\mu/\partial x_\nu)} \delta \frac{\partial A_\mu}{\partial x_\nu} \right) d^4x$$

$$= \int \left( \frac{\partial L}{\partial A_\mu} - \frac{\partial}{\partial x_\nu} \frac{\partial L}{\partial(\partial A_\mu/\partial x_\nu)} \right) \delta A_\mu d^4x,$$

and, therefore, the field equations, which are the Lagrangian equations of the system, have the form

$$\frac{\partial}{\partial x_\nu} \frac{\partial L}{\partial(\partial A_\mu/\partial x_\nu)} - \frac{\partial L}{\partial A_\mu} = 0, \quad \text{where } \mu = 1, 2, 3, 4. \qquad (16.4)$$

In order that the field equations should be invariant under Lorentz transformations the action must be invariant, and, consequently, the Lagrangian density $L$ must also be invariant.

On the other hand, the equations of the electromagnetic field must possess the property of gauge invariance, and, therefore, the Lagrangian density cannot contain the components of the four-potential directly, but only the components of the field tensor $F_{\mu\nu}$. But from the field components we can construct only two independent invariants: $E^2 - H^2$ and $(EH)^2$, so that the Lagrangian density can depend only on the two invariants, $L = L\big(E^2 - H^2$ and $(EH)^2\big)$.

If, in addition, we require that the superposition principle should hold, i.e., that the field equations should be linear, then the Lagrangian density can only be a quadratic function in the field components. In other words, up to a numerical factor the Lagrangian $L$ must coincide with $E^2-H^2$. We set

$$L=\tfrac{1}{2}(E^2-H^2)=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}, \tag{16.5}$$

which corresponds to the use of Heaviside units.

In accordance with (16.4) the field equations for this Lagrangian assumes the form

$$\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial(\partial A_\mu/\partial x_\nu)}=0,$$

and since

$$\delta L=-\tfrac{1}{2}F_{\mu\nu}\delta F_{\mu\nu}=-\tfrac{1}{2}F_{\mu\nu}\left(\delta\frac{\partial A_\nu}{\partial x_\mu}-\delta\frac{\partial A_\mu}{\partial x_\nu}\right)=F_{\mu\nu}\delta\frac{\partial A_\mu}{\partial x_\nu},$$

i.e.,

$$\frac{\partial L}{\partial(\partial A_\mu/\partial x_\nu)}=F_{\mu\nu},$$

we finally obtain the second pair of Maxwell's equations $\partial F_{\mu\nu}/\partial x_\nu=0$.

We use the Lagrangian equations (16.4) to obtain the four-dimensional form of the conservation laws for the energy and the momentum of the electromagnetic field. We consider the derivative $\partial L/\partial x_\nu$;

$$\frac{\partial L}{\partial x_\nu}=\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\sigma)}\frac{\partial(\partial A_\varrho/\partial x_\sigma)}{\partial x_\nu}+\frac{\partial L}{\partial A_\varrho}\frac{\partial A_\varrho}{\partial x_\nu}.$$

On substituting into this expression the value of $\partial L/\partial A_\varrho$ from (16.4) we obtain

$$\frac{\partial L}{\partial x_\nu}=\frac{\partial A_\varrho}{\partial x_\nu}\frac{\partial}{\partial x_\sigma}\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\sigma)}+\frac{\partial}{\partial x_\nu}\frac{\partial A_\varrho}{\partial x_\sigma}\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\sigma)}$$
$$=\frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\varrho}{\partial x_\nu}\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\mu)}\right),$$

whence it follows that

$$\frac{\partial}{\partial x_\mu}\left(L\delta_{\mu\nu}-\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\mu)}\frac{\partial A_\varrho}{\partial x_\nu}\right)=0,$$

or, on introducing the notation

$$T_{\mu\nu}=L\delta_{\mu\nu}-\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\nu)}\frac{\partial A_\varrho}{\partial x_\mu}, \tag{16.6}$$

we obtain finally

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0 \quad \text{where} \quad \mu = 1, 2, 3, 4. \tag{16.7}$$

The quantities $T_{\mu\nu}$ form a four-dimensional tensor which is called the *energy-momentum tensor* of the electromagnetic field.

The laws of conservation of energy and momentum follow from equation (16.7). Indeed, we consider the four-vector

$$P_\mu = -i\int T_{\mu 4}dr, \tag{16.8}$$

where the integration is taken over the whole volume of the field.

We evaluate $dP_\mu/dt$ in accordance with (16.7)

$$\frac{d}{dt}P_\mu = \int \frac{\partial T_{\mu 4}}{\partial x_4}dr = -\int \frac{\partial T_{\mu j}}{\partial x_j}dr.$$

The three-dimensional integral appearing in this expression reduces to a surface integral which can be taken equal to zero. Therefore, $P_\mu$ is conserved:

$$P_\mu = \text{const.} \tag{16.9}$$

It represents the energy-momentum four-vector for the field; its spatial components represent the momentum of the field, and its time-like component divided by $i$ is equal to the energy of the field.

On utilizing the formula for $L$ it is possible to express the tensor $T_{\mu\nu}$ in terms of $F_{\mu\nu}$:

$$T_{\mu\nu} = F_{\mu\varrho}F_{\nu\varrho} - \tfrac{1}{4}\,\delta_{\mu\nu}\,F^2_{\sigma\tau}, \tag{16.10}$$

or

$$T_{ik} = \tfrac{1}{2}(E^2+H^2)\delta_{ik} - E_i E_k - H_i H_k, \quad T_{k4} = -is_k \quad (i, k = 1, 2, 3),$$

$$T_{44} = -w.$$

where $w = \frac{1}{2}(E^2+H^2)$ is the energy density of the field, and $s = [\boldsymbol{E}, \boldsymbol{H}]$ is the energy flux density of the field.

The wave equations (16.3) for the potential can also be obtained from a variational principle, but in this case the Lagrangian density must be taken not in the form (16.5), but in the form

$$L = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial A_\mu/\partial x_\mu)^2,$$

or

$$L = -\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu}. \tag{16.11}$$

These two expressions differ by the quantity

$$-\frac{1}{2}\frac{\partial}{\partial x_\nu}\left(A_\nu\frac{\partial A_\mu}{\partial x_\mu}-A_\mu\frac{\partial A_\nu}{\partial x_\mu}\right),$$

which has the form of a four-dimensional divergence, and therefore lead to the same field equations. In order to obtain Maxwell's equations the subsidiary condition (16.2) must now be added to the wave equations.

On taking the Lagrangian in the form (16.11) we obtain the expression for the energy-momentum tensor

$$T_{\mu\nu} = L\delta_{\mu\nu} - \frac{\partial L}{\partial(\partial A_\sigma/\partial x_\mu)}\frac{\partial A_\sigma}{\partial x_\nu} = \frac{\partial A_\sigma}{\partial x_\nu}\frac{\partial A_\sigma}{\partial x_\mu} - \frac{1}{2}\delta_{\mu\nu}\frac{\partial A_\sigma}{\partial x_\varrho}\frac{\partial A_\sigma}{\partial x_\varrho}. \tag{16.12}$$

In the future we shall make use of the wave equations (16.3) and the subsidiary condition (16.2) as our field equations.

### 16.3. *Expansion of the Potentials into Plane Waves*

The general solution of the wave equation can be represented in the form of a superposition of plane waves with arbitrary amplitudes. We write it in the form

$$A_\mu(x) = \frac{1}{\sqrt{\Omega}}\sum_{k,\lambda}\frac{1}{\sqrt{2\omega}}e_\mu^{(\lambda)}(c_{k\lambda}e^{ikx}+c_{k\lambda}^{+}e^{-ikx}). \tag{16.13}$$

Here $\Omega$ is the normalizing volume, $kx = \boldsymbol{kr} - \omega t$ is the scalar product of the four-vector $x(\boldsymbol{r}, it)$ and the four-dimensional propagation vector $k(\boldsymbol{k}, i\omega)$ which satisfies the condition $k^2 = \boldsymbol{k}^2 - \omega^2 = 0$, $c_{k\lambda}$ are the wave amplitudes, and $e_\mu^{(\lambda)}$ are the polarization unit vectors. For a given $k$ four linearly independent solutions are possible differing by their polarizations. We choose the vectors $e^{(\lambda)}$ as the basis vectors for the coordinate system $e_\mu^{(\lambda)} = \delta_{\lambda\mu}$. If we direct the $x_3$-axis along $\boldsymbol{k}$ and the other two space axes $x_1$ and $x_2$ at right angles to $\boldsymbol{k}$ and to each other, then $\lambda = 1, 2$ will correspond to transverse, and $\lambda = 3$ to longitudinal polarization; $\lambda = 4$ corresponds to the so-called scalar polarization. The following relations hold:

$$e^{(\lambda)}k = \begin{cases} 0, & \lambda = 1, 2, \\ \omega, & \lambda = 3, \\ i\omega, & \lambda = 4, \end{cases} \tag{16.14}$$

where

$$e^{(\lambda)}e^{(\lambda')} = \delta_{\lambda\lambda'}, \quad \text{and} \quad e_\mu^{(\lambda)}e_\nu^{(\lambda)} = \delta_{\mu\nu}. \tag{16.14'}$$

From the relation $k^2 = k^2 - \omega^2 = 0$ it follows that the frequency $\omega$ can be of either sign. This fact is taken into account in (16.13) by the existence of two types of terms, containing $\exp(ikx)$ and $\exp(-ikx)$ respectively, in each of which we must take $\omega = |k|$.

We note that the sign of the frequency is relativistically invariant, as can be verified directly with the aid of Lorentz transformations. Therefore, the decomposition of $A_\mu(x)$ into terms with positive and negative frequencies is also invariant.

Since the spatial components of the potential $A_j(x)$ $(j = 1, 2, 3)$ are real, while the time-like component $A_4(x) = iA_0(x)$ is a purely imaginary one, the amplitudes $c_{k\lambda}$ of waves with positive and negative frequencies must satisfy the following relations $c_{kj}^{+} = c_{kj}^{*}$, $c_{k4}^{+} = -c_{k4}^{*}$, where $c^*$ denotes a quantity which is complex-conjugate with respect to $c$. We also use the notation $c_{k4} = ic_{k0}$, $c_{k4}^{+} = ic_{k0}^{*}$.

As we have seen earlier, the potentials $A_\mu(x)$ must in addition to (16.3) also satisfy the subsidiary condition (16.2). On substituting the expansion (16.13) into (16.2) we obtain, after utilizing (16.14),

$$\frac{\partial A_\mu}{\partial x_\mu} = \frac{i}{\sqrt{\Omega}} \sum_{k,\lambda} \frac{k_\mu e_\mu^{(\lambda)}}{\sqrt{2\omega}} (c_{k\lambda} e^{ikx} - c_{k\lambda}^{+} e^{-ikx})$$

$$= \frac{i}{\sqrt{\Omega}} \sum_{k} \sqrt{\frac{\omega}{2}} \left((c_{k3} + ic_{k4}) e^{ikx} - (c_{k3}^{+} + ic_{k4}^{+}) e^{-ikx}\right) = 0,$$

from which it follows that

$$c_{k3} + ic_{k4} = 0, \qquad c_{k3}^{+} + ic_{k4}^{+} = 0. \tag{16.15}$$

By utilizing the expansion (16.13) it is possible to write the field tensor in the form

$$F_{\mu\nu} = \frac{i}{\sqrt{\Omega}} \sum_{k,\lambda} \frac{1}{\sqrt{2\omega}} (c_{k\lambda} e^{ikx} - c_{k\lambda}^{+} e^{-ikx}) (e_\nu^{(\lambda)} k_\mu - e_\mu^{(\lambda)} k_\nu).$$

The electric and the magnetic fields are equal to

$$\boldsymbol{E} = \frac{i}{\sqrt{\Omega}} \sum_{k;\,\lambda=1,2} \sqrt{\frac{\omega}{2}} (c_{k\lambda} e^{ikx} - c_{k\lambda}^{+} e^{-ikx}) \boldsymbol{e}^{(\lambda)},$$

$$\boldsymbol{H} = \frac{i}{\sqrt{\Omega}} \sum_{k;\,\lambda=1,2} \frac{1}{\sqrt{2\omega}} [\boldsymbol{k}, \boldsymbol{e}^{(\lambda)}] (c_{k\lambda} e^{ikx} - c_{k\lambda}^{+} e^{-ik\varkappa}).$$

The energy-momentum tensor defined according to (16.12) can be expressed in terms of the plane wave amplitudes in the following manner:

$$T_{\mu\nu} = \frac{1}{2\Omega} \sum_{k,k',\lambda} \frac{1}{\sqrt{\omega\omega'}} (c_{k\lambda} e^{ikx} - c_{k\lambda}^{+} e^{-ikx})$$
$$\times (c_{k'\lambda} e^{ik'x} - c_{k'\lambda}^{+} e^{-ik'x}) (\tfrac{1}{2} k_o k'_\sigma \delta_{\mu\nu} - k'_\mu k_\nu).$$

The energy and the momentum of the field, which are related to $T_{\mu\nu}$ by expressions (16.8), are equal to

$$\begin{aligned} H &= \sum_{k;\,\lambda=1,2} \frac{\omega}{2} (c_{k\lambda} c_{k\lambda}^{+} + c_{k\lambda}^{+} c_{k\lambda}), \\ \boldsymbol{P} &= \sum_{k;\,\lambda=1,2} \frac{\boldsymbol{k}}{2} (c_{k\lambda} c_{k\lambda}^{+} + c_{k\lambda}^{+} c_{k\lambda}). \end{aligned} \tag{16.16}$$

We see that the electric and magnetic fields, and also the energy and momentum of the field, do not depend on the longitudinal or scalar waves. This is connected with the existence of conditions (16.15). If we had not used these conditions, we would have obtained for $H$ and $\boldsymbol{P}$ the same expressions (16.16) in which, however, $\lambda$ would assume the values 1, 2, 3, 4. The terms corresponding to the polarizations $\lambda = 3, 4$, cancel because of the conditions (16.15).

## 16.4. *Quantization of the Electromagnetic Field*

The expressions for the energy and the momentum of the electromagnetic field enable us to establish the quantization rules for it. These rules must lead to the correct corpuscular picture, which consists of the energy and the momentum of the field being sums of the energies or the momenta of the individual particles—the photons.

In order to arrive at such a picture it is necessary to regard the amplitudes $c_{k\lambda}$ of the plane waves in the expansions for the potentials (16.13) not as ordinary numbers, but as operators satisfying definite commutation relations.

We assume that these relations are

$$[c_{k\lambda}, c^+_{k'\lambda'}] = \delta_{kk'}\delta_{\lambda\lambda'}, \quad [c_{k\lambda}, c_{k'\lambda'}] = 0, \quad [c^+_{k\lambda}, c^+_{k'\lambda'}] = 0, \\ \lambda, \lambda' = 1, 2, 3, 4, \tag{16.17}$$

or, in abbreviated form, $[c_\lambda, c^+_{\lambda'}] = \delta_{\lambda\lambda'}$, $[c_\lambda, c_{\lambda'}] = 0$, $[c^+_\lambda, c^+_{\lambda'}] = 0$, where $c_\lambda \equiv c_{k\lambda}$ and $[a, b] = ab - ba$. Moreover, in accordance with the fact that $A$ and $A_0$ are real we must assume that the operators $c_\lambda$ and $c^+_\lambda$ with $\lambda = 1, 2, 3$, and also the operators $c_0$ and $c^+_0$ are Hermitian conjugate.

We shall show that from the quantization rules (16.17) it follows that the eigenvalues of the operator $N_\lambda \equiv N_{k\lambda} = c^+_\lambda c_\lambda$ with $\lambda = 1, 2, 3$ are equal to $N_\lambda = 0, 1, 2, \ldots$

First of all, it is evident that the eigenvalues $N_\lambda$ are not negative. Indeed, from the equation $c^+_\lambda c_\lambda \Phi = N_\lambda \Phi$, where $\Phi$ is the eigenfunction of the operator $N_\lambda$ belonging to the eigenvalue $N_\lambda$ it follows that

$$N_\lambda = \frac{(c^+_\lambda c_\lambda \Phi, \Phi)}{(\Phi, \Phi)} = \frac{(c_\lambda \Phi, c_\lambda \Phi)}{(\Phi, \Phi)} \geqslant 0,$$

where $(\Phi_1, \Phi_2)$ denotes the scalar product of the wave functions $\Phi_1$ and $\Phi_2$.

We consider the function $\varphi = c_\lambda \Phi$ on the assumption that $N_\lambda > 0$. It differs from zero, since $c^+_\lambda \varphi = c^+_\lambda c_\lambda \Phi = N_\lambda \Phi \neq 0$, and it satisfies

$$N_\lambda \varphi = c^+_\lambda c_\lambda c_\lambda \Phi = (c_\lambda c^+_\lambda - 1) c_\lambda \Phi = (N_\lambda - 1)\varphi.$$

Therefore, $\varphi$ is the eigenfunction of the operator $N_\lambda$ belonging to the eigenvalue $N_\lambda$-1. Similarly, we can show that $N_\lambda$-2, $N_\lambda$-3, ... are also eigenvalues of the operator $N_\lambda$. But such a series must, evidently, break off since the eigenvalues $N_\lambda$ cannot be negative. This is only possible in the case that $N_\lambda$ is either a positive integer or zero.

It can be easily shown that all integers are eigenvalues of the operator $N_\lambda$. This follows from the fact that the function $\chi = c^+_\lambda \Phi$ which satisfies the equation

$$N_\lambda \chi = c^+_\lambda c_\lambda c^+_\lambda \Phi = c^+_\lambda (1 + c^+_\lambda c_\lambda) \Phi = (N_\lambda + 1)\chi,$$

is the eigenfunction of the operator $N_\lambda$ belonging to the eigenvalue $N_\lambda + 1$.

We note that the foregoing problem is equivalent to the problem of finding the eigenvalues of the energy of an oscillator. Indeed, if we set

$$\mathrm{c}_\lambda = \frac{1}{\sqrt{2\omega}}(\mathrm{p}_\lambda + i\omega \mathrm{q}_\lambda), \quad \mathrm{c}_\lambda^+ = \frac{1}{\sqrt{2\omega}}(\mathrm{p}_\lambda - i\omega \mathrm{q}_\lambda),$$

then $\mathrm{q}_\lambda$ and $\mathrm{p}_\lambda$ will satisfy the commutation relations for the momentum and coordinate operators $[\mathrm{q}_\lambda, \mathrm{p}_\lambda] = i$, while the operator $\mathrm{H}_\lambda = \omega (\mathrm{c}_\lambda^+ \mathrm{c}_\lambda + \frac{1}{2})$ will assume the form of the energy operator for the oscillator

$$\mathrm{H}_\lambda = \tfrac{1}{2}(\mathrm{p}_\lambda^2 + \omega^2 \mathrm{q}_\lambda^2).$$

We now obtain the matrix elements of the operators $\mathrm{c}_\lambda$ and $\mathrm{c}_\lambda^+$. We denote the normalized eigenfunctions of the operator $\mathrm{N}_\lambda = \mathrm{c}_\lambda^+ \mathrm{c}_\lambda$ by $\Phi_{N_\lambda}$. Then from the foregoing it follows that $\mathrm{c}_\lambda \Phi_{N_\lambda} = \mathrm{const} \cdot \Phi_{N_\lambda - 1}$ and $\mathrm{c}_\lambda^+ \Phi_{N_\lambda} = \mathrm{const} \cdot \Phi_{N_\lambda + 1}$. These relations mean that the only nonvanishing matrix elements of the operators $\mathrm{c}_\lambda$ and $\mathrm{c}_\lambda^+$ are $(\mathrm{c}_\lambda)_{N_\lambda, N_\lambda + 1}$ and $(\mathrm{c}_\lambda^+)_{N_\lambda + 1, N_\lambda}$ $(N_\lambda = 0, 1, 2, \ldots)$. Up to a phase factor $\exp(i\alpha)$, which may taken equal to unity, they are equal to

$$(\mathrm{c}_\lambda)_{N_\lambda, N_\lambda + 1} = \sqrt{N_\lambda + 1}, \quad (\mathrm{c}_\lambda^+)_{N_\lambda, N_\lambda - 1} = \sqrt{N_\lambda}, \quad \lambda = 1, 2, 3, \tag{16.18}$$

since

$$N_\lambda = (\mathrm{c}_\lambda^+ \mathrm{c}_\lambda)_{N_\lambda, N_\lambda} = (\mathrm{c}_\lambda^+)_{N_\lambda, N_\lambda - 1} (\mathrm{c}_\lambda)_{N_\lambda - 1, N_\lambda} = |(\mathrm{c}_\lambda^+)_{N_\lambda, N_\lambda - 1}|^2 = |(\mathrm{c}_\lambda)_{N_\lambda - 1, N_\lambda}|^2.$$

On utilizing these relations we obtain

$$\mathrm{c}_\lambda \Phi_{N_\lambda} = \sqrt{N_\lambda}\, \Phi_{N_\lambda - 1}, \quad \mathrm{c}_\lambda^+ \Phi_{N_\lambda} = \sqrt{N_\lambda + 1}\, \Phi_{N_\lambda + 1}. \tag{16.19}$$

The quantization rule (16.17) for $\lambda = 4$ can be rewritten in the form

$$[\mathrm{c}_0^+, \mathrm{c}_0] = 1, \tag{16.20}$$

whence it follows that the eigenvalues $N_4$ of the operator $\mathrm{c}_0 \mathrm{c}_0^+$ are equal to 0, 1, ... The nonvanishing matrix elements of $\mathrm{c}_0$ and $\mathrm{c}_0^+$ are obviously equal to

$$(\mathrm{c}_0^+)_{N_4, N_4 + 1} = \sqrt{N_4 + 1}, \quad (\mathrm{c}_0)_{N_4, N_4 - 1} = \sqrt{N_4}. \tag{16.21}$$

We now show that the quantization rules (16.17) actually lead to the correct corpuscular picture of the field. In order to do this we obtain the operators for the energy H and the momentum **P** of the field, which in accordance with (16.16) can be expressed in terms of the operators $c_\lambda$ and $c_\lambda^+$:

$$\begin{aligned} H &= \tfrac{1}{2}\sum_{k,\lambda} \omega(c_\lambda c_\lambda^+ + c_\lambda^+ c_\lambda), \\ P &= \tfrac{1}{2}\sum_{k,\lambda} \boldsymbol{k}(c_\lambda c_\lambda^+ + c_\lambda^+ c_\lambda), \end{aligned} \tag{16.22}$$

where the summation is taken over $\boldsymbol{k}$, and over all the polarizations $\lambda$ including the longitudinal and the scalar ones.

From these expressions it follows that the states for which the operators $N_\lambda = c_\lambda^+ c_\lambda$ have definite values are states of the field with a definite value of the momentum. This is connected with the fact that the amplitudes $c_\lambda$ and $c_\lambda^+$ are the coefficients in the expansion of the potential in terms of plane waves, i.e., in terms of photon states of definite momentum.

We note that the angular momentum of the field in these states does not have a definite value, since the angular momentum operator does not commute with $\boldsymbol{P}$.

We shall obtain the states of the field having definite values of angular momentum if we expand the field potentials not in terms of plane waves, but in terms of the spherical waves $A_{\omega jM\lambda}$ which describe states of definite angular momentum (cf. formulas (6.16), (6.17)). This expansion can be written in the form

$$A_\mu = \sum_{\omega, j, M, \lambda} c_{\omega jM\lambda}(A_{\omega jM\lambda})_\mu + c_{\omega jM\lambda}^+ (A_{\omega jM\lambda})_\mu^*,$$

where the expansion amplitudes $c_{\omega jM\lambda}$ and $c_{\omega jM\lambda}^+$ regarded as operators satisfy the following commutation relations:

$$\begin{gathered} [c_{\omega jM\lambda}, c_{\omega' j' M' \lambda'}^+] = \delta_{\omega\omega'}\delta_{jj'}\delta_{MM'}\delta_{\lambda\lambda'}, \\ [c_{\omega jM\lambda}, c_{\omega' j' M' \lambda'}] = 0, \qquad [c_{\omega jM\lambda}^+, c_{\omega' j' M' \lambda'}^+] = 0. \end{gathered}$$

The validity of these relations can be verified if we use the expansion of plane waves in terms of spherical waves (cf. subsection 4.4).

The operators for the energy and for the component of the angular momentum of the field are expressed in terms of the amplitudes $c_{\omega jM\lambda}$ and $c_{\omega jM\lambda}^+$ as

$$H = \tfrac{1}{2}\sum_{\omega, j, M, \lambda} \omega(c_{\omega jM\lambda}^+ c_{\omega jM\lambda} + c_{\omega jM\lambda} c_{\omega jM\lambda}^+),$$

$$M_z = \tfrac{1}{2}\sum_{\omega, j, M, \lambda} M(c_{\omega jM\lambda}^+ c_{\omega jM\lambda} + c_{\omega jM\lambda} c_{\omega jM\lambda}^+).$$

From this it can be seen that the states of the field in which the operators $N_{\omega jM\lambda} = c^{+}_{\omega jM\lambda} c_{\omega jM\lambda}$ have definite values (they are equal to 0, 1, ...), are states of definite energy, angular momentum, and component of angular momentum.

The operators for the energy, momentum, and angular momentum of the field contain all the polarizations including the longitudinal and scalar ones ($\lambda = 3, 4$). We have seen earlier that the subsidiary condition (16.2) imposed on the potentials guarantees that in classical electrodynamics all physical quantities are independent of the longitudinal and scalar oscillations. This independence must, of course, also exist in quantum electrodynamics, but the subsidiary condition in its ordinary classical form is incompatible with the conditions of quantization. Indeed, in accordance with (16.15) we have $c_{k3} = -ic_{k4}$, while in the quantization rules the operators $c_{k3}$ and $c_{k4}$ are independent.

The subsidiary condition for the quantized field can be formulated in the form of a condition imposed not on the operators $c_{k3}$ and $c_{k4}$ but on the wave function $\Phi$ (59). In particular, we shall regard as admissible only those wave functions which yield the result zero when the part of the operator $\partial A_\mu/\partial x_\mu$ which contains positive frequencies, i.e., terms of the type $\exp(ikx)$ is applied to them. On denoting this part of the operator $\partial A_\mu/\partial x_\mu$ by $(\partial A_\mu/\partial x_\mu)_+$ we write the subsidiary condition as

$$\left(\frac{\partial}{\partial x_\mu} A_\mu\right)_+ \Phi = 0, \tag{16.23}$$

or

$$\Big(\sum_k \sqrt{\omega}\,(c_{k3}+ic_{k4})\, e^{ikx}\Big)\, \Phi = 0,$$

whence it follows that the function $\Phi$ must satisfy for all $k$ the conditions

$$(c_{k3}+ic_{k4})\Phi = 0, \tag{16.24}$$

and, consequently, also the conditions

$$\Phi^*(c^{+}_{k3}+ic^{+}_{k4}) = 0. \tag{16.24'}$$

In other words, the function $\Phi$, in addition to satisfying condition (16.23), also satisfies the following condition:

$$\Phi^*\left(\frac{\partial}{\partial x_\mu} A_\mu\right)_- = 0, \tag{16.25}$$

where $(\partial A_\mu/\partial x_\mu)$ is that part of the operator $\partial A_\mu/\partial x_\mu$ which contains negative frequencies, i.e., terms of the type $\exp(-ikx)$.

On taking the scalar products of (16.23) on the left with $\Phi^*$, and of (16.25) on the right with $\Phi$, and on adding these two expressions, we obtain

$$\left(\Phi, \frac{\partial}{\partial x_\mu} A_\mu \Phi\right) \equiv \left\langle \frac{\partial}{\partial x_\mu} A_\mu \right\rangle = 0, \tag{16.26}$$

where $\langle L \rangle = (\Phi, L\Phi)$ denotes the expectation value of the operator L in the state $\Phi$.

We see that in quantum electrodynamics the quantity which vanishes is not $\partial A_\mu/\partial x_\mu$, this requirement being impossible to satisfy, but the expectation value of the operator $\partial A_\mu/\partial x_\mu$ in the state $\Phi$. As a result of this, in quantum electrodynamics the equations for the field tensor instead of being of the form (16.1) are

$$\left(\Phi, \frac{\partial}{\partial x_\nu} F_{\mu\nu} \Phi\right) = 0. \tag{16.27}$$

It follows from (16.24) and (16.24′) that

$$\left(\Phi, (c_{k3}^+ - ic_{k4}^+)(c_{k3} + ic_{k4})\Phi\right) + \left(\Phi, (c_{k3}^+ + ic_{k4}^+)(c_{k3} - ic_{k4})\Phi\right) = 0,$$

i.e.,

$$\left(\Phi, (c_{k3}^+ c_{k3} + c_{k4}^+ c_{k4})\Phi\right) = \langle c_{k3}^+ c_{k3} \rangle + \langle c_{k4}^+ c_{k4} \rangle = 0. \tag{16.28}$$

However, it does not follow from this that $N_3 = N_4 = 0$. On the contrary, as we shall see later, there exist no states of the field with $N_3 = N_4 = 0$.

We now determine the eigenvalues of the operator for the energy of the field (16.22). On noting that $c_\lambda c_\lambda^+ = c_\lambda^+ c_\lambda + 1$, and on utilizing (16.28), we find that the eigenvalues of the energy of the field are equal to

$$\mathcal{E} = \sum_{k;\lambda=1,2} \omega(N_{k\lambda} + \tfrac{1}{2}). \tag{16.29}$$

We see that if we disregard the infinite sum $\frac{1}{2} \sum_{k;\lambda=1,2} \omega$, the energy of the electromagnetic field is indeed expressed in the form of the sum of the energies of the individual particle—the photons. The integer $N_{k\lambda} \equiv N_\lambda$ represents the number of photons with propagation vector $\boldsymbol{k}$, polarization $\lambda$ ($\lambda = 1, 2$) and energy $\omega$.

The energy of the field is determined only by the transverse vibrations ($\lambda = 1, 2$). The osciliations with $\lambda = 3, 4$ and the corresponding numbers of longitudinal and scalar photons $N_{k3}$ and $N_{k4}$ do not appear in the expression for the energy.

In a similar manner we can obtain the eigenvalues of the momentum and of the component of the angular momentum of the field, which turn out to be given by $\boldsymbol{P} = \sum\limits_{k,\lambda=1,2} \boldsymbol{k}(N_{k\lambda}+\frac{1}{2})$ and $M_z = \sum\limits_{\omega,j,M,\lambda} M(N_{\omega jM\lambda}+\frac{1}{2})$, where $N_{\omega jM\lambda}$ are the eigenvalues of the operator $\mathrm{c}^+_{\omega jM\lambda}\mathrm{c}_{\omega jM\lambda}$.

We see that if we disregard the sums $\sum\limits_{k,\lambda} \frac{1}{2}\boldsymbol{k}$ and $\sum\limits_{\omega,j,M,\lambda} \frac{1}{2}M$, then the momentum, or the component of the angular momentum of the field, is equal to the sum of the momenta or of the components of the angular momenta of the individual photons.

It follows from (16.29) that the energy of the field has the lowest value when all the numbers $N_{k\lambda}$ ($\lambda = 1,2$) are zero, i.e., when no photons are present. This lowest energy state of the field is named the electromagnetic field *vacuum*. It is a characteristic feature that in accordance with (16.29) the value of the energy of the vacuum is different from zero. To each degree of freedom of the field corresponding to given values of $\boldsymbol{k}$ and $\lambda$ there corresponds in the vacuum state the *zero point energy* equal to $\frac{1}{2}\omega$.

Taken by itself, the presence of an additive constant in the expression for the energy of the field is, of course, not essential, provided we are dealing only with the eigenvalues of the corresponding operators or with their expectation values. However, it would be incorrect to assume on this basis that the zero point energy can be simply neglected. We shall see later that zero point oscillations appear in the investigation of the interaction of the electromagnetic field with electrons and lead to a number of observable effects (these effects will be discussed in Chapter VIII).

In concluding this subsection we note that the operators $\mathrm{c}_\lambda$ and $\mathrm{c}_\lambda^+$ have a simple physical meaning. If the field contains $N_\lambda$ photons of polarization $\lambda$ and $\Phi_{N_\lambda}$ denotes the corresponding wave function, then it follows from (16.19) and (16.21) that

$$\begin{aligned} &\mathrm{c}_\lambda \Phi_{N_\lambda} = \sqrt{N_\lambda}\,\Phi_{N_\lambda - 1}, \qquad \mathrm{c}_\lambda^+ \Phi_{N_\lambda} = \sqrt{N_\lambda + 1}\,\Phi_{N_\lambda + 1}, \quad \lambda = 1, 2, 3, \\ &\mathrm{c}_0 \Phi_{N_4} = \sqrt{N_4 + 1}\,\Phi_{N_4 + 1}, \qquad \mathrm{c}_0^+ \Phi_{N_4} = \sqrt{N_4}\,\Phi_{N_4 - 1}. \end{aligned} \tag{16.19'}$$

Thus, the operators $c_\lambda$ and $c_\lambda^+$ with $\lambda = 1, 2, 3$ represent *annihilation* and *creation* operators for a photon of polarization $\lambda$. For $\lambda = 4$ the roles of $c_0$ and $c_0^+$ are interchanged: $c_0$ represents a creation operator, and $c_0^+$ represents an annihilation operator for a scalar photon.

### 16.5. *Use of the Indefinite Metric*

We have already seen that the vacuum of the electromagnetic field represents a state of the field in which no transverse photons are present, i.e., the numbers $N_\lambda$ ($\lambda = 1, 2$) are equal to zero. However, insofar as the longitudinal and the scalar "photons" are concerned, their numbers $N_3$ and $N_4$ differ from zero. Let us investigate that part of the wave function of the field which describes longitudinal and scalar oscillations. This function, which we denote by $\Phi_{3,4}$, can be represented in the form

$$\Phi_{3,4} = \sum_{N_3, N_4} c(N_3, N_4)\Phi_{N_3, N_4},$$

where the function $\Phi_{N_3, N_4}$ describes states of the field with definite numbers of longitudinal and of scalar "photons", equal respectively to $N_3$ and $N_4$. On substituting this expression for $\Phi_{3,4}$ into (16.24) and on utilizing (16.19′), we obtain

$$\sum_{N_3, N_4} c(N_3, N_4)\{\sqrt{N_3}\Phi_{N_3-1, N_4} - \sqrt{N_4+1}\Phi_{N_3, N_4+1}\} = 0.$$

On making in the first term the substitution $N_3-1 \to N_3$, and in the second term the substitution $N_4+1 \to N_4$, we rewrite this relation in the form

$$\sum_{N_3 \geqslant 0,\, N_4 \geqslant 0} c(N_3+1, N_4)\sqrt{N_3+1}\,\Phi_{N_3, N_4} - \sum_{N_3 \geqslant 0,\, N_4 \geqslant 1} c(N_3, \mathrm{N}_4-1)\sqrt{N_4}\,\Phi_{N_3, N_4} = 0,$$

from which it follows that the quantities $c(N_3, N_4)$ satisfy the difference equation

$$c(N_3+1, N_4)\sqrt{N_3+1} = c(N_3, N_4-1)\sqrt{N_4},$$
$$\text{where} \quad N_3 = 0, 1, \ldots, \quad N_4 = 1, 2, \ldots,$$

and the boundary condition $c(N_3,\ 0) = 0$; $N_3 = 1, 2, \ldots$

On introducing in place of $c(N_3, N_4)$ the new unknown $b(N_3, N_4)$ so that

$$c(N_3, N_4) = \sqrt{(N_4!/N_3!)}\, b(N_3, N_4),$$

we obtain $b(N_3+1, N_4) = b(N_3, N_4-1) = b(N_3-1, N_4-2) = \dots$, whence it follows that $b(N_3, N_4)$ depends only on the difference $N_4 - N_3$. By utilizing also the boundary condition we finally obtain the desired function $c(N_3, N_4)$:

$$c(N_3, N_4) = \begin{cases} 0, & N_4 < N_3, \\ \sqrt{(N_4!/N_3!)}f(N_4-N_3), & N_4 \geqslant N_3, \end{cases} \tag{16.30}$$

where $f(N)$ is an arbitrary function of the non-negative integer $N$.

We see that various states of the field of the longitudinal and the scalar oscillations are possible, but that there exists no state in which the numbers of the longitudinal and the scalar "photons" are equal to zero.

It follows from (16.30) that the wave function cannot be normalized, since $\sum_{N_3, N_4} |c(N_3, N_4)|^2 = \infty$, and consequently, the usual definition of expectation values in the state $\Phi_{3,4}$ has, strictly speaking, no meaning.

The longitudinal and the scalar "photons" do not appear in the expressions for the energy of the field, and have no physical meaning in the case of the free electromagnetic field. Their appearance in the theory leads only to a formal difficulty, which can be eliminated by altering the quantization rule for the scalar potential, and by using a definition of the expectation value of an operator which differs from the customary one. According to the usual definition the expectation value of any operator L in the state $\Psi$ is equal to

$$\langle \mathrm{L} \rangle = (\Psi, \mathrm{L}\Psi) \equiv \int \Psi^* \mathrm{L} \Psi \, dq,$$

where $\Psi^*$ is the function complex-conjugate to $\Psi$ and $dq$ is the product of the differentials of the variables of which $\Psi$ is a function; moreover, it is assumed that $\Psi$ satisfies the normalization condition

$$(\Psi, \Psi) \equiv \int \Psi^* \Psi \, dq = 1.$$

However, it is possible to adopt a different definition of the expectation value of an operator, and of the norm of a wave function (150):

$$\langle \mathrm{L} \rangle = (\Psi, \mathrm{L}\Psi) \equiv \int \Psi^+ \mathrm{L} \Psi \, dq, \qquad (\Psi, \Psi) \equiv \int \Psi^+ \Psi \, dq = \pm 1, \tag{16.31}$$

where $\Psi^+ = \Psi^* \eta$ and $\eta$ is a certain Hermitian operator.

With such a definition of the expectation value the norm of the wave function may be negative. In this case we speak of the indefinite metric in Hilbert space.

Obviously, the expectation value of any operator which corresponds to some physical quantity must be real.

We now show that the expectation value of L will be real if the operator L satisfies the condition

$$\mathrm{L} = \bar{\mathrm{L}}, \tag{16.32}$$

where $\bar{\mathrm{L}} = \eta^{-1}\mathrm{L}^{+}\eta$ and $\mathrm{L}^{+}$ is the operator which is Hermitian conjugate to the operator L. For the sake of simplicity, we examine the case of discrete variables. In this case we have

$$\langle \mathrm{L} \rangle = \sum_{i,k,l} \Psi_i^* \eta_{ik} \mathrm{L}_{kl} \Psi_l \,.$$

Since, by definition, $\eta$ is a Hermitian operator, then the quantity which is complex-conjugate to $\langle \mathrm{L} \rangle$, can be represented in the following form:

$$\langle \mathrm{L} \rangle^* = \sum_{i,k,l} \Psi_l^* (\mathrm{L}^{+})_{lk} \eta_{ki} \Psi_i = \sum_{i,k,l} \Psi_l^* \eta_{lk} (\bar{\mathrm{L}})_{ki} \Psi_i = \langle \bar{\mathrm{L}} \rangle \,.$$

If condition (16.32) is satisfied, then $\langle \mathrm{L} \rangle^* = \langle \mathrm{L} \rangle$, i.e., $\langle \mathrm{L} \rangle$ is real.

We refer to the operator $\bar{\mathrm{L}} = \eta^{-1}\mathrm{L}^{+}\eta$ as the operator conjugate to L, and to an operator satisfying condition (16.32) as a self-conjugate operator.

When the indefinite metric is used in Hilbert space, with the norm of a vector defined not by $\int \Psi^* \Psi dq$, but by $\int \Psi^* \eta \Psi dq$, the self-conjugate operators in the sense of (16.32) play the same role as Hermitian operators do in the usual scheme, which defines the norm as $\int \Psi^* \Psi dq = 1$.

The general definition of an expectation value in the form (16.31) based on the indefinite metric can be utilized for carrying out the quantization of the electromagnetic field. In this way we eliminate the difficulty noted earlier which is associated with the introduction of longitudinal and scalar "photons", and which consists of the impossibility of employing the usual definition of expectation values in the state $\Phi_{3,4}$ which describes longitudinal and scalar oscillations.

The method of quantization utilizing the indefinite metric is developed in the following manner (80):

We start with the expansion of the potential $A_\mu(x)$ into plane waves in the form

$$A_j(x) = \frac{1}{\sqrt{\Omega}} \sum_{k,\lambda} \frac{1}{\sqrt{2\omega}} (c_{k\lambda} e^{ikx} + \bar{c}_{k\lambda} e^{-ikx}) e_j^{(\lambda)}, \quad \text{where} \quad j = 1, 2, 3,$$
$$A_0(x) = \frac{1}{\sqrt{\Omega}} \sum_{k,\lambda} \frac{1}{\sqrt{2\omega}} (c_{k\lambda} e^{ikx} + \bar{c}_{k\lambda} e^{-ikx}) e_0^{(\lambda)} \qquad (16.33)$$

and we assume $c_{k\lambda} \equiv c_\lambda$ and $\bar{c}_{k\lambda} \equiv \bar{c}_\lambda$ to be operators conjugate in the sense of (16.32) and satisfying the following quantization rules:

$$[c_j, \bar{c}_j] = 1, \quad [c_0, \bar{c}_0] = -1. \qquad (16.34)$$

The Hermitian operator $\eta$ which appears in (16.32) is defined by the conditions

$$\bar{c}_j = c_j^+, \quad j = 1, 2, 3,$$
$$\bar{c}_0 = -c_0^+,$$

i.e.,

$$c_j^+ \eta = \eta c_j^+, \quad j = 1, 2, 3,$$
$$c_0^+ \eta = -\eta c_0^+. \qquad (16.35)$$

We note that the expansion (16.33) and the quantization rules (16.34) differ from (16.13) and (16.17) formally only by the replacement of $c^+$ by $\bar{c}$. Therefore, in the future, we shall, for the sake of uniformity, utilize the notation (16.13), keeping in mind that all the formulas will remain valid if we replace $c^+$ by $\bar{c}$.

It follows from (16.34) and (16.35) that the eigenvalues of the operators $N_j = \bar{c}_j c_j = c_j^+ c_j$ $(j = 1, 2, 3)$ are equal to $N_j = 0, 1, 2, \ldots$, while the eigenvalues of the operators $\bar{c}_0 c_0 = -c_0^+ c_0$ are equal to $N_0 = 0, -1, -2, \ldots$; $N_1$ and $N_2$ represent the numbers of transverse photons, while $N_3$ and $N_0$ represent the numbers of longitudinal and of scalar "photons."

As can be easily shown, the operator $\eta$ commutes with $c_j c_j$ and $c_0 c_0$. Therefore, we may assume that the operator $\eta$ is diagonal in the representation in which $\bar{c}_j c_j$ and $\bar{c}_0 c_0$ are diagonal. It follows from (16.35) that the diagonal elements of $\eta$ can be represented in the form

$$(N_1, N_2, N_3, N_0 | \eta | N_1, N_2, N_3, N_0) = \eta_{N_1} \eta_{N_2} \eta_{N_3} \eta_{N_0},$$

where $\eta_{N_\lambda}$ satisfy the conditions

$$\eta_{N_j+1} = \eta_{N_j}, \quad j = 1, 2, 3,$$
$$\eta_{N_0+1} = -\eta_{N_0}.$$

These conditions show that the matrix elements of the operator $\boldsymbol{\eta}$ have the form

$$(N_1, N_2, N_3, N_0 | \boldsymbol{\eta} | N_1', N_2', N_3', N_0') = (-1)^{N_0} \delta_{N_1 N_1'} \delta_{N_2 N_2'} \delta_{N_3 N_3'} \delta_{N_0 N'}. \quad (16.36)$$

We denote by $\Phi_{N_\lambda}$ the eigenfunction which describes the state which contains $N_\lambda$ photons of polarization $\lambda$ ($\lambda = 1, 2, 3, 0$). It follows from (16.31) and (16.36) that these functions must be normalized as follows:

$$\begin{aligned} (\Phi_{N_j}, \Phi_{N_j'}) &= \delta_{N_j, N_j'}, \quad j = 1, 2, 3, \\ (\Phi_{N_0}, \Phi_{N_0'}) &= (-1)^{N_0} \delta_{N_0, N_0'}. \end{aligned} \quad (16.37)$$

By utilizing (16.35) and (16.19′) we can show that the result of operating with the operators $\mathrm{c}_\lambda$ and $\bar{\mathrm{c}}_\lambda$ on $\Phi_{N_j}$ and $\Phi_{N_0}$ is given by the following formulas:

$$\begin{aligned} \mathrm{c}_j \Phi_{N_j} &= \sqrt{N_j} \Phi_{N_j-1}, \quad j = 1, 2, 3, \\ \bar{\mathrm{c}}_j \Phi_{N_j} &= \sqrt{N_j+1} \Phi_{N_j+1}, \\ \mathrm{c}_0 \Phi_{N_0} &= -\sqrt{N_0} \Phi_{N_0-1}, \\ \bar{\mathrm{c}}_0 \Phi_{N_0} &= \sqrt{N_0+1} \Phi_{N_0+1}. \end{aligned} \quad (16.38)$$

We see that if the quantization is carried out in accordance with relations (16.34), then both $\mathrm{c}_j$ (j = 1, 2, 3), and $\mathrm{c}_0$ will be photon annihilation operators, while $\bar{\mathrm{c}}_j$ and $\bar{\mathrm{c}}_0$ will be photon creation operators in contrast to the quantization based on the quantization rules (16.17) in which the operator $\mathrm{c}_0$ was the creation operator for the scalar "photon."

We now determine the form of the allowable wave functions $\Phi$ which describe the states of the electromagnetic field. The subsidiary condition (16.23) retains its form and we now have $(\partial \mathrm{A}_\mu / \partial x_\mu)_+ \Phi = 0$. In place of (16.25) we must write $\Phi^+ (\partial \mathrm{A}_\mu / \partial x_\mu)_- = 0$. It follows from (16.23) that

$$(\mathrm{c}_3 - \mathrm{c}_0) \Phi = 0. \quad (16.39)$$

We seek $\Phi$ in the form

$$\Phi = \sum_{N_1, N_2, N_3, N_0} b(N_1, N_2, N_3, N_0)\Phi_{N_1, N_2, N_3, N_0},$$

where $\Phi_{N_1, N_2, N_3, N_0}$ is the normalized wave function which describes the state of the field containing $N_1$ and $N_2$ transverse, $N_3$ longitudinal and $N_0$ scalar photons, while $b(N_1, N_2, N_3, N_0)$ are numerical coefficients. Conditions (16.38) together with (16.37) lead to the relations

$$\sqrt{N_3}\, b(N_1, N_2, N_3-1, N_0) + \sqrt{N_0}\, b(N_1, N_2, N_3, N_0-1) = 0$$

(the quantities $b$ with negative arguments are equal to zero), whence it follows that these states of the free electromagnetic field are possible:

$$\begin{aligned} \Phi_0 &= \Phi_{N_1, N_2, 0, 0}, \\ \Phi_1 &= \Phi_{N_1, N_2, 1, 0} - \Phi_{N_1, N_2, 0, 1}, \\ \Phi_2 &= \Phi_{N_1, N_2, 2, 0} - \sqrt{2}\Phi_{N_1, N_2, 1, 1} + \Phi_{N_1, N_2, 0, 2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{16.40}$$

or in general $\Phi_n = \frac{1}{\sqrt{n!}}(c_3 - \bar{c}_0)^n \Phi_0$, where $n$ is the total number of longitudinal and scalar "photons."

We see that in contrast to quantization based on the quantization rules (16.17), in the case of quantization based on conditions (16.34) and the indefinite metric, a state of the field is possible in which no longitudinal or scalar "photons" are present (the state $\Phi_0$).

On forming an arbitrary superposition of states (16.40) we obtain the general state of the electromagnetic field $\Phi = \Phi_0 + a_1\Phi_1 + a_2\Phi_2 + \ldots$, where $a_1$, $a_2$, ... are arbitrary numbers. A definite choice of these numbers corresponds to a definite choice of the gauge for the potentials with the condition (16.39) being taken into account.

The function $\Phi$ is normalizable. Indeed, by utilizing (16.37) and (16.40) it can be easily shown that for all $i$ and j, with the exception $i = j = 0$, we have $(\Phi_i, \Phi_j) = 0$. Therefore,

$$(\Phi, \Phi) = (\Phi_0, \Phi_0), \tag{16.40'}$$

independently of the values of $a_1, a_2, \ldots$

Thus, the norm of $\Phi$ contains only that state in which no longitudinal or scalar "photons" are present.

If we neglect the zero point energy, then the operator for the energy of the field in terms of the variables $\bar{c}_\lambda$ and $c_\lambda$ has the form

$$H = \sum_k \omega(\bar{c}_1 c_1 + \bar{c}_2 c_2 + \bar{c}_3 c_3 - \bar{c}_0 c_0).$$

From this, on making use of (16.40′), we can easily conclude that the states $\Phi_1$, $\Phi_2$, ... make no contribution to the expectation value of the energy of the field, which, as should have been expected, is determined only by the transverse vibrations. We can therefore completely leave the states $\Phi_1$, $\Phi_2$, ... out of consideration, and assume that in the free electromagnetic field only transverse photons are present, while the numbers of longitudinal and scalar "photons" are equal to zero:

$$N_3 = N_0 = 0.$$

In this case vacuum may be defined as the state which satisfies the conditions

$$c_\lambda \Phi_{0,0,0,0} = 0, \qquad \lambda = 1, 2, 3, 0. \tag{16.41}$$

This definition is convenient because it is symmetric with respect to the transverse and the longitudinal degrees of freedom of the electromagnetic field, but we must keep in mind that it corresponds to a certain particular choice of the gauge for the potentials.

## § 17. Commutators of the Electromagnetic Field

### 17.1. *Commutation Relations for the Potentials and the Field Components*

If we know the quantization rules for $c_{k\lambda}$ and $c^+_{k\lambda}$, we can determine the commutation relations which must be satisfied by the operators for the potentials and for the field components.

We first of all evaluate the commutator for the potentials $[A_\mu(x), A_\nu(x')]$. For this we use the expansion (16.13):

$$[A_\mu(x), A_\nu(x')] = \frac{1}{2\Omega} \sum_{k,\lambda,k',\lambda'} \frac{1}{\sqrt{\omega\omega'}} \{[c_{k\lambda}, c_{k'\lambda'}] e^{i(kx+k'x')}$$

$$+ [c_{k\lambda}, c^+_{k'\lambda'}] e^{i(kx-k'x')} + [c^+_{k\lambda}, c_{k'\lambda'}] e^{-i(kx-k'x')}$$

$$+ [c^+_{k\lambda}, c^+_{k'\lambda'}] e^{-i(kx+k'x')}\} e^{(\lambda)}_\mu e^{(\lambda')}_\nu .$$

In accordance with (16.17) and (16.14′) this expression may be rewritten in the form

$$[A_\mu(x), A_\nu(x')] = \frac{1}{2\Omega}\sum_k \frac{1}{\omega}\left(e^{ik(x-x')} - e^{-ik(x-x')}\right)\delta_{\mu\nu}.$$

On going over from the summation over $k$ to integration over $k$-space in accordance with the formula

$$\sum_k f(k) = \frac{\Omega}{(2\pi)^3}\int f(k)\,dk,$$

we finally obtain

$$[A_\mu(x), A_\nu(x')] = -iD(x-x')\delta_{\mu\nu}, \tag{17.1}$$

where

$$D(x) = \frac{1}{(2\pi)^3}\int e^{ikr}\frac{\sin\omega t}{\omega}\,dk. \tag{17.2}$$

It may be easily seen that the function $D(x)$ is invariant under Lorentz transformations. Indeed, expression (17.2) for $D(x)$ may be written as

$$D(x) = \frac{i}{(2\pi)^3}\int e^{ikx}\xi(k)\delta(k^2)\,d^4k, \tag{17.2′}$$

where

$$\xi(k) = k_0/|k_0| \quad \text{and} \quad d^4k = dk\,dk_0.$$

But $\xi(k)$ is an invariant, since (17.2), because of the presence of the $\delta$-function, contains only propagation vectors which satisfy the condition $k^2 = 0$, while, as has been noted previously, the sign of the frequency is in this case relativistically invariant.

The invariance of the function $D(x)$ shows that the quantization rules (17.1) are invariant under Lorentz transformations.

We note some properties of the $D$-function.

It may be seen from (17.2) that $D(x)$ is an even function of the coordinates, and an odd function of the time

$$D(x) \equiv D(r, t) = D(-r, t) = -D(r, -t).$$

On carrying out the integration over the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ in (17.2) we obtain

$$D(x) = \frac{1}{8\pi^2 r}\int_0^\infty \{e^{i\omega(r-t)} + e^{-i\omega(r-t)} - e^{i\omega(r+t)} - e^{-i\omega(r+t)}\}\, d\omega$$
$$= \frac{1}{4\pi r}\{\delta(r-t) - \delta(r+t)\},$$

i.e.,

$$D(x) = \frac{1}{2\pi}\,\xi(x)\,\delta(r^2-t^2), \tag{17.3}$$

where $\xi(x) = t/|t|$.

Expression (17.3) shows that the function $D(x)$ vanishes everywhere, except at points lying on the light cone $x^2 = r^2 - t^2 = 0$; at these points $D(x)$ has a $\delta$-like singularity.

From the integral representation (17.2) it follows that $D(x)$ is the singular solution of $\Box D(x) = 0$, which satisfies

$$D(\boldsymbol{r}, 0) = 0, \quad \frac{\partial}{\partial t} D(x)|_{t=0} = \delta(\boldsymbol{r}). \tag{17.4}$$

On differentiating the commutation relations (17.1) with respect to $t'$, and on setting $t' = t$ in the result, we obtain

$$\left[\mathrm{A}_\mu(\boldsymbol{r}, t), \frac{\partial}{\partial t}\mathrm{A}_\nu(\boldsymbol{r}', t)\right] = i\delta_{\mu\nu}\,\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{17.1'}$$

On differentiating (17.1) with respect to $x$ and $x'$ and on utilizing the definition of the field tensor, we can easily obtain the commutation relations for the field components

$$\begin{aligned}
[\mathrm{F}_{\mu\nu}(x), F_{\mu'\nu'}(x')] &= i\left\{\delta_{\nu\nu'}\frac{\partial^2}{\partial x_\mu \partial x'_{\mu'}} - \delta_{\nu\mu'}\frac{\partial^2}{\partial x_\mu \partial x'_{\nu'}}\right. \\
&\qquad \left. - \delta_{\mu\nu'}\frac{\partial^2}{\partial x_\nu \partial x'_{\mu'}} + \delta_{\mu\mu'}\frac{\partial^2}{\partial x_\nu \partial x'_{\nu'}}\right\} D(x-x'), \\
[\mathrm{E}_j(x), \mathrm{E}_k(x')] = [\mathrm{H}_j(x), \mathrm{H}_k(x')] &= i\left\{\frac{\partial^2}{\partial x_j \partial x'_k} - \delta_{jk}\frac{\partial^2}{\partial t\,\partial t'}\right\} D(x-x'), \\
[\mathrm{E}_j(x), \mathrm{H}_k(x')] &= ie_{jkl}\frac{\partial^2}{\partial t\,\partial x'_l} D(x-x'); \quad j, k, l = 1, 2, 3,
\end{aligned} \tag{17.5}$$

where $e_{jkl}$ is the unit antisymmetric tensor of the third rank. We see that the field components may fail to commute with each other only in the case when the corresponding space-time points lie on the light cone $(x-x')^2 = 0$.

As is well known, the commutation relations admit the following physical interpretation. If the operators L and M which correspond to certain physical quantities $L$ and $M$ do not commute, but satisfy the commutation relations $[\mathrm{L}, \mathrm{M}] = G$, $G \neq 0$, then it can be asserted that there exist no states of the system with well defined values of the quantities $L$ and $M$, i.e., with well defined eigenvalues of the operators L and M. The states of the system can be characterized only by ranges of values of $L$ and $M$. The intervals $\Delta L$ and $\Delta M$, within which these values lie must satisfy the uncertainty relation $\Delta L \cdot \Delta M \geqslant |G|$.

Since the commutation relations for fields contain the singular function $D(x)$, we cannot obtain definite physical results directly from them. Such results can be obtained, if we average the fields over certain space-time regions. It can be shown that such averaged values of the field components commute with each other if the time-like or the space-like intervals over which averaging is carried out coincide. In particular, the values of the fields, averaged over the same space-time region, commute. However, if the regions over which averaging is carried out can be connected by light signals even partially, then the averaged fields do not commute with each other (31), (87).

In place of $\mathrm{c}_\lambda$ and $\mathrm{c}_\lambda^+$ we introduce the new variables $\mathrm{N}_\lambda$ and $\varphi_\lambda$:

$$c_\lambda = e^{-i\varphi_\lambda}\sqrt{\mathrm{N}_\lambda}, \qquad \mathrm{c}_\lambda^+ = \sqrt{\mathrm{N}_\lambda}\, e^{i\varphi_\lambda}. \tag{17.6}$$

Here $\mathrm{N}_\lambda = \mathrm{c}_\lambda^+ \mathrm{c}_\lambda$ is the operator for the number of photons; the operator $\varphi_\lambda$ may be interpreted as the phase operator.

We obtain the commutation relations between the operators $\mathrm{N}_\lambda$ and $\varphi_\lambda$. On substituting (17.6) into (16.17) we obtain

$$[\mathrm{N}_\lambda, e^{i\varphi_\lambda}] = e^{i\varphi_\lambda}. \tag{17.6'}$$

This relation will be satisfied if $\mathrm{N}_\lambda$ and $\varphi_\lambda$ satisfy the condition

$$[\mathrm{N}_\lambda, \varphi_\lambda] = -i. \tag{17.7}$$

Indeed, the following relation can be obtained from (17.7):

$$[\mathrm{N}_\lambda, \varphi_\lambda^n] = -in\varphi_\lambda^{n-1},$$

which immediately yields (17.6′) if we note that

$$e^{i\varphi} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\varphi)^n.$$

The commutation relations (17.7) yield $\Delta \mathrm{N}_\lambda \Delta \varphi_\lambda \geqslant 1$. The physical meaning of this inequality consists of the fact that the electromagnetic field cannot be simultaneously characterized by definite numbers of photons of different kinds, and also by definite phase relations.

If the difference in phase between two waves has a well defined value, then only the total number of photons associated with these two waves can be specified. But the manner in which the photons are distributed between the two waves remains completely undetermined.

The classical electromagnetic field is characterized by large expectation values of the numbers of photons. In this case, even for very small values of $\Delta\varphi_\lambda$, the expectation value of the number of photons $\langle N_\lambda \rangle$ is considerably greater than $\Delta N_\lambda$, i.e., the states of the field are practically characterized both by definite phase relations and by definite intensities (photon numbers).

## 17.2. *Chronological and Normal Products of Components of the Potential*

An important role will be played by products of the operators for the potentials arranged in a definite order, in the future investigation of the interaction between the electromagnetic field and electrons. These ordered products bear the names of *chronological* (*time-ordered*) and *normal products*.

In the chronological product of the operators $\mathrm{A}_\mu(x)$ and $\mathrm{A}_\nu(x')$, which we denote by $T(\mathrm{A}_\mu(x)\,\mathrm{A}_\nu(x'))$, the factors are arranged in chronological order, viz.: the operator corresponding to the smaller value of time is placed on the right and the operator corresponding to the greater value of time is placed on the left. In other words,

$$T(\mathrm{A}_\mu(x)\,\mathrm{A}_\nu(x')) = \begin{cases} \mathrm{A}_\mu(x)\,\mathrm{A}_\nu(x'), & t > t', \\ \mathrm{A}_\nu(x')\,\mathrm{A}_\mu(x), & t < t'. \end{cases} \tag{17.8}$$

We note that this definition is relativistically invariant. Indeed, if $x$ and $x'$ are separated by a time-like interval, then this assertion is obvious, since in this case the concepts of a later or of an earlier time have absolute significance. However, if $x$ and $x'$ are separated by a space-like interval and $t-t'$ can change sign under Lorentz transformations, then the operators $A_\mu(x)$ and $A_\nu(x')$ commute, and, therefore, can be arranged in all coordinate systems in the order which is required for the chronological product.

By utilizing the expansion for the potentials (16.13) into plane waves we can represent the chronological product $T\big(A_\mu(x)A_\nu(x')\big)$ in the following form:

$$T\big(A_\mu(x)A_\nu(x')\big) = \frac{1}{2\Omega}\sum_{k,\lambda,k',\lambda'}\frac{1}{\sqrt{\omega\omega'}}\{c_{k\lambda}c_{k'\lambda'}e^{i(kx+k'x')}+c_{k\lambda}c^{+}_{k'\lambda'}e^{i(kx-k'x')}$$

$$+c^{+}_{k\lambda}c_{k'\lambda'}e^{-i(kx-k'x')}+c^{+}_{k\lambda}c^{+}_{k'\lambda'}e^{-i(kx+k'x')}\}\,e^{(\lambda)}_{\mu}e^{(\lambda')}_{\nu},\qquad t>t'. \tag{17.8'}$$

On interchanging $x$ and $x'$, we obtain the chronological product for $t<t'$.

We now define the normal product of the operators $A_\mu(x)$ and $A_\nu(x')$. In this product, which we denote by $N\big(A_\mu(x)A_\nu(x')\big)$, the photon creation operators $c^{+}_{k\lambda}$ are placed to the left of the annihilation operators $c_{k\lambda}$. If we represent the operator $A_\mu(x)$ as a sum of two terms with positive $\big(A^{(+)}_\mu(x)\big)$ and negative $\big(A^{(-)}_\mu(x)\big)$ frequencies, then we have

$$\begin{aligned}N\big(A_\mu(x)A_\nu(x')\big) &= A^{(+)}_\mu(x)A^{(+)}_\nu(x')+A^{(-)}_\mu(x)A^{(-)}_\nu(x')\\ &\quad + A^{(-)}_\mu(x)A^{(+)}_\nu(x')+A^{(-)}_\nu(x')A^{(+)}_\mu(x)\\ &= \frac{1}{2\Omega}\sum_{k,\lambda,k',\lambda'}\frac{1}{\sqrt{\omega\omega'}}\{c_{k\lambda}c_{k'\lambda'}e^{i(kx+k'x')}+c^{+}_{k'\lambda'}c_{k\lambda}e^{i(kx-k'x')}\\ &\quad +c^{+}_{k\lambda}c_{k'\lambda'}e^{i(k'x'-kx)}+c^{+}_{k\lambda}c^{+}_{k'\lambda'}e^{-i(kx+k'x')}\}\,e^{(\lambda)}_{\mu}e^{(\lambda')}_{\nu}.\end{aligned} \tag{17.9}$$

A remarkable relation exists between the chronological and the normal products—their difference contains no photon creation or annihilation operators, and is an ordinary function ($c$-number). Indeed, on utilizing the expansion of the potentials into plane waves and the quantization rules (16.17) we obtain for this difference the following expression:

$$T\big(A_\mu(x)\,A_\nu(x')\big)-N\big(A_\mu(x)\,A_\nu(x')\big) = \frac{1}{2\Omega}\sum_{k}\frac{1}{\omega}\,e^{ik(r-r')-i\omega|t-t'|}\,\delta_{\mu\nu},$$

which is valid both for $t>t'$, and for $t<t'$.

We call the difference between the chronological and the normal products of two operators the *pairing* or the *contraction* of the two operators.

By making the transition in the preceding expression for the pairing from the sum to an integral over $k$-space we obtain

$$T\big(\mathrm{A}_\mu(x)\mathrm{A}_\nu(x')\big) - N\big(\mathrm{A}_\mu(x)\mathrm{A}_\nu(x')\big) \equiv \mathrm{A}_\mu^a(x)\mathrm{A}_\nu^a(x') = D^c(x-x')\delta_{\mu\nu}, \tag{17.10}$$

where the pairing of the operators $\mathrm{A}_\mu(x)$ and $\mathrm{A}_\nu(x)$ is denoted by $\mathrm{A}_\mu^a(x)\mathrm{A}_\nu^a(x)$, and the function $D^c(x)$ is

$$D^c(x) = \frac{1}{2(2\pi)^3}\int e^{ikr-i\omega|t|}\frac{dk}{\omega}. \tag{17.11}$$

We now show that the pairing of the operators $\mathrm{A}_\mu(x)$ and $\mathrm{A}_\nu(x)$ defines the expectation value of the chronological product of these operators in the vacuum state.

In order to obtain the vacuum expectation value of $T(\mathrm{A}_\mu(x)\mathrm{A}_\nu(x')$ we must know in accordance with (17.8′) the vacuum expectation values of products of the operators $\mathrm{c}_{k\lambda}$ and $\mathrm{c}_{k\lambda}^+$. Since in the vacuum state the numbers of transverse photons are equal to zero, we have

$$\langle \mathrm{c}_{k\lambda}^+\mathrm{c}_{k\lambda}\rangle_0 = 0, \quad \langle \mathrm{c}_{k\lambda}\mathrm{c}_{k\lambda}^+\rangle_0 = 1, \quad \lambda = 1, 2,$$

where $\langle \mathrm{L}\rangle_0$ denotes the vacuum expectation value of the operator L. It is obvious that for these polarizations the following relations also hold:

$$\begin{aligned} &\langle \mathrm{c}_{k\lambda}^+\mathrm{c}_{k'\lambda'}\rangle_0 = 0, \quad \langle \mathrm{c}_{k\lambda}\mathrm{c}_{k'\lambda'}\rangle_0 = 0, \\ &\langle \mathrm{c}_{k\lambda}^+\mathrm{c}_{k'\lambda'}^+\rangle_0 = 0, \quad \langle \mathrm{c}_{k\lambda}\mathrm{c}_{k'\lambda'}^+\rangle_0 = \delta_{kk'}\delta_{\lambda\lambda'}; \quad \lambda, \lambda' = 1, 2. \end{aligned} \tag{17.12}$$

As regards the longitudinal and the scalar polarizations, if that gauge is chosen for the potentials which corresponds to the definition of the vacuum (16.41), the numbers of longitudinal and scalar photons in the vacuum state will also be equal to zero. However, here we shall not utilize this gauge for the potentials and, in contrast to subsection 16.5, we shall not even specialize the definition for the expectation value in the state $\Phi_{3,4}$, but shall assume that the vacuum expectation value of the operator $\mathrm{c}_{k3}^+\,\mathrm{c}_{k'3}$ has been defined in some way:

$$\langle \mathrm{c}_{k3}^+\mathrm{c}_{k'3}\rangle_0 = (ke^{(3)})^2\chi(k)\delta_{kk'}, \tag{17.12′}$$

where $\chi(k)$ is, generally speaking, some arbitrary function of $k$. Moreover, we shall assume that for all $\lambda$ the following relations will hold:

$$\langle \mathrm{c}_{k\lambda}\mathrm{c}_{k'\lambda'}\rangle_0 = \langle \mathrm{c}_{k\lambda}^+\mathrm{c}_{k'\lambda'}^+\rangle_0 = 0.$$

From relation (16.28), which we can use since it is based only on the subsidiary condition (16.23), and also from formula (17.12′) it follows that

$$\langle c_{k4}^{+} c_{k4} \rangle_0 = -\langle c_{k3}^{+} c_{k3} \rangle_0 = (ke^{(4)})^2 \chi(k).$$

Further, it follows from (16.24), (16.24′), and (16.14) that

$$\langle c_{k3}^{+} c_{k4} \rangle_0 = \langle c_{k4}^{+} c_{k3} \rangle_0 = i \langle c_{k3}^{+} c_{k3} \rangle_0 = i(ke^{(3)})^2 \chi(k) = (ke^{(3)})(ke^{(4)}) \chi(k).$$

All these expressions, and also expression (17.12), valid for $\lambda = 1, 2$, can be written in the form

$$\begin{aligned} &\langle c_{k\lambda}^{+} c_{k'\lambda'} \rangle_0 = (ke^{(\lambda)})(k'e^{(\lambda')}) \chi(k) \delta_{kk'}, \qquad \langle c_{k\lambda} c_{k'\lambda'} \rangle_0 = 0, \\ &\langle c_{k\lambda} c_{k'\lambda'}^{+} \rangle_0 = \{ \delta_{\lambda\lambda'} + (ke^{(\lambda)})(k'e^{(\lambda')}) \chi(k) \} \delta_{kk'}, \qquad \langle c_{k\lambda}^{+} c_{k'\lambda'}^{+} \rangle_0 = 0, \qquad (17.13) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda; \lambda' = 1, 2, 3, 4. \end{aligned}$$

On utilizing these formulas we obtain, in accordance with (17.8′),

$$\begin{aligned} \langle T(A_\mu(x) A_\nu(x')) \rangle_0 = {} & \frac{1}{2\Omega} \sum_{k,\lambda,\lambda'} \frac{1}{\omega} e^{ik(r-r')-i\omega|t-t'|} \delta_{\lambda\lambda'} e_\mu^{(\lambda)} e_\nu^{(\lambda')} \\ & + \frac{1}{\Omega} \sum_{k,\lambda,\lambda'} \frac{1}{\omega} (ke^{(\lambda)})(ke^{(\lambda')}) \cos k(x-x') e_\mu^{(\lambda)} e_\nu^{(\lambda')} \chi(k), \end{aligned}$$

or, since $e_\mu^{(\lambda)} e_\nu^{(\lambda)} = \delta_{\mu\nu}$, we have

$$\begin{aligned} \langle T(A_\mu(x) A_\nu(x')) \rangle_0 = {} & \delta_{\mu\nu} \frac{1}{2\Omega} \sum_k \frac{1}{\omega} e^{ik(r-r')-i\omega|t-t'|} \\ & + \frac{1}{\Omega} \sum_k k_\mu k_\nu \frac{1}{\omega} \cos k(x-x') \chi(k). \end{aligned}$$

Here the first sum is equal to $D^c(x-x')$, while the second sum can evidently be written in the form $(\partial^2/\partial x_\mu \partial x_\nu)\varphi(x-x')$, where

$$\varphi(x) = -\frac{1}{\Omega} \sum_k \frac{1}{\omega} \cos kx \chi(k).$$

Therefore, we finally obtain the following expression for $\langle T(A_\mu(x) A_\nu(x')) \rangle_0$, which is valid both for $t > t'$, and also for $t < t'$:

$$D_{\mu\nu}^c(x-x') \equiv \langle T(A_\mu(x) A_\nu(x')) \rangle_0 = D^c(x-x') \delta_{\mu\nu} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \varphi(x-x'). \tag{17.14}$$

The second term containing the arbitrary function $\varphi(x)$ is due to the arbitrary gauge for the potentials. If we use the gauge in which $N_3 = N_4 = 0$ then this term will vanish. It is clear that no physical quantity can depend on the arbitrary function $\varphi(x)$. This is the meaning of the property of gauge invariance.

On repeating the calculations just given it can be easily shown that the vacuum expectation value of the normal products of the operators $\mathrm{A}_\mu(x)$ and $\mathrm{A}_\nu(x')$ is equal to

$$\langle N(\mathrm{A}_\mu(x)\mathrm{A}_\nu(x'))\rangle_0 = \frac{\partial^2}{\partial x_\mu \partial x_\nu}\varphi(x-x'). \tag{17.15}$$

We expand the function $D^c(x)$ into the four-dimensional Fourier integral:

$$D^c(x) = \frac{1}{(2\pi)^4}\int D^c(k)e^{ikx}d^4k, \tag{17.16}$$

in which all four components of $k$ are independent and $d^4k = d\mathbf{k}\,dk_0$. In order to do this we use the following integral representation of the function $(1/\omega)\exp(-i\omega|t|)$:

$$\frac{1}{\omega}e^{-i\omega|t|} = \frac{1}{\pi i}\int\limits_C \frac{e^{-ik_0t}}{k^2}dk_0, \tag{17.17}$$

where $k^2 = \mathbf{k}^2 - k_0^2 = \omega^2 - k_0^2$, and the integration is carried out in the complex plane of $k_0$ along the contour $C$ shown in Fig. 8. In order to show the validity of formula (17.17) we note that for $t > 0$ the contour of integration can be completed by the semicircle of infinite radius situated in the lower half-plane. Then the integral will be determined by its residue at the pole $k_0 = \omega$ and will be equal to the left hand side of (17.17). However, if $t < 0$, then the contour of integration must be completed by the semicircle in the upper half-plane. In this case the integral will be determined by the residue at the point $k_0 = -\omega$ and will again be equal to the left hand side of formula (17.17).

Instead of integrating in (17.17) along the contour $C$ which passes above the pole $k_0 = \omega$ and below the pole $k_0 = -\omega$, we can integrate along the real axis of $k_0$, by displacing the former pole into the lower, and the latter pole into the upper half-plane (Fig. 9). For this we must replace the denominator $k^2$ of the integrand by $k^2 - i\eta$, where $\eta$ is an infinitesimal positive number, since in this case the poles will be given by the points $k_0 = \pm\omega\big(1-(i\eta/2\omega^2)\big)$ situated as shown in Fig. 9.

On substituting formula (17.17) into (17.11) and on comparing it with (17.16) we obtain the Fourier component $D^c(k)$:

$$D^c(k) = \frac{1}{i(k^2-i0)}, \tag{17.18}$$

where $i0$ denotes $i|\eta|$ for $|\eta| \to 0$. This expression can also be written in the form

$$D^c(k) = \pi\delta_+(-k^2), \tag{17.18'}$$

where the singular function $\delta_+(x)$ is defined by the integral

$$\delta_+(x) = \frac{1}{\pi}\int\limits_0^\infty e^{i\xi(+i|\eta|)}\,d\xi = \frac{i}{\pi}\frac{1}{x+i0}, \qquad |\eta| \to 0. \tag{17.18}$$

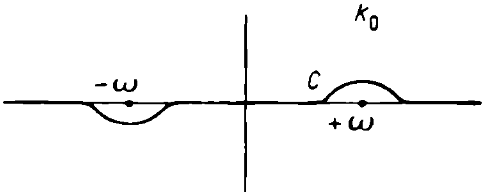

Fig. 8.

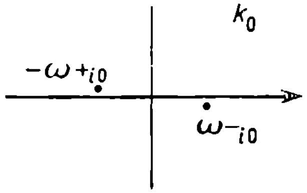

Fig. 9.

We note that $\delta_+(x) = \delta(x)+(i/\pi)P(1/x)$. This equation, which follows from (17.19), means that integrals containing the $\delta_+$-function are to be evaluated according to the following rule:

$$\int\limits_{-\infty}^{+\infty} f(x)\,\delta_+(x)\,dx = \int\limits_C f(x)\frac{i}{\pi x}\,dx = f(0) + \frac{i}{\pi}P\int\limits_{-\infty}^{+\infty} f(x)\frac{dx}{x}, \tag{17.19'}$$

where the contour of integration $C$ coincides with the real axis, but passes above the pole $x = 0$, while $P$ denotes the principal value of the integral.

On substituting (17.18′) into (17.16) we obtain

$$D^c(x) = \frac{1}{(2\pi)^4}\int d^4k\,e^{ikx}\int\limits_0^\infty e^{-i\xi(k^2-i|\eta|)}\,d\xi$$

$$= \frac{1}{(2\pi)^4}\int\limits_0^\infty d\xi\,e^{i\frac{x^2+i|\eta|}{4\xi}}\int e^{-ik'^2\xi}\,d^4k',$$

where $k' = k-(x/2\xi)$. On noting that

$$\int_{-\infty}^{+\infty} e^{\pm i\xi a^2}\,da = \sqrt{\frac{\pi}{\xi}}\,e^{\pm i\frac{\pi}{4}}$$

and that, therefore,

$$\int e^{-ik'^2\xi}\,d^4k' = (\pi^2/i\xi^2),$$

we obtain

$$D^c(x) = \frac{1}{(4\pi)^2 i}\int_0^\infty e^{i\cdot\frac{x^2+i|\eta|}{4\xi}}\,\frac{d\xi}{\xi^2} = \frac{1}{4\pi^2 i}\int_0^\infty e^{iz(x^2+i|\eta|)}\,dz,$$

i.e.,

$$D^c(x) = \frac{1}{4\pi i}\,\delta_+(x^2). \tag{17.20}$$

If we know $D^c(k)$, we can easily obtain the Fourier component $D^c_{\mu\nu}(k)$ of the function $D^c_{\mu\nu}(x)$ defined by formula (17.14). Since differentiation of any function with respect to $x_\mu$ corresponds to the multiplication of its Fourier component by $ik_\mu$, we can write $D^c_{\mu\nu}(k)$ in the form (115)

$$D^c_{\mu\nu}(k) = \frac{1}{i(k^2-i0)}\left\{\delta_{\mu\nu}+(d_l(k^2)-1)\frac{k_\mu k_\nu}{k^2}\right\}, \tag{17.21}$$

where $d_l(k^2)$ is an arbitrary function of the invariant $k^2$ corresponding to the arbitrary function $\varphi(x)$. On setting $d_l(k^2) = 1$ in this expression we obtain $D^c_{\mu\nu}(k) = D^c(k)\delta_{\mu\nu}$. When $d_l(k^2) = 0$ the function $D^c_{\mu\nu}(k)$ will have the property of "transversality":

$$D^c_{\mu\nu}(k)k_\mu = 0, \quad d_l(k^2) = 0,$$

or in the coordinate representation

$$\frac{\partial}{\partial x_\mu}D^c_{\mu\nu}(x) = 0.$$

It has been pointed out previously that physical quantities do not depend on the arbitrary function $\varphi(x)$. Therefore we can say that the property of gauge invariance of physical quantities means that they are independent of the quantity $d_l(k^2)$.

We note, that since $k^2 D^c(k) = -i$, the function $D^c(x)$ satisfies the equation

$$\Box D^c(x) = i\delta(x). \tag{17.22}$$

Thus, $D^c(x)$ is the Green's function of the wave equation.

### 17.3. *Singular Functions Associated with the Operators $\Box$ and $(\Box - m^2)$*

We have discussed the two singular functions $D(x)$ and $D^c(x)$ associated with the electromagnetic field. The function $D^c(x)$ can be expressed in the form of a four-dimensional integral

$$D^c(x) = \frac{1}{(2\pi)^4 i} \int\limits_C \frac{e^{ikx}}{k^2} d^4k, \tag{17.23}$$

where the integration over $k_0$ is carried out over the contour $C_0$ shown in Fig. 10.

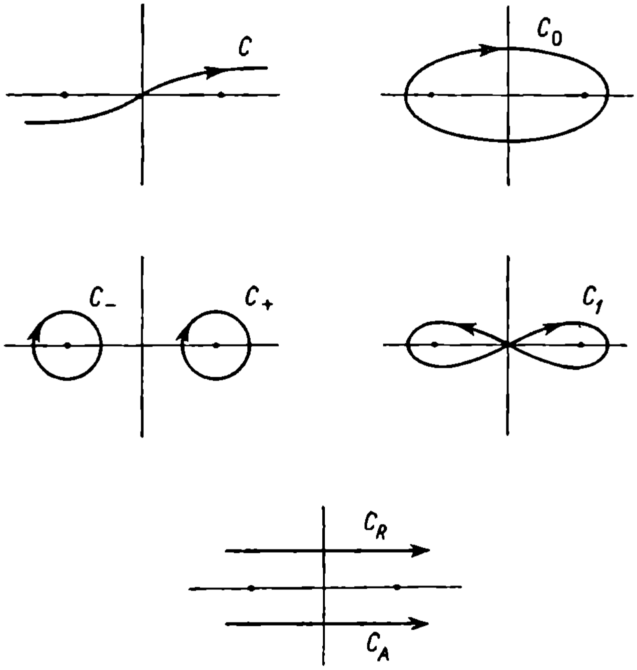

Fig. 10.

The function $-iD(x)$ can also be represented in the form of a four-dimensional integral with the same integrand as in $D^c(x)$, viz.,

$$D(x) = \frac{1}{(2\pi)^4} \int\limits_{C_0} \frac{e^{ikx}}{k^2} d^4k, \tag{17.24}$$

where the integration over $k_0$ is carried out along the contour $C_0$ shown in Fig. 10. Indeed, in carrying out the integration over $k_0$ with the aid of the residue theorem we obtain

$$D(x) = -\frac{i}{(2\pi)^3}\int dk\left\{\frac{e^{ikr+i\omega t}}{2\omega}-\frac{e^{ikr-i\omega t}}{2\omega}\right\} = \frac{1}{(2\pi)^3}\int e^{ikr}\frac{\sin\omega t}{\omega}dk.$$

We obtain other singular solutions of the wave equation associated with the electromagnetic field if we investigate the integral

$$F(x) = \frac{1}{(2\pi)^4}\int\limits_{\mathfrak{L}} e^{ikx}\frac{d^4k}{k^2},$$

in which the integration with respect to $k_0$ is carried out over the contour $\mathfrak{L}$ which coincides with one of the contours $C_+$, $C_-$, $C_1$, $C_R$, $C_A$, shown in Fig. 10. Moreover, if $\mathfrak{L}$ surrounds one or both poles of the integrand, then $F(x)$ will be a solution of the homogeneous wave equation, while if the contour extends to infinity we shall obtain a Green's function of the wave equation, i.e., a solution of the inhomogeneous wave equation with a $\delta$-function on the right-hand side.

We first of all discuss the singular functions $D_\pm(x)$ for which $\mathfrak{L}$ has the form of the contours $C_+$ or $C_-$:

$$D_\pm(x) = \frac{1}{(2\pi)^4}\int\limits_{C_\pm}\frac{e^{ikx}}{k^2}d^4k. \tag{17.25}$$

Integration with respect to $k_0$ yields

$$\begin{aligned} D_+(x) &= -\frac{1}{2i(2\pi)^3}\int e^{ikx}\frac{dk}{\omega},\\ D_-(x) &= \frac{1}{2i(2\pi)^3}\int e^{-ikx}\frac{dk}{\omega}. \end{aligned} \tag{17.25'}$$

From this it follows that with $k_0 = \omega|k|$,

$$D_-(x) = -D_+(-x). \tag{17.25''}$$

It is clear that

$$D(x) = D_+(x)+D_-(x). \tag{17.26}$$

We now show that the functions $D_\pm(x)$ define the expectation values of products of the components of the potential in the vacuum state. By making use of the expansion of $A_\mu(x)$ into plane waves, and relations (17.13) we obtain

$$\langle A_\mu(x) A_\nu(x')\rangle_0 = \frac{1}{2\Omega} \sum_{k,\lambda,\lambda'} \frac{1}{\omega} e^{ik(x-x')} e_\mu^{(\lambda)} e_\nu^{(\lambda')} \delta_{\lambda\lambda'}$$

$$+\frac{1}{2\Omega} \sum_{k,\lambda,\lambda'} \frac{1}{\omega} \chi(k) \, (ke^{(\lambda)}) \, (ke^{(\lambda')}) \left\{e^{ik(x-x')}+e^{-ik(x-x')}\right\} e_\mu^{(\lambda)} e_\nu^{(\lambda')}.$$

On noting that $e_\mu^{(\lambda)} e_\nu^{(\lambda)} = \delta_{\mu\nu}$, and on going over from summation to integration over $k$, we obtain

$$\langle A_\mu(x) A_\nu(x')\rangle_0 = -iD_+(x-x')\delta_{\mu\nu}+\frac{\partial^2}{\partial x_\mu \partial x_\nu}\varphi(x-x'),$$

$$\langle A_\nu(x') A_\mu(x)\rangle_0 = iD_-(x-x')\delta_{\mu\nu}+\frac{\partial^2}{\partial x_\mu \partial x_\nu}\varphi(x-x').$$

From this it follows that the vacuum expectation value of the anti-commutator $\{A_\mu(x), A_\nu(x')\} \equiv A_\mu(x) A_\nu(x')+A_\nu(x') A_\mu(x)$ is equal to

$$\langle\{A_\mu(x), A_\nu(x')\}\rangle_0 = D_1(x-x')\delta_{\mu\nu}+2\frac{\partial^2}{\partial x_\mu \partial x_\nu}\varphi(x-x'), \quad (17.27)$$

where

$$D_1(x) = -i\{D_+(x)-D_-(x)\}. \quad (17.28)$$

Evidently, the function $D_1(x)$ can be represented in the form of the integral

$$D_1(x) = \frac{1}{(2\pi)^4 i} \int_{C_1} \frac{e^{ikx}}{k^2} d^4k, \quad (17.28')$$

where the contour $C_1$ is shown in Fig. 10. Integration over $k_0$ with the aid of the residue theorem yields

$$D_1(x) = \frac{1}{(2\pi)^3} \int dk \left\{\frac{1}{2\omega} e^{ikr+i\omega t}+\frac{1}{2\omega} e^{ikr-i\omega t}\right\} = \frac{1}{(2\pi)^3} \int e^{ikr} \frac{\cos \omega t}{\omega} dk. \quad (17.29)$$

This expression can be rewritten in the form

$$D_1(x) = \frac{1}{(2\pi)^3} \int e^{ikx} \delta(k^2) d^4k. \quad (17.29')$$

On carrying out in (17.29′) the integration over the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$ we obtain

$$D_1(x) = \frac{1}{(2\pi)^3}\frac{\pi}{ir}\int\limits_0^\infty \{e^{i\omega(r+t)} - e^{-i\omega(r+t)} + e^{i\omega(r-t)} - e^{-i\omega(r-t)}\}\,d\omega$$
$$= \frac{2}{(2\pi)^2}\frac{1}{r^2-t^2},$$

i.e.,

$$D_1(x) = \frac{2}{(2\pi)^2}\frac{1}{x^2}. \tag{17.30}$$

We see that the function $D_1(x)$, like the function $D(x)$, is a singular solution of the wave equation, but in contrast to the latter vanishes for no finite values of $x^2$.

On comparing the integral representations (17.23), (17.24) and (17.28′) for the functions $D^c(x)$, $D(x)$, and $D_1(x)$ we can express $D^c(x)$ in terms of $D_1(x)$ and $D(x)$:

$$D^c(x) = \tfrac{1}{2}\{D_1(x) - i\xi(x)D(x)\}, \tag{17.31}$$

where

$$\xi(x) = \begin{cases} 1, & t > 0, \\ -1, & t < 0. \end{cases}$$

We also give the expression for $D^c(x)$ in terms of the functions $D_\pm(x)$:

$$D^c(x) = \frac{1}{i}\{\theta(x)D_+(x) - \theta(-x)D_-(x)\}, \tag{17.31′}$$

where $\theta(x) = \frac{1}{2}(1+\xi(x))$.

We now consider the function

$$D_R(x) = \frac{1}{(2\pi)^4}\int\limits_{C_R} \frac{e^{ikx}}{k^2}\,d^4k, \tag{17.32}$$

where the integration over $k_0$ is carried out along the contour $C_R$ (Fig. 10). When $t > 0$, the contour of integration can be completed by a semicircle in the lower half-plane, and we then obtain the function $D(x)$. However, if $t < 0$, then the contour of integration can be completed in the upper half-plane, where the integrand has no poles, and we obtain zero as the result. Therefore,

$$D_R(x) = \begin{cases} D(x), & t > 0, \\ 0, & t < 0. \end{cases} \tag{17.32'}$$

On applying the d'Alembertain operator to (17.32) and on making $C_R$ coincide with the $k_0$-axis we obtain

$$\Box D_R(x) = -\frac{1}{(2\pi)^4}\int e^{ikx}d^4k = -\delta(x). \tag{17.33}$$

Similarly, it may be easily shown that the function

$$D_A(x) = \frac{1}{(2\pi)^4}\int_{C_A} \frac{e^{ikx}}{k^2}d^4k, \tag{17.34}$$

where the integration with respect to $k_0$ is carried out along the contour $C_A$ (Fig. 10), reduces to

$$D_A(x) = \begin{cases} 0, & t > 0, \\ -D(x), & t < 0. \end{cases} \tag{17.34'}$$

and satisfies the equation

$$\Box D_A(x) = -\delta(x). \tag{17.33'}$$

It follows from (17.32') and (17.33) that the solution of the inhomogeneous wave equation $\Box A(x) = -j(x)$, which vanishes for $t = -\infty$, can be written in the form

$$A(x) = \int D_R(x-x')j(x')d^4x'; \tag{17.35}$$

i.e., the function $D_R(x)$ is the Green's function of the wave equation which leads to retarded solutions. Similarly, we can easily show that $D_A(x)$ leads to advanced solutions.

In the future study of the electron-positron field we shall need a number of singular functions which bear the same relation to the operator $\Box - m^2$ that the functions which we have just discussed bear to the d'Alembertian operator. These functions are defined by the four-dimensional Fourier integrals

$$\begin{aligned} \Delta(x) &= \frac{1}{(2\pi)^4}\int_{C_0}\frac{e^{ipx}}{p^2+m^2}d^4p, & \Delta_1(x) &= \frac{1}{(2\pi)^4 i}\int_{C_1}\frac{e^{ipx}}{p^2+m^2}d^4p, \\ \Delta^c(x) &= \frac{1}{(2\pi)^4 i}\int_{C}\frac{e^{ipx}}{p^2+m^2}d^4p, & \Delta_\pm(x) &= \frac{1}{(2\pi)^4}\int_{C_\pm}\frac{e^{ipx}}{p^2+m^2}d^4p, \end{aligned} \tag{17.36}$$

where the integration with respect to $p_0$ is carried out along the contours $C_0, C_1, C$, shown in Fig. 10. On applying to these expansions the operator $\Box - m^2$, and on making the contours coincide with the $p_0$-axis, we find that the functions $\Delta(x), \Delta_1(x), \Delta^c(x)$ satisfy the following equations:

$$(\Box - m^2)\Delta(x) = 0,$$

$$(\Box - m^2)\Delta_1(x) = 0, \tag{17.36'}$$

$$(\Box - m^2)\Delta^c(x) = i\delta(x).$$

On carrying out in (17.36) the integration with respect to $p_0$ with the aid of the residue theorem we obtain representations of the functions $\Delta(x), \Delta_1(x), \Delta^c(x)$ in the form of three-dimensional Fourier integrals. For example,

$$\Delta(x) = -\frac{i}{(2\pi)^3}\int dp \left\{ \frac{1}{2\varepsilon} e^{ipr+i\varepsilon t} - \frac{1}{2\varepsilon} e^{ipr-i\varepsilon t} \right\} = \frac{1}{(2\pi)^3}\int e^{ipr} \frac{\sin \varepsilon t}{\varepsilon} dp, \tag{17.37}$$

where $\varepsilon = \sqrt{p^2+m^2}$.

In a similar manner we obtain

$$\Delta_1(x) = \frac{1}{(2\pi)^3}\int e^{ipr} \frac{\cos \varepsilon t}{\varepsilon} dp, \tag{17.38}$$

$$\Delta^c(x) = \frac{1}{2(2\pi)^3}\int e^{ipr - i\varepsilon |t|} \frac{dp}{\varepsilon}. \tag{17.39}$$

These formulas differ from the corresponding formulas for the functions $D(x), D_1(x), D^c(x)$ by the fact that $\varepsilon$ appears in them in place of $\omega$.

On comparing the expansions just obtained we can easily show that

$$\Delta^c(x) = \frac{1}{2}\{\Delta_1(x) - i\xi(x)\Delta(x)\}. \tag{17.39'}$$

It is evident that all these functions are invariant under Lorentz transformations, and therefore depend only on $x^2$ for $x^2 > 0$ (for $x^2 < 0$ these functions depend on two quantities: $x^2$ and $\xi(x)$). From this it follows, in particular, that since the function $\Delta(x)$ vanishes for $t = 0$, then it vanishes everywhere outside the light cone when $x^2 = r^2 - t^2 > 0$.

By utilizing (17.37) and (17.38) we can easily show that the functions $\Delta(x)$ and $\Delta_1(x)$ may be written in the form

$$\begin{aligned} \Delta(x) &= \frac{i}{(2\pi)^3}\int e^{ipx}\xi(p)\,\delta(p^2+m^2)\,d^4p, \\ \Delta_1(x) &= \frac{1}{(2\pi)^3}\int e^{ipx}\,\delta(p^2+m^2)\,d^4p. \end{aligned} \tag{17.40}$$

Thus the Fourier components of the functions $\Delta(x)$ and $\Delta_1(x)$ are given by

$$\begin{aligned} \Delta(p) &= 2\pi i\xi(p)\,\delta(p^2+m^2), \\ \Delta_1(p) &= 2\pi\,\delta(p^2+m^2). \end{aligned} \tag{17.40'}$$

The Fourier component of the function $\Delta^c(x)$ can be obtained in the same way as the component of the function $D^c(x)$ if we displace the $+\varepsilon$ pole into the lower and the $-\varepsilon$ pole into the upper half-plane, and then integrate in (17.36) along the real axis. Such a displacement is evidently equivalent to a replacement of $p^2$ by $p^2-i|\eta|$ where $|\eta|\to 0$. From this it follows that the Fourier component of the function $\Delta^c(x)$ is equal to

$$\Delta^c(p) = \frac{1}{i(p^2+m^2-i0)} = \pi\delta_+(-p^2-m^2). \tag{17.41}$$

By utilizing formula (17.19) it is possible to obtain in a manner analogous to the way this was done for the function $D^c(x)$ the following integral representations for the functions $\Delta(x), \Delta_1(x), \Delta^c(x)$:

$$\begin{aligned} \Delta(x) &= \frac{\xi(x)}{4\pi^2}\int_{-\infty}^{\infty} e^{-i\left(x^2z-\frac{m^2}{4z}\right)}dz, \\ \Delta_1(x) &= \frac{1}{2\pi^2}\int_0^{\infty}\sin\left(x^2z-\frac{m^2}{4z}\right)dz, \\ \Delta^c(x) &= \frac{1}{4\pi^2 i}\int_0^{\infty} e^{i\left(x^2z-\frac{m^2}{4z}\right)}dz. \end{aligned} \tag{17.42}$$

We now show that the functions $\Delta^c(x), \Delta(x)$, and $\Delta_1(x)$ can be expressed explicitly in terms of Bessel functions. We start from the integral representation (17.36) of the function $\Delta^c(x)$. Since this function is relativistically invariant and, consequently, depends only upon $x^2 = r^2 - t^2$, it is sufficient to discuss the two cases when $t = 0$ and $x^2 > 0$ and when $r^2 = 0$ and $x^2 < 0$. In the first case on carrying out the integration over the angle between $\boldsymbol{p}$ and $\boldsymbol{r}$, and then on setting $p = m\sinh y$, $\varepsilon = m\cosh y$ we bring the integral (17.39) into

$$\Delta^c(x) = \frac{m}{8\pi^2 ir}\int_{-\infty}^{\infty} e^{irm\sinh y}\sinh y\,dy.$$

Further, on utilizing the following representation for the Hankel function (204),

$$H_0^{(2)}(r) = \frac{i}{\pi}\int_{-\infty}^{\infty} e^{-ir\cosh y}\,dy \tag{17.43}$$

and setting $r \to -ir$, $y \to y - i\pi/2$ in this expression, we obtain

$$H_0^{(2)}(-ir) = \frac{i}{\pi}\int_{-\infty}^{\infty} e^{ir\sinh y}dy.$$

A comparison of this formula with the one given earlier for $\Delta^c(x)$ shows that

$$\Delta^c(x) = \frac{i}{8\pi r}\frac{d}{dr}H_0^{(2)}(-imr).$$

But

$$\frac{d}{dz}H_0^{(2)}(z) = -H_1^{(2)}(z),$$

and therefore

$$\Delta^c(x) = \frac{im^2}{8\pi}\cdot\frac{H_1^{(2)}(-imr)}{-imr} = \frac{im^2}{8\pi}\,\frac{H_1^{(2)}(-im\sqrt{x^2})}{-im\sqrt{x^2}}, \qquad x^2 > 0.$$

When $x \to 0$ the term $(i/4\pi)\delta(-x^2)$ arising from the differentiation of the function $H_0^{(2)}$ must be added to the right hand side of this expression. Indeed, when $|z| \ll 1$ we have

$$H_0^{(2)}(z) = J_0(z) - iN_0(z) \approx -(2i/\pi)\ln(2/\gamma z),$$

and since

$$\frac{d}{dz}\ln z = \frac{1}{z} - \pi i\delta(z),$$

the additional term

$$\frac{i}{4\pi}\frac{d}{d(x^2)}H_0^{(2)}(-im\sqrt{x^2})=\frac{i}{4\pi}\delta(-x^2)$$

will appear in $\Delta^c(x)$ when $x^2\to 0$. Thus, when $x^2>0$ we have

$$\Delta^c(x)=\frac{i}{4\pi}\left\{\delta(-x^2)+\frac{m^2}{2}\frac{H_1^{(2)}(-im\sqrt{x^2})}{-im\sqrt{x^2}}\right\}. \tag{17.44}$$

If $x^2<0$, then we can take $r=0$ in (17.39); on setting $p=m\sinh z$, $\varepsilon=m\cosh z$, we obtain

$$\Delta^c(x)=\frac{m^2}{4\pi^2}\int\limits_0^\infty e^{-im|t|\cosh^2 z}\sinh^2 z\,dz.$$

This integral can be reduced with the aid of (17.43) to

$$\Delta^c(x)=\frac{i}{8\pi}\left\{\frac{d^2}{dt^2}H_0^{(2)}(m|t|)+m^2H_0^{(2)}(m|t|)\right\},$$

and since

$$\frac{d^2}{dz^2}H_0^{(2)}(z)+H_0^{(2)}(z)=\frac{1}{z}H_1^{(2)}(z),$$

it can be easily shown that we again obtain formula (17.44) which is, consequently, valid for arbitrary $x^2$.

If we have an expression for $\Delta^c(x)$, we can find with the aid of formula (17.39′) the functions $\Delta(x)$ and $\Delta_1(x)$. On separating in (17.44) the real and the imaginary parts we obtain

$$\Delta(x)\frac{x_0}{|x_0|}=-\frac{1}{2\pi}\delta(-x^2)-\frac{m^2}{4\pi}\frac{J_1(-im\sqrt{x^2})}{-im\sqrt{x^2}}\theta(-x^2),$$

$$\Delta_1(x)=\begin{cases}\dfrac{m^2}{2\pi^2}\dfrac{K_1(m\sqrt{x^2})}{m\sqrt{x^2}}, & x^2>0,\\[2ex] \dfrac{m^2}{4\pi^2}\dfrac{N_1(m\sqrt{-x^2})}{m\sqrt{-x^2}}, & x^2<0.\end{cases} \tag{17.45}$$

When $x^2\to 0$ these formulas assume the form

$$\begin{aligned}\Delta(x)\frac{x_0}{|x_0|}&=-\frac{1}{2\pi}\left\{\delta(-x^2)+\frac{m^2}{4}\theta(-x^2)\right\},\\ \Delta_1(x)&=\frac{1}{4\pi^2}\left\{-\frac{2}{x^2}+m^2\ln\frac{\gamma m\sqrt{|x^2|}}{2}-\frac{m^2}{2}\right\},\end{aligned} \tag{17.45′}$$

where $\gamma=1.781$.

We also give expressions for the functions $\Delta_{\pm}(x)$, $\Delta_R(x)$, and $\Delta_A(x)$, which are analogous to the functions $D_{\pm}(x)$, $D_R(x)$, $D_A(x)$:

$$\begin{aligned} \Delta_{\pm}(x) &= \frac{1}{(2\pi)^4} \int\limits_{C_{\pm}} \frac{e^{ipx}}{p^2+m^2}\, d^4p, \\ \Delta_R(x) &= \frac{1}{(2\pi)^4} \int\limits_{C_R} \frac{e^{ipx}}{p^2+m^2}\, d^4p, \\ \Delta_A(x) &= \frac{1}{(2\pi)^4} \int\limits_{C_A} \frac{e^{ipx}}{p^2+m^2}\, d^4p, \end{aligned} \tag{17.46}$$

where the contours of integration $C_{\pm}$, $C_R$, $C_A$ over $p_0$ are shown in Fig. 10.

It can be easily shown that

$$\Delta_{\pm}(x) = \mp \frac{1}{2i(2\pi)^3} \int e^{\pm ipx} \frac{dp}{\varepsilon} = \tfrac{1}{2}\{\Delta(x) \pm i\Delta_1(x)\}. \tag{17.47}$$

From (17.47) it can be seen that similarly to (17.25″) we have

$$\Delta_+(x) = -\Delta_-(-x). \tag{17.47'}$$

The function $\Delta^c(x)$ can be expressed in terms of the functions $\Delta_{\pm}(x)$ as follows:

$$\Delta^c(x) = \frac{1}{i}\{\theta(x)\Delta_+(x) - \theta(-x)\,\Delta_-(x)\}. \tag{17.48}$$

By repeating the calculations given earlier with reference to the function $D_R(x)$ we can show that

$$\Delta_R(x) = \begin{cases} \Delta(x), & t > 0, \\ 0, & t < 0 \end{cases} \tag{17.49}$$

and also that

$$(\Box - m^2)\,\Delta_R(x) = -\delta(x). \tag{17.49'}$$

In conclusion we note that all the functions considered above can be expressed in terms of one analytic function. In order to do this we consider the function $\Delta_+(x)$. From (17.36) it can be easily seen that the integral defining it is still meaningful if $x$ is replaced by $z = x - i\eta$, where $\eta$ is a four-vector for which $\eta^2 < 0, \eta_0 > 0$. This means that $\Delta_+$ is a function of $z^2$ analytic in the plane of this variable with a cut along the negative real semi-axis $(-\infty, 0)$ (81a, 211a).

Moreover, the region $x^2 > 0$ corresponds in the $z^2$ plane to the positive real semi-axis, while the regions $x^2 < 0, x_0 \gtrless 0$ correspond, respectively, to the upper and lower edges of the cut $(z^2 = x^2 + i0)$. Indeed, from the preceding formulas an explicit expression for $\Delta_+$ follows:

$$\Delta_+(z) \equiv F(z^2) = \frac{im}{4\pi^2} K_+(m\sqrt{z^2})/\sqrt{z^2}. \tag{17.50}$$

The remaining functions can be expressed in terms of $\Delta_+$, if relations (17.47), (17.47′), (17.48) are utilized. In particular, we obtain

$$i\Delta^c(x) = F(x^2+i0), \tag{17.51}$$

$$\frac{x_0}{|x_0|}\Delta(x) = F(x^2+i0) - F(x^2-i0). \tag{17.52}$$

## § 18. Quantization of the Electron-Positron Field

### 18.1. *Variational Principle for the Dirac Equation. Energy-Momentum Tensor of the Electron-Positron Field*

In Chapter II we studied the properties of a single electron. In order to study an arbitrary system of noninteracting electrons and positrons we can, just as in the case of a system of photons, go over from the description by means of the wave function in configuration space to the description by means of the wave function in particle number space. In the case of a system of photons such a transition is equivalent to the quantization of the electromagnetic field. In the case of a system of electrons it is equivalent to the quantization of the electron-positron field, i.e., of the field of the wave functions $\psi(x)$ and $\bar{\psi}(x)$ satisfying the Dirac equations. In this case the Dirac equations, like Maxwell's equations, play the role of field equations, and not of equations for an individual particle, while the quantization of the electron-positron field means that the wave functions $\psi(x)$ and $\bar{\psi}(x)$ are regarded as operators satisfying certain commutation relations and operating on the state vector, or on the wave function of the system in particle number space. As regards the dependence of these operators on the coordinates and the time, it is determined by the Dirac equations.

In order to emphasize the field nature of the Dirac equations and to be able to treat them similarly to the equations of the electromagnetic field, we show, first of all, that the Dirac equations can be obtained from a variational principle: $\delta \int L d^4 x = 0$, if for the Lagrangian density we take

$$L = -\tfrac{1}{2}\left(\bar{\psi}\gamma_\nu \frac{\partial \psi}{\partial x_\nu} - \frac{\partial \bar{\psi}}{\partial x_\nu}\gamma_\nu \psi\right) - m\bar{\psi}\psi \tag{18.1}$$

and in varying $L$ regard the functions $\psi$ and $\bar{\psi}$ as independent.

On treating $\psi$, $\bar{\psi}$ and $\partial\psi/\partial x_\mu$, $\partial\bar{\psi}/\partial x_\mu$ as generalized coordinates and velocities of a certain dynamical system—the electron-positron field—we obtain from the variational principle the following Lagrangian equations:

$$\begin{aligned} &\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial[\partial\psi_\lambda/\partial x_\mu]} - \frac{\partial L}{\partial \psi_\lambda} = 0, \\ &\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial[\partial\bar{\psi}_\lambda/\partial x_\mu]} - \frac{\partial L}{\partial \bar{\psi}_\lambda} = 0, \qquad \lambda = 1, 2, 3, 4. \end{aligned} \tag{18.1'}$$

On substituting into these expressions

$$\begin{aligned} &\frac{\partial L}{\partial[\partial\psi_\lambda/\partial x_\mu]} = -\frac{1}{2}\bar{\psi}_\nu(\gamma_\nu)_{\mu\lambda}, \qquad \frac{\partial L}{\partial[\partial\bar{\psi}_\lambda/\partial x_\mu]} = \frac{1}{2}(\gamma_\mu)_{\lambda\nu}\psi_\nu, \\ &\frac{\partial L}{\partial\psi_\lambda} = \frac{1}{2}\frac{\partial\bar{\psi}_\mu}{\partial x_\lambda}(\gamma_\nu)_{\mu\lambda} - m\bar{\psi}_\lambda, \qquad \frac{\partial L}{\partial\bar{\psi}_\lambda} = -\frac{1}{2}(\gamma_\nu)_{\lambda\mu}\frac{\partial\psi_\mu}{\partial x_\nu} m\psi_\lambda \end{aligned} \tag{18.2}$$

or, in abbreviated form,

$$\begin{aligned} &\frac{\partial L}{\partial[\partial\psi/\partial x_\mu]} = -\frac{1}{2}\bar{\psi}\gamma_\mu, \qquad \frac{\partial L}{\partial[\partial\bar{\psi}/\partial x_\mu]} = \frac{1}{2}\gamma_\mu\psi, \\ &\frac{\partial L}{\partial\psi} = \frac{1}{2}\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\nu - m\bar{\psi}, \qquad \frac{\partial L}{\partial\bar{\psi}} = -\frac{1}{2}\gamma_\nu\frac{\partial\psi}{\partial x_\nu} - m\psi, \end{aligned}$$

we obtain the Dirac equations

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + m\right)\psi = 0, \qquad \bar{\psi}\left(\gamma_\mu\frac{\partial}{\partial x_\mu} - m\right) = 0. \tag{18.3}$$

If we know the Lagrangian density, then we can, just as in the case of the electromagnetic field, define *the energy-momentum tensor* of the electron-positron field.

We begin with expression (16.6) for the energy-momentum tensor of the electromagnetic field. As can be seen from the derivation of this expression, it can be applied directly to define the energy-momentum tensor of the electron-positron field, if in it we replace the components of the potential $A_\mu$ by the components of the Dirac functions $\psi_\mu$ and $\bar{\psi}_\mu$ which we regard as generalized coordinates of the electron-positron field. Thus, we arrive at the following definition of the energy-momentum tensor of the electron-positron field:

$$T_{\mu\nu} = L\delta_{\mu\nu} - \left( \frac{\partial L}{\partial[\partial\psi_\lambda/\partial x_\nu]} \frac{\partial \psi_\lambda}{\partial x_\mu} + \frac{\partial L}{\partial[\partial\bar{\psi}_\lambda/\partial x_\nu]} \frac{\partial \bar{\psi}_\lambda}{\partial x_\mu} \right).$$

On substituting into this expression formulas (18.2), and on noting that the functions $\psi$ and $\bar{\psi}$ satisfying the Dirac equation make the Lagrangian density vanish, we finally obtain

$$T_{\mu\nu} = \frac{1}{2}\left( \bar{\psi}\gamma_\nu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \bar{\psi}}{\partial x_\mu} \gamma_\nu \psi \right). \tag{18.4}$$

From this we see that the energy density and the momentum density of the field are respectively given by

$$\begin{aligned} w &= -T_{44} = \frac{1}{2i}\left( -\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right) \\ \boldsymbol{p} &= \frac{1}{2i}\left( \psi^* \frac{\partial \psi}{\partial \boldsymbol{r}} - \frac{\partial \psi^*}{\partial \boldsymbol{r}} \psi \right). \end{aligned} \tag{18.4'}$$

The energy-momentum tensor (18.4) in virtue of the Dirac equations satisfies the condition $(\partial/\partial x_\nu)\, T_{\mu\nu} = 0$, and this, as we already know, implies conservation of the four-dimensional vector

$$P_\mu = -i \int T_{\mu 4}\, d\boldsymbol{r}. \tag{18.5}$$

This vector can be interpreted as the four-dimensional energy-momentum vector of the electron-positron field.

The total energy and momentum of the field are obviously equal to integrals of $w$ and $\boldsymbol{p}$ over all space:

$$\begin{aligned} H &= \int w\, d\boldsymbol{r} = \frac{i}{2}\int \left( \bar{\psi}\gamma_4 \frac{\partial \psi}{\partial t} - \frac{\partial \bar{\psi}}{\partial t}\gamma_4 \psi \right) d\boldsymbol{r}, \\ \boldsymbol{P} &= \int \boldsymbol{p}\, d\boldsymbol{r} = -i \int \bar{\psi}\gamma_4 \frac{\partial \psi}{\partial \boldsymbol{r}}\, d\boldsymbol{r}. \end{aligned} \tag{18.6}$$

We note that the tensor $T_{\mu\nu}$ is not symmetric, but that we can construct the symmetric tensor $\theta_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$, which, like the tensor $T_{\mu\nu}$, satisfies the equation of continuity $(\partial \theta_{\mu\nu}/\partial x_\nu) = 0$.

### 18.2. *Quantization Rules for the Electron-Positron Field*

We now proceed to establish the quantization rules for the electron-positron field. As has been noted earlier, we must, just as in the case of the electromagnetic field, regard the components of the field $\psi$ as operators which operate in particle number space, and which obey definite quantization rules. However, these rules must differ from the rules for the quantization of the electromagnetic field (16.17). Indeed, from the latter it follows that the number of particles in a definite state may be arbitrary, while the Dirac equation describes particles which obey Fermi-Dirac statistics, and therefore their number in any given state may be equal either to unity, or to zero (the Pauli exclusion principle).

In order to quantize the electron-positron field, we start with the general series expansion of $\psi$ in terms of the orthonormal system of functions $\psi_r^{(+)}$ and $\psi_r^{(-)}$ which are solutions of the Dirac equations with positive and negative frequencies respectively. These may be solutions which correspond either to free motion, or to motion in an arbitrary, but sufficiently weak, stationary field. In a strong field it is impossible to carry out the separation into positive and negative frequencies (cf. § 12). The wave functions with positive frequencies correspond to electron states, while the wave functions with negative frequencies correspond to positron states. We write the expansion of $\psi$ in the form

$$\begin{aligned} \psi &= {\sum_r}' \mathrm{a}_r \psi_r^{(+)} + {\sum_r}' \mathrm{b}_r^+ \psi_r^{(-)}, \\ \bar{\psi} &= {\sum_r}' \mathrm{a}_r^+ \bar{\psi}_r^{(+)} + {\sum_r}' \mathrm{b}_r \bar{\psi}_r^{(-)}, \end{aligned} \tag{18.7}$$

where $\psi_r^{(\pm)}$ are normalized in accordance with

$$\int \psi_r^{(\eta)} \psi_{r'}^{(\eta')*} \, dr = \delta_{rr'} \delta_{\eta\eta'}, \quad \eta, \eta' = \pm,$$

and we assume that the expansion amplitudes $\mathrm{a}_r$, $\mathrm{a}_r^+$, $\mathrm{b}_r$, $\mathrm{b}_r^+$, are not ordinary numbers, but operators, just as we did in the case of the expansion amplitudes $\mathrm{c}_{k\lambda}$ in (16.13) for the potential of the electromagnetic field.

We assume that $a_r$ and $a_r^+$, and also $b_r$ and $b_r^+$ are Hermitian conjugate operators which satisfy the following commutation relations:

$$\begin{aligned} &\{a_r, a_{r'}^+\} = \delta_{rr'}, && \{b_r, b_{r'}^+\} = \delta_{rr'}, \\ &\{a_r, a_{r'}\} = \{a_r^+, a_{r'}^+\} = 0, && \{b_r, b_{r'}\} = \{b_r^+, b_{r'}^+\} = 0, \\ &\{a_r, b_{r'}\} = \{a_r, b_{r'}^+\} = \{a_r^+, b_{r'}\} = \{a_r^+, b_{r'}^+\} = 0, \end{aligned} \tag{18.8}$$

where $\{A, B\} = AB+BA$ is the anticommutator of the operators A and B. These relations, which are the quantization rules for the electron-positron field, lead, as we shall see later, to the correct corpuscular picture of the field.

By making use of the quantization rules we determine the form of the operators $a_r, a_r^+, b_r, b_r^+$. We first of all find the eigenvalues of the operators

$$n_r^{(+)} = a_r^+ a_r, \quad \text{and} \quad n_r^{(-)} = b_r^+ b_r.$$

On squaring $n_r^{(+)}$ and $n_r^{(-)}$, and on noting that $a_r^2 = b_r^2 = 0$, we obtain

$$n_r^{(+)2} = a_r^+ a_r a_r^+ a_r = a_r^+ a_r (1 - a_r a_r^+) = a_r^+ a_r,$$

$$n_r^{(-)2} = b_r^+ b_r b_r^+ b_r = b_r^+ b_r (1 - b_r b_r^+) = b_r^+ b_r,$$

i.e., $n_r^{(+)2} = n_r^{(+)}$, $n_r^{(-)2} = n_r^{(-)}$, whence it follows that the operators $n_r^{(+)}$ and $n_r^{(-)}$ have only two different eigenvalues equal to unity and zero.

We make use of the representation in which the matrices $n_r^{(+)}$ and $n_r^{(-)}$ are diagonal; then the matrices $a_r, a_r^+, b_r, b_r^+$ can be taken to be two-rowed ones, and we can write

$$(n_r^{(+)})_{ii} = \sum_{k=1}^{2} (a_r^+)_{ik} (a_r)_{ki} = \sum_{k=1}^{2} |(a_r)_{ki}|^2 = \begin{cases} 1, & i = 1, \\ 0, & i = 2 \end{cases}$$

(it is assumed that the eigenvalue $n_r^{(+)} = 1$ corresponds to the subscript $i = 1$, while the eigenvalue $n_r^{(+)} = 0$ corresponds to the subscript $i = 2$). Since $a_r^2 = a_r^{+2} = 0$, i.e.,

$$(a_r^2)_{ik} = \sum_{j=1}^{2} (a_r)_{ij} (a_r)_{jk} = 0,$$

$$(a_r^{+2})_{ik} = \sum_{j=1}^{2} (a_r^+)_{ij} (a_r^+)_{jk} = 0,$$

we finally obtain

$$\mathrm{a}_r = \eta_r \alpha_r, \qquad \mathrm{a}_r^+ = \eta_r^+ \alpha_r^+,$$
$$\alpha_r = \begin{pmatrix} 0, & 0 \\ 1, & 0 \end{pmatrix}, \qquad \alpha_r^+ = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}, \tag{18.9}$$

where $\eta_r$ is a certain matrix which does not operate on the variables of the $r$th electron state and which satisfies the condition $\eta_r^+ \eta_r = 1$.

By proceeding in a similar manner we can show that the matrices $\mathrm{b}_r$ and $\mathrm{b}_r^+$ are of the form

$$\mathrm{b}_r = \xi_r \alpha_r, \qquad \mathrm{b}_r^+ = \xi^+ \alpha_r^+, \tag{18.9'}$$

where $\xi_r$ is a certain matrix which does not operate on the variables of the $r$th positron state, and which satisfies the condition $\xi_r^+ \xi_r = 1$.

In order to obtain the form of the matrices $\eta_r$ and $\xi_r$ we arrange all the electron and positron states in a definite (arbitrary, but fixed once and for all) order, and we label these states uniquely by the subscript $r$. The matrices $\eta_r$ and $\xi_r$ must be chosen in such a way that the following anticommutation relations hold:

$$\{\mathrm{a}_r, \mathrm{a}_{r'}\} = \{\mathrm{b}_{r''}, \mathrm{b}_{r'''}\} = \{\mathrm{a}_r, \mathrm{b}_{r''}\} = 0.$$

For this it is sufficient to assume that the matrices $\xi_r$ and $\eta_r$ (we shall in future denote both kinds of matrices by $\zeta_r$) operate only on the variables of those states which precede the state $r$, i.e.,

$$\zeta_r = \zeta_r(1, 2, \ldots, r-1),$$

and which have the form of the product

$$\zeta_r = \zeta(1)\,\zeta(2) \ldots \zeta(\mathrm{r}-1), \tag{18.10}$$

where $\zeta\,(n)$ operates only on the variables of the state $n$.

We now consider the anticommutator

$$\{\mathrm{a}_r, \mathrm{a}_{r+1}\} = \zeta_r \alpha_r \zeta_{r+1} \alpha_{r+1} + \zeta_{r+1} \alpha_{r+1} \zeta_r \alpha_r = 0.$$

Since the matrices $\alpha_r$ and $\alpha_{r+1}$ refer to different states, i.e., operate in different spaces, they commute with each other. Therefore, the anticommutator $\{\mathrm{a}_r, \mathrm{a}_{r+1}\}$ can be rewritten, on taking (18.10) into account, in the form

$$\alpha_r \zeta(r) + \zeta(r) \alpha_r = 0. \tag{18.11}$$

Thus, the matrix $\zeta(r)$ anticommutes with $a_r$; similarly it can be easily shown that $\zeta(r)$ also anticommutes with $a_r^+$. From (18.11) it follows that

$$[\zeta(r), a_r^+ a_r] = 0.$$

Therefore, the matrix $\zeta(r)$ may be assumed to be diagonal in the representation in which the matrix $a_r^+ a_r$ is diagonal, and we may write

$$(a_r)_{21}\big(\zeta(r)\big)_{11} + \big(\zeta(r)\big)_{22}(a_r)_{21} = 0,$$

i.e.,

$$\big(\zeta(r)\big)_{11} = -\big(\zeta(r)\big)_{22},$$

where the subscripts 1 and 2 correspond to the eigenvalues of $n_r^{\pm}$ equal to 1 and 0. This relation, together with the relation $\zeta_r^+ \zeta_r = 1$ shows that $\zeta(r)$ has the following form:

$$\zeta(r) = (-1)^{\mathrm{d}_r^+ \mathrm{d}_r} = 1 - 2\mathrm{d}_r^+ \mathrm{d}_r,$$

where $\mathrm{d}_r$ denotes both the operators $\mathrm{a}_r$, and the operators $\mathrm{b}_r$. By making use of (18.10) we finally obtain

$$\zeta_r = \prod_{r'=1}^{r-1} (1 - 2\mathrm{d}_{r'}^+ \mathrm{d}_{r'}) = \prod_{r'=1}^{r-1} (1 - 2\mathrm{n}_{r'}^{(\pm)}), \tag{18.12}$$

where $\mathrm{n}_r^{(\pm)} = \mathrm{d}_r^+ \mathrm{d}_r$. We now show that this quantity represents the number of particles (electrons or positrons) in the state $r$.

Thus, $\zeta_r$ is equal to $+1$ or $-1$ depending on whether the number of occupied electron and positron states preceding the state $r$ turns out to be even or odd.

We now show that the quantization rules (18.8) lead to the correct corpuscular picture. In order to do this we obtain the energy of the electron-positron field. This quantity, like the field components, is an operator defined by means of formulas (18.6) and (18.7) in which the operators $\mathrm{a}_r$, $\mathrm{a}_r^+$, $\mathrm{b}_r$, $\mathrm{b}_r^+$ obey the quantization rules (18.8).

By making use of the orthonormal properties of the system of functions $\psi_r^{(+)}, \psi_r^{(-)}$, we can easily show that the field energy is given by

$$\mathrm{H} = \sum_r \varepsilon_r^{(+)} a_r^+ a_r - \sum_r \varepsilon_r^{(-)} \mathrm{b}_r \mathrm{b}_r^+,$$

where $\varepsilon_r^{(+)} = \omega_r^{(+)}$, $\varepsilon_r^{(-)} = -\omega_r^{(-)}$ and $\omega_r^{\pm}$ are the frequencies of the corresponding solutions of the Dirac equation. By introducing into these expressions the quantities $\mathrm{n}_r^{(+)} = \mathrm{a}_r^+ \mathrm{a}_r$ and $\mathrm{n}_r^{(-)} = \mathrm{b}_r^+ \mathrm{b}_r$, we rewrite H in the form

$$\mathrm{H} = \sum_r \varepsilon_r^{(+)} \mathrm{n}_r^{(+)} + \sum_r \varepsilon_r^{(-)} \mathrm{n}_r^{(-)} - \sum_r \varepsilon_r^{(-)}. \tag{18.13}$$

This expression is not positive definite. In other words, the energy of the electron-positron field, both in classical theory, and also after the quantization rules (18.8) have been imposed on the quantities $\mathrm{a}_r$ and $\mathrm{b}_r$, can assume both positive and negative values.

However, we can define the vacuum state of the electron-positron field in such a way that the energy of the field with respect to the vacuum state is always positive definite. This state is defined as the state of lowest energy, in which all the numbers $n_r^{(+)}$ and $n_r^{(-)}$ are equal to zero:

$$\begin{aligned} \langle \mathrm{a}_r^+ \mathrm{a}_r \rangle_0 = 0, \quad \langle \mathrm{b}_r^+ \mathrm{b}_r \rangle_0 = 0, \\ \langle \mathrm{a}_r \mathrm{a}_r^+ \rangle_0 = 1, \quad \langle \mathrm{b}_r \mathrm{b}_r^+ \rangle_0 = 1, \end{aligned} \tag{18.14}$$

where $\langle \mathrm{L} \rangle_0$ denotes the vacuum expectation value of the operator L.

According to (18.13) the energy of the vacuum state is equal to $E_0 = -\sum_r \varepsilon_r^{(-)}$, where the summation is taken over all the individual states of the electron with negative frequencies. Therefore, the energy of the quantized field with respect to the vacuum state is given by

$$E = H - E_0 = \sum_r \varepsilon_r^{(+)} n_r^{(+)} + \sum_r \varepsilon_r^{(-)} n_r^{(-)}. \tag{18.15}$$

We see that the numbers $n_r^{(+)}$ and $n_r^{(-)}$, which can have the values 0 and 1, appear in the expression for the energy of the field in the form of factors multiplying the energies of the electron and the positron. Therefore, these numbers should be interpreted as the numbers of electrons and positrons having respectively the energies $\varepsilon_r^{(+)}$ and $\varepsilon_r^{(-)}$.

Thus, the quantization rules do indeed lead to the correct corpuscular picture, since the energy of the field is equal to the sum of the energies of the individual particles—the electrons and the positrons.

We note that in accordance with (18.15) the energy of the electron-positron field attains its minimum value (equal to zero) for $n_r^{(+)} = n_r^{(-)} = 0$ only in the case when $\varepsilon_r^{(+)} > 0$, $\varepsilon_r^{(-)} > 0$. These inequalities hold both for free electrons, and for electrons in a weak external field. In a strong external field, as we have already seen in § 12, when the depth of the well exceeds a certain value $V'_0$, a negative energy $\varepsilon^{(+)} < 0$ can appear. In this case the energy of the field will be a minimum, and equal to $\varepsilon^{(+)}$ not when all the $n_r^{(+)}$ vanish, but under the condition that the state of energy $\varepsilon^{(+)}$ is occupied, i.e., $n_{\varepsilon^{(+)}}^{(+)} = 1$. In this state the field has the charge $e$. When $V_0 > V_{0c}$ the energy $\varepsilon^{(+)}$ becomes less than $-m$, and a state of the field with negative energy and of zero charge is possible, in which in addition to an electron of energy $\varepsilon^{(+)} < 0$ there also exists a positron of energy $\varepsilon^{(-)} > 0$. Moreover, the positron, for which the external field is repulsive, can escape to infinity, while the electron charge will cancel a part of the charge which gives rise to the external field, and will therefore reduce it. These considerations lead to the conclusion that the external field cannot be too strong (the depth of the potential well cannot exceed $V_{0c}$ in the example of § 12). The complete solution of this problem requires a more detailed investigation of the interaction between the electron and the source of the external field, taking into account the energy and the charge states of the latter. But this already goes beyond the limits of applicability of the very concept of an external field.

We now show that the matrices $\mathrm{a}_r(\mathrm{b}_r)$ and $\mathrm{a}_r^+(\mathrm{b}_r^+)$ have a simple physical meaning, and represent annihilation and creation operators for an electron (positron) in the state $r$. Indeed, if we denote by $\Phi_{n_r^{(+)}}$ the wave function (or the state vector), which describes $n_r^{(+)}$ electrons in the state $r$, we can write

$$\begin{aligned} \mathrm{a}_r \Phi_{n_r^{(+)}} &= \Phi_{n_r^{(+)}-1}, \qquad n_r^{(+)} = 1, \\ \mathrm{a}_r^+ \Phi_{n_r^{(+)}} &= \Phi_{n_r^{(+)}+1}, \qquad n_r^{(+)} = 0. \end{aligned} \tag{18.16}$$

In a similar manner we can show that $b_r$ and $b_r^+$ are annihilation and creation operators for a positron in the state $r$.

We write the nonvanishing matrix elements of the particle annihilation and creation operators as

$$\begin{aligned} (\mathrm{a}_r)_{n_r^{(+)}-1,\, n_r^{(+)}} &= \zeta_r \sqrt{n_r^{(+)}}, \\ (\mathrm{a}_r^+)_{n_r^{(+)}+1,\, n_r^{(+)}} &= \zeta_r \sqrt{n_r^{(+)}+1}, \\ (\mathrm{b}_r)_{n_r^{(-)}-1,\, n_r^{(-)}} &= \zeta_r \sqrt{n_r^{(-)}}, \\ (\mathrm{b}_r^+)_{n_r^{(-)}+1,\, n_r^{(-)}} &= \zeta_r \sqrt{n_r^{(-)}+1}. \end{aligned} \tag{18.17}$$

If we write the operators $\psi$ and $\overline{\psi}$ in the form

$$\psi = \mathrm{A}+\mathrm{B}^+, \qquad \overline{\psi} = \mathrm{A}^+ + \mathrm{B}, \tag{18.18}$$

where A and $B^+$ comprise the terms in the expansion of the operator $\psi$ which respectively contain positive and negative frequencies, while $A^+$ and B play a similar role in the case of the operator $\overline{\psi}$, then in accordance with (18.17) we can conclude that A and B are electron and positron annihilation operators, while $A^+$ and $B^+$ are electron and positron creation operators. In other words, $\psi$ is an electron annihilation operator and a positron creation operator, while $\overline{\psi}$ is an electron creation operator and a positron annihilation operator.

We return to the expansion (18.7) and examine in greater detail the very important case when the external field is absent. In this case we can start with the expansion of $\psi$ into plane waves, describing states of definite momentum $p$ and polarization $r$:

$$\begin{aligned}\psi &= \frac{1}{\sqrt{\Omega}} \sum_{p;\, r=1,2} \{a_r(p) u^r(p) e^{ipx} + b_r^+(p) v^r(-p) e^{-ipx}\},\\ \overline{\psi} &= \frac{1}{\sqrt{\Omega}} \sum_{p;\, r=1,2} \{a_r^+(p) \bar{u}^r(p) e^{-ipx} + b_r(p) \bar{v}^r(-p) e^{ipx}\},\end{aligned} \tag{18.19}$$

where the spinor amplitudes $u(p)$ and $v(-p)$ satisfy the equations

$$(i\hat{p}+m)u(p) = 0,$$
$$(-i\hat{p}+m)v(-p) = 0$$

and the orthogonality and normalization conditions

$$\begin{aligned}\sum_{\varrho=1}^{4} u_\varrho^r(p) u_\varrho^{r'}(p)^* &= \delta_{rr'},\\ \sum_{\varrho=1}^{4} v_\varrho^r(-p) v_\varrho^{r'}(-p)^* &= \delta_{rr'}.\end{aligned} \tag{18.19'}$$

We note that without loss of generality we can assume that the solution $v^r(-p)\exp(-ipx)$ is charge-conjugate with respect to $u^r(p)\exp(ipx)$, i.e.,

$$v(-p) = C\bar{u}(p), \quad \bar{v}(-p) = C^+u(p).$$

The energy and the momentum of the free quantized electron-positron field with respect to vacuum can be represented as

$$\begin{aligned}E &= \sum_{p;\, r=1,2} \varepsilon_p (n_{pr}^{(+)} + n_{pr}^{(-)}),\\ \boldsymbol{P} &= \sum_{p;\, r=1,2} \boldsymbol{p}(n_{pr}^{(+)} + n_{pr}^{(-)}),\end{aligned} \tag{18.20}$$

where $n_{pr}^{(+)}$ and $n_{pr}^{(-)}$ are the numbers of electrons and positrons of momentum $p$ and polarization $r$.

We note in concluding this section that the quantization of the electron-positron field in accordance with the rules (18.8) is in accord with the quantum mechanical description of a system of electrons by means of a wave function which is antisymmetric with respect to the coordinates and spins of each pair of electrons.

## § 19. Anticommutators of the Electron-Positron Field. Chronological and Normal Products of Field Components. Current Density

### 19.1. *Commutation Relations for Field Components*

We now establish the commutation relations satisfied by the field operators $\psi$ and $\bar{\psi}$. In order to do this we use the quantization rules (18.8) for the operators $\mathrm{a}_r, \mathrm{a}_r^+, \mathrm{b}_r, \mathrm{b}_r^+$.

From (18.8) it follows, first of all, that

$$\{\psi_\alpha(x), \psi_\beta(x')\} = 0, \quad \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')\} = 0, \tag{19.1}$$

i.e., the operators $\psi_\alpha$ and $\psi_\beta$ anticommute, and so do the operators $\bar{\psi}_\alpha$ and $\bar{\psi}_\beta$.

Further, we determine the anticommutator $\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}$. On making use of the expansion of $\psi$ into plane waves, we obtain

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = \frac{1}{\Omega} \sum_{p;\, r=1,2} u_\alpha^r(p)\bar{u}_\beta^r(p)\, e^{ip(x-x')} + \frac{1}{\Omega} \sum_{p;\, r=1,2} v_\alpha^r(-p)\bar{v}_\beta^r(-p)\, e^{-ip(x-x')}. \tag{19.1'}$$

We evaluate the following sums which appear in the preceding expressions:

$$\begin{aligned} -iS_{\alpha\beta}^+(x-x') &= \frac{1}{\Omega} \sum_{p;\, r=1,2} u_\alpha^r(p)\bar{u}_\beta^r(p)\, e^{ip(x-x')}, \\ -iS_{\alpha\beta}^-(x-x') &= \frac{1}{\Omega} \sum_{p;\, r=1,2} v_\alpha^r(-p)\bar{v}_\beta^r(-p)\, e^{-ip(x-x')}. \end{aligned} \tag{19.2}$$

On noting that in accordance with (10.28) and (10.33)

$$\sum_{r=1,2} u_\alpha^r(p)\bar{u}_\beta^r(p) = -\frac{1}{2\varepsilon}(i\hat{p}-m)_{\alpha\beta},$$

$$\sum_{r=1,2} v_\alpha^r(-p)\bar{v}_\beta^r(-p) = -\frac{1}{2\varepsilon}(i\hat{p}+m)_{\alpha\beta},$$

we can write the functions $S^{\pm}_{\alpha\beta}(x)$ in the form

$$-iS^{+}_{\alpha\beta}(x) = -\frac{1}{\Omega}\sum_{p}\frac{1}{2\varepsilon}(i\hat{\mathrm{p}}-m)_{\alpha\beta}e^{ipx}$$
$$= -\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)_{\alpha\beta}\frac{1}{\Omega}\sum_{p}\frac{1}{2\varepsilon}e^{ipx},$$
$$-iS^{-}_{\alpha\beta}(x) = \frac{1}{\Omega}\sum_{p}\frac{1}{2\varepsilon}(-i\hat{\mathrm{p}}-m)_{\alpha\beta}e^{-ipx}$$
$$= \left(\gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)_{\alpha\beta}\frac{1}{\Omega}\sum_{p}\frac{1}{2\varepsilon}e^{-ipx}.$$

On making the transition from summation to integration over $\boldsymbol{p}$ and on utilizing the definition of the functions $\Delta_{+}(x)$ we obtain finally

$$S^{\pm}(x) = -\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)\Delta_{\pm}(x) = \frac{1}{i}(\hat{\mathrm{p}}+im)\Delta_{\pm}(x), \qquad (19.3)$$

where $\hat{\mathrm{p}} = -i\gamma_\mu[\partial/\partial x_\mu]$. In accordance with (17.47) the functions $S^{\pm}(x)$ can also be written in the form

$$S^{\pm}_{\alpha\beta}(x) = -\tfrac{1}{2}(i\hat{\mathrm{p}}-m)_{\alpha\beta}\big(\Delta(x)\pm i\Delta_1(x)\big). \qquad (19.3')$$

On utilizing these formulas, and also relation (19.1′), we obtain the following relation for the anticommutator $\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}$:

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = -iS_{\alpha\beta}(x-x'), \qquad (19.4)$$

where

$$S_{\alpha\beta}(x) = S^{+}_{\alpha\beta}(x)+S^{-}_{\alpha\beta}(x) = -\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)_{\alpha\beta}\Delta(x) \qquad (19.5)$$

and $\Delta(x)$ is defined by formula (17.37).

We note that from (19.4) the following relation is immediately obtained:

$$\{\psi_\alpha(\boldsymbol{r}, t), \psi^{+}_{\beta}(\boldsymbol{r}', t)\} = \delta_{\alpha\beta}\,\delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad (19.6)$$

in which both operators $\psi_\alpha$ and $\psi^{+}_{\beta}$ refer to the same instant of time.

The functions $S^{\pm}_{\alpha\beta}(x)$ have a simple physical meaning: they are vacuum expectation values of the products of the field components $\psi_\alpha(x)\bar{\psi}_\beta(x')$ and $\bar{\psi}_\beta(x')\,\psi_\alpha(x)$. Indeed, by utilizing the expansions (18.19) and formulas (18.14), we can easily show that

$$\begin{aligned}\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 &= -iS^{+}_{\alpha\beta}(x-x'),\\ \langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0 &= -iS^{-}_{\alpha\beta}(x-x').\end{aligned} \qquad (19.7)$$

From these expressions and from (17.47) it follows that

$$\langle[\psi_\alpha(x), \bar{\psi}_\beta(x')]\rangle_0 = S^{(1)}_{\alpha\beta}(x-x'), \qquad (19.8)$$

where

$$S^{(1)}_{\alpha\beta}(x) = -\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{\alpha\beta} \Delta_1(x). \tag{19.9}$$

## 19.2. *Chronological and Normal Products of Operators of the Electron-Positron Field*

We now define the chronological and the normal products of the operators of the electron-positron field, analogous to the corresponding products of the operators of the electromagnetic field, which are needed for our subsequent development.

If $\varphi_1(x_1)$ and $\varphi_2(x_2)$ are any two components of the field operators $\psi$ and $\bar{\psi}$, then their *chronological* product denoted by $T(\varphi_1(x_1)\varphi_2(x_2))$, is defined as follows:

$$T\big(\varphi_1(x_1)\varphi_2(x_2)\big) = \begin{cases} \varphi_1(x_1)\varphi_2(x_2), & t_1 > t_2, \\ -\varphi_2(x_2)\varphi_1(x_1), & t_1 < t_2. \end{cases} \tag{19.10}$$

We note that in contrast to the chronological product of operators of the electromagnetic field the chronological product of operators of the electron-positron field changes sign when the operators $\varphi_1$ and $\varphi_2$ are commuted. This is in accordance with the difference in the commutation rules for the electromagnetic and the electron-positron fields.

The chronological product of an arbitrary number of field components $\varphi_1(x_1)$, $\varphi_2(x_2)$, ..., $\varphi_n(x_n)$ is defined by the formula

$$T\big(\varphi_1(x_1)\varphi_2(x_2)\dots\big) = \delta_P \varphi_{i_1}(x_{i_1})\varphi_{i_2}(x_{i_2})\dots\varphi_{i_n}(x_{i_n}), \tag{19.11}$$

where the operators $\varphi_{i_1}(x_{i_1})$, $\varphi_{i_2}(x_{i_2})$ ... are arranged in chronological order, i.e., so that $t_{i_1} > t_{i_2} > \dots t_{i_n}$ and $\delta_P$ is equal to $+1$ or $-1$ depending on whether the permutation $(1, 2, \dots) \to (i_1, i_2, \dots, i_n)$ is even or odd.

The chronological product of field operators defined in this manner is relativistically invariant. Indeed, if the interval between the points is time-like then this assertion is obvious, since in this case the inequality $t_1 > t_2$ in the $K$ system leads to the inequality $t'_1 > t'_2$ in the other system $K'$. We therefore consider the case when the interval between $x_1$ and $x_2$ is space-like, with the inequality $t_1 > t_2$ holding in the $K$ coordinate system, and the inequality $t'_1 < t'_2$ holding in the $K'$ coordinate system. Then in the $K$ system the $T$-product of the operators $\varphi_1(x_1)$ and $\varphi_2(x_2)$ can be written in the form

$$T\big(\varphi_1(x_1)\varphi_2(x_2)\big) = \varphi_1(x_1)\varphi_2(x_2), \qquad t_1 > t_2.$$

It would appear that on the basis of this definition we should write the $T$-product in the $K'$ system in the following form:

$$T\big(\varphi_1'(x_1')\varphi_2'(x_2')\big) = \varphi_1'(x_1')\varphi_2'(x_2'), \quad t_1' < t_2',$$

where $\varphi_1'$ and $\varphi_2'$ are the values of the field operators in the $K'$ system. At first sight this expression contradicts the definition of the $T$-product, since if in the $K'$ system $t_1' < t_2'$, then in accordance with (19.10) the $T$-product in the $K'$ system should be written in the form

$$T\big(\varphi_1'(x_1')\varphi_2'(x_2')\big) = -\varphi_2'(x_2')\varphi_1'(x_1'), \quad t_1' < t_2'.$$

However, actually the two expressions are identical, since for a space-like interval the operators $\varphi_1'$ and $\varphi_2'$ anticommute.

We consider also the *normal* product of field operators in which the particle annihilation operators are placed to the right of the particle creation operators. If $\psi = \mathrm{A} + \mathrm{B}^+$, $\bar{\psi} = \mathrm{A}^+ + \mathrm{B}$, where A and B are respectively the electron and positron annihilation operators, while $\mathrm{A}^+$ and $\mathrm{B}^+$ are the creation operators for these particles, then the normal product of the operators $\psi_1, \psi_2, \ldots$, denoted by $N(\psi_1 \psi_2 \ldots)$ is defined by the distributive law $N\big(\psi_1\psi_2(\mathrm{A}+\mathrm{B}^+)\psi_3 \ldots\big) = N(\psi_1\psi_2\mathrm{A}\psi_3\ldots) + N(\psi_1\psi_2\mathrm{B}^+\psi_3 \ldots)$ and by the rule, according to which

$$N(Z_1 Z_2 \ldots Z_n) = \delta_P Z_{i_1} Z_{i_2} \ldots Z_{i_n}, \tag{19.12}$$

where each of the operators $Z_1, Z_2, \ldots$ is either a particle creation operator ($\mathrm{A}^+$, $\mathrm{B}^+$) or a particle annihilation operator (A, B), with these operators arranged in the right hand side of (19.12) in such a way that the creation operators are placed to the left of the annihilation operators and $\delta_P$ is equal to $+1$ or $-1$ depending on whether the permutation $(1, 2, \ldots, n) \to (i_1, i_2, \ldots, i_n)$ is even or odd.

We give examples of $N$-products:

$$\begin{aligned}
N\big(\psi(x)\bar{\psi}(y)\big) &= N\big[\big(\mathrm{A}(x)+\mathrm{B}^+(x)\big)\big(\mathrm{A}^+(y)+\mathrm{B}(y)\big)\big] \\
&= -\mathrm{A}^+(y)\,\mathrm{A}(x)+\mathrm{A}(x)\,\mathrm{B}(y) \\
&\qquad +\mathrm{B}^+(x)\,\mathrm{A}^+(y)+\mathrm{B}^+(x)\,\mathrm{B}(y), \\
N\big(\bar{\psi}(y)\psi(x)\big) &= \mathrm{A}^+(y)\,\mathrm{A}(x)+\mathrm{B}(y)\,\mathrm{A}(x) \\
&\qquad +\mathrm{A}^+(y)\,\mathrm{B}^+(x)-\mathrm{B}^+(x)\,\mathrm{B}(y), \\
N\big(\psi(x)\psi(y)\big) &= \psi(x)\psi(y), \\
N\big(\bar{\psi}(x)\bar{\psi}(y)\big) &= \bar{\psi}(x)\bar{\psi}(y).
\end{aligned} \tag{19.13}$$

From this and from the field quantization rules (18.8) it follows that

$$N\big(\psi(x)\bar{\psi}(y)\big)+N\big(\bar{\psi}(y)\psi(x)\big) = 0.$$

We shall show that, just as in the case of the electromagnetic field, the difference between the chronological and the normal products of the operators of the electron-positron field, which we shall call the pairing of these operators, does not contain any particle annihilation or creation operators and represents a $c$-number.

We consider the pairing of the operators $\psi_\alpha(x)$ and $\psi_\beta(x')$. By utilizing the expansions (18.19) and the quantization rules (18.8), we obtain

$$T\big(\psi_\alpha(x)\bar{\psi}_\beta(x')\big)-N\big(\psi_\alpha(x)\bar{\psi}_\beta(x')\big) = \begin{cases} -iS^+_{\alpha\beta}(x-x'), & t' < t, \\ \ \ iS^-_{\alpha\beta}(x-x'), & t' > t, \end{cases} \tag{19.14}$$

where the functions $S^\pm_{\alpha\beta}(x)$ are defined by formulas (19.2). In accordance with (19.3) and (17.39′) this expression can be written in the form

$$T\big(\psi_\alpha(x)\bar{\psi}_\beta(x')\big)-N\big(\psi_\alpha(x)\bar{\psi}_\beta(x')\big) = S^c_{\alpha\beta}(x-x'), \tag{19.14′}$$

where

$$S^c_{\alpha\beta}(x) = -\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{\alpha\beta} \Delta^c(x), \tag{19.15}$$

and $\Delta^c(x)$ is defined by formulas (17.36) and (17.39).

By proceeding in a similar manner we can show that the pairing of the operators $\psi_\alpha$ and $\psi_\beta$, and also of the operators $\bar{\psi}_\alpha$ and $\bar{\psi}_\beta$ vanishes. Thus, the following relations hold:

$$\begin{aligned} \psi^a_\alpha(x)\bar{\psi}^a_\beta(y) &= S^c_{\alpha\beta}(x-y), \\ \bar{\psi}^a_\beta(y)\psi^a_\alpha(x) &= -S^c_{\alpha\beta}(x-y), \\ \psi^a_\alpha(x)\psi^a_\beta(y) &= 0, \\ \bar{\psi}^a_\alpha(x)\bar{\psi}^a_\beta(y) &= 0, \end{aligned} \tag{19.16}$$

where $\varphi^a_1\varphi^a_2$ denotes the pairing of the operators $\varphi_1$ and $\varphi_2$ (in place of the superscript $a$ we shall also use other letters of the Latin alphabet).

Since the expectation value of the normal product of any field operators $\varphi_1$ and $\varphi_2$ is obviously equal to zero in the vacuum state $\langle N(\varphi_1\varphi_2)\rangle_0 = 0$, then the expectation value in this state of their chronological product coincides with the pairing of the operators $\varphi_1$ and $\varphi_2$

$$\langle T(\varphi_1\varphi_2)\rangle_0 = \varphi^a_1\varphi^a_2.$$

In particular, we have

$$\langle T(\psi_\alpha(x)\bar{\psi}_\beta(y))\rangle_0 = S^c_{\alpha\beta}(x-y). \tag{19.17}$$

We expand the function $S^c(x)$ into the Fourier integral

$$S^c_{\alpha\beta}(x) = 1/(2\pi)^4 \int S^c_{\alpha\beta}(p)e^{ipx}d^4p.$$

By utilizing the expression (19.15) for the function $S^c_{\alpha\beta}(x)$ and (17.36), we obtain

$$S^c_{\alpha\beta}(p) = i\,\frac{(i\hat{p}-m)_{\alpha\beta}}{p^2+m^2-i0}\,. \tag{19.18}$$

This formula will play a particularly important role in subsequent developments.

Since $-(i\hat{p}-m)(i\hat{p}+m) = p^2+m^2$, the function $S^c_{\alpha\beta}(p)$ can be represented in the following symbolic form:

$$S^c(p) = \frac{-i}{i\hat{p}+m}, \qquad m \to m - i0. \tag{19.18'}$$

In this expression we do not explicitly write in the denominator the imaginary quantity $-i|\eta|$, $|\eta| \to 0$, but we regard the mass $m$ as complex with an infinitesimal negative imaginary part.

From formulas (19.15) and (17.36′) it follows that $S^c(x)$ satisfies the following equation:

$$(i\hat{\mathrm{p}}+m)S^c(x) = -\,i\delta(x), \quad \hat{\mathrm{p}} = -i\gamma_\mu\frac{\partial}{\partial x_\mu}\,. \tag{19.19}$$

Thus, $iS^c(x)$ is the Green's function for the Dirac equation.

We also note that the function $S^c(x)$ can be expressed in terms of $S^+(x)$ and $S^-(x)$ or in terms of $S(x)$ and $S^{(1)}(x)$ in analogy wtih $\Delta^c(x)$:

$$S^c(x) = -i\big(\theta(x)S^+(x)-\theta(-x)S^-(x)\big) = \tfrac{1}{2}\big(S^{(1)}(x)-i\xi(x)S(x)\big). \tag{19.20}$$

In evaluating the pairings of operators we have expanded $\psi$ and $\bar{\psi}$ in terms of plane waves. However, we could have started with the general expansion (18.7) of the operator $\psi$ in terms of the complete system of functions $\psi^{(+)}_r$ and $\psi^{(-)}_r$, which represent the electron eigenfunctions in some stationary external field. Such an expansion has to be used if the external field cannot be regarded as a perturbation, but must be taken into account exactly in the electron wave functions. On repeating the calculations leading to (19.14′) it can be easily shown that in this case the pairing of the operators $\psi_\alpha(x)$ and $\psi_\beta(x')$ is defined by the following general formula:

$$\psi^a_\alpha(x)\bar{\psi}^a_\beta(x') \equiv S^{(e)}_{c\alpha\beta}(x, x') = \begin{cases} \sum\limits_r \psi^{(+)}_{r\alpha}(x)\bar{\psi}^{(+)}_{r\beta}(x'), & t' < t, \\ -\sum\limits_r \psi^{(-)}_{r\alpha}(x)\bar{\psi}^{(-)}_{r\beta}(x'), & t' > t. \end{cases} \tag{19.21}$$

We note that the function $iS^{(e)}_c(x, x')$ is the Green's function for the Dirac equation for an electron in an external field.

### 19.3. *Electric Current Density*

In § 9 we have introduced the electron current density vector $s_\mu(x)$. On multiplying this vector by the electron charge $e$ we obtain the four-dimensional electric current density vector for a single electron:

$$j_\mu = ie\bar{\psi}\gamma_\mu\psi. \tag{19.22}$$

If in this expression we interpret $\psi$ and $\bar{\psi}$ as the operators of the electron-positron field, then the vector $j_\mu$ can be interpreted as the electric current density associated with the electron-positron field.

We note that the electric current density vector is related by a simple expression to the Lagrangian of the electron-positron field:

$$j_\mu = -ie\left(\frac{\partial L}{\partial[\partial\psi_\lambda/\partial x_\mu]}\psi_\lambda - \bar{\psi}_\lambda\frac{\partial L}{\partial[\partial\bar{\psi}_\lambda/\partial x_\mu]}\right). \tag{19.23}$$

Indeed, on substituting into this expression the expressions for the derivatives $\partial L/\partial(\partial\psi_\lambda/\partial x_\mu)$ and $\partial L/\partial(\partial\bar{\psi}_\lambda/\partial x_\mu)$ from (18.2), we obtain (19.22).

Formula (19.22) shows that the charge density $\varrho = -ij_4$ and the three-dimensional current density $\mathbf{j}$ are defined by the following expressions:

$$\varrho = e\bar{\psi}\gamma_4\psi = e\psi^+\psi, \quad \mathbf{j} = ie\bar{\psi}\boldsymbol{\gamma}\psi = e\psi^+\boldsymbol{\alpha}\psi. \tag{19.24}$$

As may be easily seen with the aid of the Dirac equations (18.3) they satisfy the equation of continuity

$$\frac{\partial}{\partial x_\mu}\mathrm{j}_\mu = 0. \tag{12.22'}$$

The total charge $Q$ associated with the field is given by

$$Q = \int \varrho\, dr = e\int \psi^+\psi\, dr. \tag{19.25}$$

On substituting into this expression in place of $\psi$ and $\bar{\psi}$ the expansions (18.7) and on utilizing the quantization rules (18.8) and also the orthonormal properties of the functions $\psi_r^{(+)}$ and $\psi_r^{(-)}$ we can easily show that $Q = e\left(\sum_r \mathrm{a}_r^+ \mathrm{a}_r + \sum_r \mathrm{b}_r \mathrm{b}_r^+\right) = e\left(\sum_r n_r^{(+)} + \sum_r (1-n_r^{(-)})\right)$, where $n_r^{(+)}$ and $n_r^{(-)}$ are the numbers of electrons and of positrons in the corresponding states. On setting $n_r^{(+)} = n_r^{(-)} = 0$ we obtain the charge in the vacuum state $Q_0 = e\sum_r 1$, where the summation is taken over states of negative frequencies.

This quantity is, evidently, infinite. Therefore, just as in the case of our discussion of the energy of the electron-positron field only the charge of the field with respect to vacuum has a meaning. This quantity, equal to

$$q = Q - Q_0 = e\sum_r n_r^{(+)} - e\sum_r n_r^{(-)}, \tag{19.26}$$

represents the sum of the charges of the individual particles—the electrons and the positrons.

The foregoing expressions for the energy and for the charge of the vacuum can be interpreted in the following manner: The vacuum represents a state in which all the negative energy levels of an individual electron are filled. This infinite "negative sea" of electron is not directly observable in itself, but under the action of various external fields an electron can make a transition from a state of negative energy to a state of positive energy (this will obviously require an energy not less than $2m$), giving rise to a "hole" in the infinite negative sea of electrons, which will behave as a particle of positive energy, with its charge opposite in sign, but equal in absolute value, to the electron charge. Such a "hole" in the infinite negative electron distribution can be interpreted as a positron, while the creation of the "hole," i.e., the transition of an electron from a state of negative energy to a state of positive energy, can be interpreted as the creation of an electron-positron pair.

In the theory of the quantized electron-positron field there exists a possibility of defining the current density in a somewhat different manner, viz., in such a way that the charge in the vacuum state is equal to zero. Indeed, with the aid of the operators $\bar{\psi}_\alpha(x)\psi_\beta(x')$ and $-\psi_\beta(x')\bar{\psi}_\alpha(x)$ which differ only by a $c$-number, we can construct the four-vector

$$\mathrm{j}_\mu(x) = \frac{ie}{2}(\gamma_\mu)_{\alpha\beta}\left(\bar{\psi}_\alpha(x)\psi_\beta(x) - \psi_\beta(x)\bar{\psi}_\alpha(x)\right), \tag{19.27}$$

which, like the vector (19.22), satisfies the equation of continuity (19.22′). Therefore, this vector can be regarded, with as much justification as the vector (19.22), as the electric current density associated with the electron-positron field. As can be easily shown, the total charge of the field will in this case be given by

$$q = -i\int \mathrm{j}_4 dr = \frac{e}{2}\sum_r [\mathrm{a}_r^+, \mathrm{a}_r] - \frac{e}{2}\sum_r [\mathrm{b}_r^+, \mathrm{b}_r] = e\sum_r n_r^{(+)} - e\sum_r n_r^{(-)}.$$

This expression does not contain the infinite charge of the vacuum, and coincides with expression (19.26).

Formula (19.27) may be written in the following abbreviated form:

$$\mathrm{j}_\mu(x) = \frac{ie}{2}[\bar{\psi}(x), \gamma_\mu \psi(x)] = \frac{ie}{2}\big(\bar{\psi}(x)\gamma_\mu\psi(x) - \psi(x)\tilde{\gamma}_\mu\bar{\psi}(x)\big), \quad (19.27')$$

where $\tilde{\gamma}_\mu$ is the matrix transposed with respect to $\gamma_\mu$. On introducing the charge-conjugate field operators

$$\psi^c(x) = \mathrm{C}\bar{\psi}(x), \qquad \bar{\psi}^c(x) = \mathrm{C}^{-1}\psi(x)$$

and on noting that in accordance with (10.8) we have

$$\psi(x)\tilde{\gamma}_\mu\bar{\psi}(x) = \bar{\psi}^c(x)\gamma_\mu\psi^c(x)$$

we can put formula (19.27′) in the form

$$\mathrm{j}_\mu(x) = \frac{ie}{2}\big(\bar{\psi}(x)\gamma_\mu\psi(x) - \bar{\psi}^c(x)\gamma_\mu\psi^c(x)\big). \quad (19.28)$$

This expression will remain unaltered if we replace the operators $\psi(x)$ and $\bar{\psi}(x)$ by the charge-conjugate operators and at the same time reverse the sign of the charge $e$.

We now show that the current density can be written in the form of a normal product:

$$\mathrm{j}_\mu(x) = ieN\big(\bar{\psi}(x)\gamma_\mu\psi(x)\big). \quad (19.29)$$

In order to do this we substitute into (19.27) the expressions

$$\psi_\alpha(x) = \mathrm{A}_\alpha(x) + \mathrm{B}_\alpha^+(x) \quad \text{and} \quad \bar{\psi}_\beta(x) = \mathrm{A}_\beta^+(x) + \mathrm{B}_\beta(x),$$

where A($x$) and B($x$) stand for the terms in the expansion (18.19) containing positive and negative frequencies, respectively:

$$\mathrm{j}_\mu(x) = \frac{ie}{2}(\gamma_\mu)_{\alpha\beta}\big(\overline{\psi}_\alpha(x)\psi_\beta(x) - \psi_\beta(x)\overline{\psi}_\alpha(x)\big)$$

$$= \frac{ie}{2}(\gamma_\mu)_{\alpha\beta}\big(A^+_\alpha(x)A_\beta(x) + \mathrm{A}^+_\alpha(x)\mathrm{B}^+_\beta(x) + \mathrm{B}_\alpha(x)\mathrm{A}_\beta(x)$$

$$+ \mathrm{B}_\alpha(x)\mathrm{B}^+_\beta(x) - \mathrm{A}_\beta(x)\mathrm{A}^+_\alpha(x) - \mathrm{A}_\beta(x)\mathrm{B}_\alpha(x) - \mathrm{B}^+_\beta(x)\mathrm{A}^+_\alpha(x) - \mathrm{B}^+_\beta(x)\mathrm{B}_\alpha(x)\big).$$

On utilizing the quantization rules (18.8) and formulas (10.28), (10.33), we can easily show that

$$\{\mathrm{A}^+_\alpha(x),\, \mathrm{B}^+_\beta(x)\} = \{\mathrm{A}_\alpha(x),\, \mathrm{B}_\beta(x)\} = 0,$$

$$\{\mathrm{A}^+_\alpha(x),\, \mathrm{A}_\beta(x)\} = \frac{1}{\Omega}\sum_{\boldsymbol{p};\, r=1,2} \overline{u}^r_\alpha(\boldsymbol{p})u^r_\beta(\boldsymbol{p}) = \frac{1}{\Omega}\sum_{\boldsymbol{p}}\frac{1}{2\varepsilon}(i\hat{p}-m)_{\alpha\beta},$$

$$\{\mathrm{B}_\alpha(x),\, \mathrm{B}^+_\beta(x)\} = \frac{1}{\Omega}\sum_{\boldsymbol{p};\, r=1,2} \overline{v}^r_\alpha(-\boldsymbol{p})v^r_\beta(-\boldsymbol{p}) = \frac{1}{\Omega}\sum_{\boldsymbol{p}}\frac{1}{2\varepsilon}(i\hat{p}+m)_{\alpha\beta}.$$

Therefore we have

$$\mathrm{j}_\mu(x) = ie(\gamma_\mu)_{\alpha\beta}\big(\mathrm{A}^+_\alpha(x)\mathrm{A}_\beta(x) + \mathrm{B}_\alpha(x)\mathrm{A}_\beta(x) + \mathrm{A}^+_\alpha(x)\mathrm{B}^+_\beta(x) - \mathrm{B}^+_\beta(x)\mathrm{B}_\alpha(x)\big)$$

$$+ ie(\gamma_\mu)_{\alpha\beta}\frac{1}{2\Omega}\left(\sum_{\boldsymbol{p}}\frac{1}{2\varepsilon}(i\hat{p}+m)_{\alpha\beta} - \sum_{\boldsymbol{p}}\frac{1}{2\varepsilon}(i\hat{p}-m)_{\alpha\beta}\right).$$

The first term in this expression represents, in accordance with (19.13), the normal product of the operators $\overline{\psi}_\alpha(x)$ and $\psi_\beta(x)$, while the second vanishes, since $(\gamma_\mu)_{\alpha\alpha} = 0$.

## § 20. General Properties of Wave Fields

### 20.1. *Wave Functions of a Field and the Lorentz Group*

In the preceding sections we have studied in detail the properties of the electromagnetic and the electron-positron fields. The theory of these fields can be generalized to the case of wave fields describing particles of arbitrary spin. Such a general theory of wave fields may be of importance in connection with the existence of mesons and of other particles. On the other hand, a general theory of wave fields will enable us to understand more clearly the uniqueness of the structure of the theory of the electromagnetic and the electron-positron fields within the framework of present physical ideas. Therefore, in this and in the following sections we shall discuss certain general properties of wave fields.

The field is defined at the point $x(x_1, x_2, x_3, x_4)$ by one or by several functions $\psi_i(x)$, which are called the wave functions of the field. In principle, the number of such functions may be infinitely great. The wave functions of the field satisfy certain differential equations— the field equations—which must be invariant under Lorentz transformations.

Since arguments associated with relativistic invariance play an important role in the theory of wave fields, we shall recall here the main properties of Lorentz transformations.

We shall give the name of the *Lorentz group* to the set of transformations

$$x'_\mu = l_{\mu\nu} x_\nu \qquad (\mu, \nu = 1, 2, 3, 4) \tag{20.1}$$

or, in abbreviated form, $x' = lx$, which leave invariant the quadratic form $s^2 = -x_\mu^2 = -x_\mu'^2$, which have a determinant equal to unity, and which do not reverse the direction of the time.

The Lorentz group is sometimes referred to as the *proper* Lorentz group in contrast to the *full* Lorentz group which, in addition to the elements contained in the proper Lorentz group, also contains elements associated with the inversion of spatial coordinates. We can say that the full Lorentz group represents the set of all the transformations (20.1) which preserve the magnitude of the interval, which have a determinant equal to $\pm 1$ and which do not reverse the direction of the time.

The wave functions of the field $\psi_i(x)$ are defined in each coordinate system, with the functions $\psi_i(x')$ in the system $K'$ being expressed linearly in terms of the functions $\psi_i(x)$ in the system $K$: $\psi'_i(x') = S_{ij}\psi_j(x)$, $x' = lx$, or, in abbreviated form,

$$\psi'(x') = S\psi(x). \tag{20.2}$$

The linearity of this transformation is associated with the equivalence of inertial coordinate systems. We note that the arguments of the functions $\psi'_i$ and $\psi_j$ are related by the Lorentz transformation.

The matrix S is defined by the Lorentz transformation which transforms the coordinate system $K$ into the coordinate system $K'$:

$$S = S(l). \tag{20.2'}$$

Since the transformation from the system $K$ to the system $K'$, followed by a subsequent transformation from $K'$ to a new system $K''$, is equivalent to a direct transformation from $K$ to $K''$ the following relation holds:

$$\mathrm{S}(\mathrm{l}')\,\mathrm{S}(\mathrm{l}) = \mathrm{S}(\mathrm{l}'\mathrm{l}), \tag{20.3}$$

where $\mathrm{l}'$ is the Lorentz transformation which transforms the coordinate system $K'$ into the coordinate system $K''$. This equation shows that the set of transformations S of the wave functions $\psi$ gives a *representation* of the proper Lorentz group[1] in wave function space. We denote this space by $R$ and the representation by $D_R$ (the number of dimensions of the space $R$ is equal to the number of the functions $\psi$ and may be either finite or infinite).

If for the wave functions of the field we choose some linear combinations of the $\psi$, $\varphi_i = A_{ij}\psi_j$ or, in abbreviated form, $\varphi = \mathrm{A}\psi$, then the new wave functions will transform like $\varphi_i' = A_{ij}\psi_j' = A_{ij}S_{jk}\psi_k = A_{ij}S_{jk}\,A_{kl}^{-1}\,\varphi_l$ or, in abbreviated form, $\varphi' = \mathrm{ASA}^{-1}\varphi$. Thus, by making a new choice of the wave functions we go over from the representation of the Lorentz group $\mathrm{l} \to \mathrm{S}$ to the representation $l \to \mathrm{ASA}^{-1}$. Since both these representations are associated with the same field they are equivalent from a physical point of view; therefore such representations are said to be *equivalent.*

If the wave functions $\psi_i$ can be divided into two classes: $\psi_1, \psi_2, \ldots, \psi_k$ and $\psi_{k+1}, \psi_{k+2}, \ldots, \psi_n$ in such a way that under Lorentz transformations the functions of each class go over into linear combinations of functions of the same class, i.e.,

$$\psi_i' = \sum_{j=1}^{k} S_{ij}\psi_j \qquad (i = 1, 2, \ldots, k),$$

$$\psi_i' = \sum_{j=k+1}^{n} \mathrm{S}_{ij}\psi_j \qquad (i = k+1, k+2, \ldots, n).$$

---

[1] We recall that if $G$ is any group, and if we set in correspondence with each element $g_i$ of the group an operator $\mathrm{T}_{g_i}$ in such a way that $\mathrm{T}_{g_i}\mathrm{T}_{g_k} = \mathrm{T}_{g_i g_k}$ then the correspondence $g \to \mathrm{T}_g$ is said to be *a representation of the group G.* The number of dimensions of the space in which the operators $\mathrm{T}_g$ operate is said to be the *dimension* of the representation.

then the representation $D_R$ is said to be *reducible*. It is clear that every reducible representation can be made up of representations of lower dimensions. If the latter are in turn reducible, they can be further broken up into representations of still lower dimensions. By repeating this process we finally arrive at the *irreducible* representations into which the representation $D_R$ can be decomposed. Thus, in order to characterize the representation $D_R$ it is sufficient to specify the irreducible representations of which it is composed, and to specify the number of times that any such representation occurs in $D_R$.

### 20.2. *Irreducible Finite-Dimensional Representations of the Lorentz Group*

In order to obtain all the irreducible representations of the Lorentz group it is sufficient to discuss the infinitesimal Lorentz transformations, i.e., the transformations which are "infinitely close" to the identity transformation:

$$x'_\mu = x_\mu + \varepsilon_{\mu\nu} x_\nu, \qquad |\varepsilon_{\mu\nu}| \ll 1. \tag{20.4}$$

Since the interval $x_\mu^2$ is an invariant, then in the first approximation the matrix $\varepsilon_{\mu\nu}$ must be $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$, which is antisymmetric. Therefore, the number of independent parameters defining an infinitesimal Lorentz transformation is equal to six.

We denote by $\mathrm{S}(\varepsilon_{\mu\nu})$ the matrix S which in some representation of the Lorentz group $D_R$ corresponds to the transformation (20.4). On expanding $\mathrm{S}(\varepsilon_{\mu\nu})$ into a series in powers of $\varepsilon_{\mu\nu}$, and on restricting ourselves to the linear terms we obtain

$$\mathrm{S}(\varepsilon_{\mu\nu}) = 1 + \sum_{\mu>\nu} \varepsilon_{\mu\nu} \mathrm{I}_{\mu\nu} = 1 + \tfrac{1}{2} \varepsilon_{\mu\nu} \mathrm{I}_{\mu\nu}, \qquad \mathrm{I}_{\mu\nu} = -\mathrm{I}_{\nu\mu}, \tag{20.4'}$$

where 1 is the unit matrix. The matrices $\mathrm{I}_{\mu\nu}$ are called the *infinitesimal* operators. They completely determine the representation of the group.

The derivation of the infinitesimal operators is based on the following theorem: The commutator of two infinitesimal operators

$$[\mathrm{I}_{\mu\nu}, \mathrm{I}_{\lambda\varrho}] = \mathrm{I}_{\mu\nu}\mathrm{I}_{\lambda\varrho} - \mathrm{I}_{\lambda\varrho}\mathrm{I}_{\mu\nu}$$

is equal to the sum of infinitesimal operators multiplied by certain coefficients $c$, which are the same for all the representations of the group (185):

$$[\mathrm{I}_{\mu\nu}, \mathrm{I}_{\lambda\varrho}] = \sum_{\sigma\tau} c_{\mu\nu\lambda\varrho;\sigma\tau} \mathrm{I}_{\sigma\tau}.$$

The coefficients $c$ can be obtained if at least one representation of the group is known. By choosing the Lorentz group itself as a representation of the Lorentz group we can show that the infinitesimal operators of the Lorentz group satisfy the relations

$$[\mathrm{I}_{\mu\nu}, \mathrm{I}_{\lambda\varrho}] = \delta_{\mu\varrho}\mathrm{I}_{\nu\lambda} - \delta_{\nu\varrho}\mathrm{I}_{\mu\lambda} - \delta_{\mu\lambda}\mathrm{I}_{\nu\varrho} + \delta_{\nu\lambda}\mathrm{I}_{\mu\varrho} \tag{20.5}$$

or

$$\begin{aligned} [\mathrm{I}_{kl}, \mathrm{I}_{mn}] &= \delta_{km}\mathrm{I}_{lm} - \delta_{ln}\mathrm{I}_{km} - \delta_{km}\mathrm{I}_{ln} + \delta_{lm}\mathrm{I}_{kn}, \\ [\mathrm{I}_{4k}, \mathrm{I}_{mn}] &= \delta_{km}\mathrm{I}_{4n} - \delta_{kn}\mathrm{I}_{4m}, \\ [\mathrm{I}_{4k}, \mathrm{I}_{4m}] &= -\mathrm{I}_{km}, \end{aligned} \tag{20.5'}$$

where $k, l, m, n = 1, 2, 3$.

We note that $\mathrm{I}_{lm} (l, m = 1, 2, 3)$ represent the infinitesimal operators of the rotation group. For the wave field defined by the Dirac equation the infinitesimal operators can be expressed in terms of the matrices $\gamma$ (cf. § 9):

$$\mathrm{I}_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).$$

Relations (20.5) enable us to determine the form of the operators $\mathrm{I}_{\mu\nu}$. Indeed, we introduce the operators

$$\mathrm{I}_l = -\frac{i}{2}\,\mathrm{e}_{lmn}\mathrm{I}_{mn} \qquad (l, m, n = 1, 2, 3), \tag{20.6}$$

where $\mathrm{e}_{lmn}$ is the antisymmetric unit tensor of the third rank. Then we have

$$\mathrm{I}_{mn} = i\mathrm{e}_{mnp}\mathrm{I}_p$$

and the relations (20.5′) assume the form

$$\begin{aligned} [\mathrm{I}_p, \mathrm{I}_r] &= i\mathrm{e}_{prs}\mathrm{I}_s, \\ [\mathrm{I}_{4p}, \mathrm{I}_{4r}] &= -i\mathrm{e}_{prs}\mathrm{I}_s, \\ [\mathrm{I}_{4p}, \mathrm{I}_r] &= i\mathrm{e}_{prs}\mathrm{I}_{4s}. \end{aligned}$$

From this it follows that the operators

$$\mathrm{X}_p = \tfrac{1}{2}(\mathrm{I}_p + i\mathrm{I}_{4p}), \qquad \mathrm{Y}_p = \tfrac{1}{2}(\mathrm{I}_p - i\mathrm{I}_{4p}) \tag{20.7}$$

satisfy the following commutation relations:

$$\begin{aligned} [\mathrm{X}_p, \ \mathrm{X}_s] &= i\mathrm{e}_{pst}\mathrm{X}_t, \\ [\mathrm{Y}_p, \ \mathrm{Y}_s] &= i\mathrm{e}_{pst}\mathrm{Y}_t, \\ [\mathrm{X}_p, \ \mathrm{Y}_s] &= 0. \end{aligned} \tag{20.7'}$$

We see that the operators $X_p$ commute with the operators $Y_s$, and that both the former and the latter satisfy the same commutation relations. These relations coincide with the commutation relations for the operators of the components of angular momentum. From this it follows that in their properties the operators $X_p$ and $Y_s$ do not differ from the operators of the components of angular momentum. In particular, it is possible to choose a representation in which the operators $X_3$ and $X^2 = X_1^2+X_2^2+X_3^2$, and also $Y_3$ and $Y^2 = Y_1^2+Y_2^2+Y_3^2$ will be diagonal. We denote the eigenvalues of the operators $X_3$ and $Y_3$ by $m_j$ and $m_k$; as is well known, they can assume the values

$$m_j = -j, -j+1, \dots j; \quad m_k = -k, -k+1, \dots k, \tag{20.8}$$

where $j$ and $k$ are positive integers or half-integers. The numbers $j$ and $k$ determine the eigenvalues of the squares of the "angular momenta" $X^2$ and $Y^2$, which are equal to $j(j+1)$ and $k(k+1)$.

Thus, the infinitesimal operators of the Lorentz group are completely determined by the two numbers, $j$ and $k$, which can independently take on integral or half-integral values.

As is well known, rotations of coordinates in ordinary three-dimensional space correspond to certain linear transformations of the eigenfunctions of the operators representing the square of the angular momentum and one of its components. These transformations form a representation of the rotation group. We can now say that the transformations of the Lorentz group correspond to certain linear transformations of eigenfunctions of the operators $X_3$, $X^2$, $Y_3$, $Y^2$, which form a representation of the Lorentz group. This representation, which turns out to be irreducible, is denoted by $\tau \sim (j, k)$. The dimension of the representation $\tau \sim (j, k)$ is equal to the number of eigenfunctions of the operators $X_3$, $X^2$, $Y_3$, $Y^2$,; i.e., $(2j+1)(2k+1)$.

We note that if we go over from the wave functions $\psi$ to the complex-conjugate functions $\psi^*$, then the operators $I_{\mu\nu}$ are replaced by the operators $I_{\mu\nu}^*$, if $\mu, \nu = 1, 2, 3$, and by $-I_{\mu\nu}^*$, if one of the subscripts $\mu, \nu = 4$, while the operators $X_p$, $Y_p$ are replaced by the operators $X_p = -Y_p^*$, $Y_p = -X_p^*$. These operators differ from $Y_p$ and $X_p$ only by a unitary transformation. Therefore, if the functions $\psi$ transform according to the representation $\tau \sim (j, k)$, then the functions $\psi^*$ will transform according to the representation $\tau^* \sim (k, j)$.

Let the functions $\psi_i^{(1)}$ and $\psi_i^{(2)}$ transform according to the representations $\tau^{(1)} \sim (j_1, k_1)$ and $\tau^{(2)} \sim (j_2, k_2)$; then the functions $\psi_i^{(1)}\psi_i^{(2)}$ will also transform according to some representation of the Lorentz group which, however, will not be irreducible. It is called a *direct* product of the representations $\tau^{(1)}$ and $\tau^{(2)}$ and is denoted by $\tau^{(1)}\times\tau^{(2)}$. The direct product of representations can be decomposed into irreducible representations in accordance with

$$(j_1, k_1)\times(j_2, k^2) = \sum (j, k) \tag{20.9}$$

where $j$ and $k$ assume the values

$$j = |j_1-j_2|, \quad |j_1-j_2|+1, \dots, j_1+j_2,$$
$$k = |k_1-k_2|, \quad |k_1-k_2|+1, \dots, k_1+k_2.$$

In particular, we have

$$(j, 0)\times(0, k) = (j, k). \tag{20.9'}$$

An irreducible representation of the Lorentz group $\tau \sim (j, k)$ can be regarded, generally speaking, as a reducible representation of the rotation group. In order to decompose the representation $\tau \sim (j, k)$ into irreducible representations of the rotation group, we consider the operators $\mathrm{I}_p$ representing the infinitesimal operators of spatial rotations. According to (20.7) they are related to the operators $\mathrm{X}_p$ and $\mathrm{Y}_p$ by $\mathrm{I}_p = \mathrm{X}_p+\mathrm{Y}_p$. Since the operators $\mathrm{X}_p$ and $\mathrm{Y}_1$ commute, the eigenvalues $m$ of the operator $\mathrm{I}_3$ are equal to the sum of the eigenvalues $m_j$ and $m_k$ of the operators $\mathrm{X}_3$ and $\mathrm{Y}_3$, while the eigenfunctions $\chi_m$ of the operator $\mathrm{I}_3$ are equal to the product of the eigenfunctions $\varphi_{j,m_j}$ and $\varphi_{k,m_k}$ of the operators $\mathrm{X}_3$ and $\mathrm{Y}_3$

$$\chi_m = \varphi_{j,m_j}\varphi_{k,m_k}, \qquad m = m_j+m_k. \tag{20.10}$$

From this by utilizing (20.8) it can be easily shown that $m$ may assume the values

$$m = -l, -l+1, \dots, l,$$

where

$$l = |j-k|, |j-k|+1, \dots, j+k.$$

Spatial rotations correspond to linear transformations of the $2j+1$ functions $\varphi_{j,m_j}$. These transformations give an irreducible representation of the rotation group of dimension $2j+1$, which we denote by $D_j$.

In a similar manner the functions $\varphi_{k,m_k}$ transform under rotations in accordance with the representation $D_k$.

If one of the numbers $k$ or $j$ is equal to zero the representation $\tau \sim (j, k)$ regarded as a representation of the rotation group reduces to the representation $D_j$ or $D_k$:

$$(j, 0) = D_j \quad \text{and} \quad (0, k) = D_k.$$

It follows from (20.10) that the representation $\tau \sim (j, k)$ regarded as a representation of the rotation group is equal to the direct product of the representations $D_j$ and $D_k$ which decomposes into representations $D_l$ with $l = |j-k|, |j-k|+1, \ldots, j+k$,

$$(j, k) = D_j \times D_k = \sum D_l \quad \text{where} \quad l = |j-k|, \ldots, j+k. \tag{20.11}$$

The number of these representations is equal to $2j+1$, if $j \leqslant k$, and to $2k+1$, if $k \leqslant j$.

We can say that the representation $\tau \sim (j, k)$ regarded as a representation of the rotation group is equivalent to representations according to which the eigenfunctions of one of the components of the angular momentum of a system consisting of two particles with angular momenta $j$ and $k$ transform under rotations.

We must differentiate between single-valued and double-valued representations of the Lorentz group. In the case of the single-valued representations a spatial rotation through the angle $2\pi$ corresponds to the identity operator 1, while in the case of the double-valued representations it corresponds to the operator $-1$. It can be shown that the representation $\tau \sim (j, k)$ will be single-valued if the numbers $2j$ and $2k$ have the same parity, and double-valued if the numbers $2j$ and $2k$ are of opposite parity. The single-valued representations are also called *tensor* representations, while the double-valued representations are called *spinor* representations. Quantities which transform according to tensor representations are called *tensors*, while quantities transforming according to spinor representations are called *spinors.*

We give several examples. The representation $\tau \sim (0, 0)$ corresponds to a scalar. A vector transforms in accordance with the representation $\tau \sim (\frac{1}{2}, \frac{1}{2})$. A symmetric tensor of the second rank with a trace equal to zero transforms according to the representation $\tau \sim (1, 1)$ (the vanishing of the sum of the diagonal elements of the tensor is necessary because this sum is itself an invariant, and if it differs from zero then the tensor does not transform according to an irreducible transformation). The antisymmetric tensor of the second rank transforms according to the representations $\tau \sim (1, 0)$ and $\tau \sim (0, 1)$. The Dirac bispinor $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ which was investigated in subsection 9.3 transforms according to the representations $\tau \sim (\frac{1}{2}, 0)$ and $\tau \sim (0, \frac{1}{2})$.

Quantities that transform according to the irreducible representations of the Lorentz group $\tau \sim (j, k)$ can be divided into four classes: (1) the class $+1$ with integral $j$ and $k$, (2) the class $-1$ with half-integral $j$ and $k$, (3) the class $+\varepsilon$ with integral $j$ and half-integral $k$, and (4) the class $-\varepsilon$ with half-integral $j$ and integral $k$.

If a quantity $\psi$ which transforms according to the irreducible representation $D_R$ can be represented in the form of a sum of quantities belonging to the same class, then we say that the quantity $\psi$ also belongs to this class.

It can be easily shown that tensors with an even number of indices belong to the class $+1$, while those with an odd number of indices belong to the class $-1$; in particular, a vector belongs to the class $-1$.

Multiplication of quantities belonging to different classes is carried out in accordance with Table 8.

TABLE 8

| | $+1$ | $-1$ | $+\varepsilon$ | $-\varepsilon$ |
|---|---|---|---|---|
| $+1$ | $+1$ | $-1$ | $+\varepsilon$ | $-\varepsilon$ |
| $-1$ | $-1$ | $+1$ | $-\varepsilon$ | $+\varepsilon$ |
| $+\varepsilon$ | $+\varepsilon$ | $-\varepsilon$ | $+1$ | $-1$ |
| $-\varepsilon$ | $-\varepsilon$ | $+\varepsilon$ | $-1$ | $+1$ |

(20.12)

If a quantity $\psi$ that transforms according to the representation $\tau \sim (j, k)$ belongs to the class $+1$ or to the class $-1$, then the complex-conjugate quantity that transforms according to the representation $\tau^* \sim (k, j)$ belongs to the same class as $\psi$. However, if $\psi$ belongs to the class $+\varepsilon$ (or $-\varepsilon$), then $\psi^*$ belongs to the class $-\varepsilon$ (or $+\varepsilon$).

### 20.3 *Energy-Momentum Tensor and Angular Momentum Tensor*

Just as in the case of the electromagnetic and the electron-positron fields we assume that the wave field, regarded as a generalized dynamical system (176), can be made to correspond to a *Lagrangian* represented by the volume integral of a Lagrangian density $L(\psi, \partial\psi/\partial x_\mu)$. The latter depends on the wave functions of the field $\psi$ and on their first derivatives with respect to the coordinates and the time, but does not explicitly depend on $x$.

The integral of the Lagrangian with respect to time is called *action.* It can be represented in the form of an integral of $L$ over the four-dimensional volume $\Omega$:

$$J = \int_\Omega L(\psi, \psi_{,\mu}) d^4x, \tag{20.13}$$

where $\psi_{,\mu} = \partial\psi/\partial x_\mu$.

We assume that the principle of least action holds, according to which the action must be an extremum for actual motion, i.e., for those $\psi$ which satisfy the field equations. Since these equations must be relativistically invariant, the function $L$ must also be relativistically invariant.

The total variation of the action resulting from varying both the wave functions of the field and the boundaries of the region of integration is given by

$$\delta J = \int_\Omega \left(\frac{\partial L}{\partial\psi}\delta\psi + \frac{\partial L}{\partial\psi_{,\mu}}\delta\psi_{,\mu}\right) d^4x + \oint_\Sigma L\delta x_\mu d\sigma_\mu,$$

where the second integral is taken over the hypersurface $\Sigma$ bounding $\Omega$, while $\delta x_\mu$ represent the variation of the boundary coordinates. On integrating by parts the second term in the volume integral we can rewrite $\delta J$ in the form

$$\delta J = \int_\Omega \left(\frac{\partial L}{\partial\psi} - \frac{\partial}{\partial x_\mu}\frac{\partial L}{\delta\psi_{,\mu}}\right)\delta\psi\, d^4x + \oint_\Sigma \left(\frac{\partial L}{\partial\psi_{,\mu}}\delta\psi + L\delta x_\mu\right) d\sigma_\mu. \tag{20.14}$$

We consider first of all the variation of the action in the case of fixed boundaries and assume that the variation of the wave functions at the boundary is equal to zero. In this case the second integral vanishes, and by equating to zero the variation of the action we obtain the Lagrangian equations for the wave functions of the field $\psi_i$:

$$\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial(\partial\psi_i/\partial x_\mu)} - \frac{\partial L}{\partial\psi_i} = 0. \tag{20.15}$$

The actual "motion" of the field must satisfy these equations.

We now consider the variation of the action in the case of actual motion, by assuming that the region of integration $\Omega$ together with the wave field is subjected as a whole to an infinitesimal translation or to an infinitesimal rotation. The action will obviously be unaltered in this case, i.e., the variation of the action $\delta J$ will be equal to zero.

In the case of an infinitesimal translation, i.e., of the infinitesimal transformation of coordinates

$$x_\mu \to x'_\mu = x_\mu + \varepsilon_\mu,$$

where $\varepsilon_\mu$ is an infinitesimal four-vector, the wave functions are subjected to the infinitesimal transformation

$$\psi(x_\mu) \to \psi'(x_\mu) = \psi(x_\mu - \varepsilon_\mu) = \psi(x_\mu) - \varepsilon_\mu \psi_{,\mu},$$

whence it follows that

$$\delta\psi = \psi' - \psi = -\varepsilon_\mu \psi_{,\mu}.$$

On substituting this value for the variation of the wave function together with $\delta x_\mu = \varepsilon_\mu$ into the surface integral (20.14) we obtain

$$\delta J = \oint_\Sigma \varepsilon_\mu T_{\mu\nu}\, d\sigma_\nu = 0, \tag{20.16}$$

where

$$T_{\mu\nu} = L\delta_{\mu\nu} - \psi_{,\mu}\frac{\partial L}{\partial \psi_{,\nu}} \equiv L\delta_{\mu\nu} - \frac{\partial \psi_i}{\partial x_\mu}\frac{\partial L}{\partial(\partial\psi_i/\partial x_\nu)}. \tag{20.17}$$

This quantity is a four-dimensional tensor of the second rank to which we shall give the name of the *energy-momentum tensor.*

It follows from (20.16) that

$$\oint_\Sigma T_{\mu\nu}\, d\sigma_\nu = 0. \tag{20.16'}$$

On assuming that the volume $\Omega$ is bounded by two hypersurfaces oriented orthogonal to the time axis, i.e., $d\sigma_1 = d\sigma_2 = d\sigma_3 = 0$, and $d\sigma_4 = -i\boldsymbol{dr}$, where $\boldsymbol{dr}$ is an element of three-dimensional volume, we obtain from (20.16′) the conservation laws

$$-i\int T_{\mu 4}\, d\boldsymbol{r} = P_\mu = \text{const}, \tag{20.18}$$

where the integration is carried out over the total volume of the field, and the contributions from the surfaces at infinity are assumed to vanish.

Formula (20.18) shows that $P_\mu$ is a four-vector. It is called the *energy-momentum four-vector* of the field. The spatial components of $P_\mu$ define the momentum of the field **P**, while the time component divided by $i$ is equal to the energy of the field $H$.

The energy density of the field is evidently equal to $w = -T_{44}$. This expression can be obtained by analogy with classical mechanics if we use the well-known expression for the energy of the system

$$H = \sum_i p_i \dot{q}_i - L,$$

where $L$ is the Lagrangian and $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is the momentum canonically conjugate to the coordinate $q_i$. In field theory we must replace $q_i$ by $\psi_i$ and $p_i$ by $\partial L/\partial \dot{\psi}_i$, and in this way we obtain the following expression for the energy density:

$$w = \dot{\psi}(\partial L/\partial \dot{\psi}_i) - L = -T_{44}.$$

We note that the energy density, as a rule, is not positive definite, and that the condition $-T_{44} > 0$ holds only for certain special fields.

By applying Gauss' theorem to the surface integral (20.16′) we obtain

$$\partial T_{\mu\nu}/\partial x_\nu = 0. \tag{20.19}$$

Thus, the four-dimensional divergence of the energy-momentum tensor vanishes. By utilizing this relation it is possible to prove the conservation of the vector $P_\mu$, as was shown in § 16.

We note that if we add to $T_{\mu\nu}$ a quantity of the form $\partial \psi_{\mu\nu\lambda}/\partial x_\lambda$ where $\psi_{\mu\nu\lambda}$ is any third rank tensor antisymmetric in the indices $\nu$ and $\lambda$, then the four-dimensional divergence of the tensor

$$T'_{\mu\nu} = T_{\mu\nu} + (\partial/\partial x_\lambda)\psi_{\mu\nu\lambda}$$

will vanish just as in the case of the tensor $T_{\mu\nu}$, where

$$(\partial T'_{\mu} / \partial x_\nu) = 0.$$

Therefore the replacement of the tensor $T_{\mu\nu}$ by the tensor $T'_{\mu\nu}$ will not lead to any change in the total energy or the total momentum of the field. In particular, $\psi_{\mu\nu\lambda}$ can be chosen in such a way that the tensor $T'_{\mu\nu}$ will be symmetric.

We now consider the variation of the action resulting from an infinitesimal rotation which corresponds to the infinitesimal Lorentz transformation given by

$$x'_\mu = x_\mu + \varepsilon_{\mu\nu} x_\nu,$$

where $\varepsilon_{\mu\nu}$ is an infinitesimal antisymmetric tensor. The wave functions in this case transform in accordance with (20.4′),

$$\psi'(x_\mu) = \psi(x_\mu - \varepsilon_{\nu\mu} x_\nu) + \tfrac{1}{2}\varepsilon_{\mu\nu} \mathrm{I}_{\mu\nu} \psi(x_\mu),$$

where $\mathrm{I}_{\mu\nu}$ are the infinitesimal operators of the Lorentz group; from this it follows that

$$\delta\psi = \psi'(x_\mu) - \psi(x_\mu) = -\varepsilon_{\mu\nu} x_\nu \psi_{,\mu} + \tfrac{1}{2}\varepsilon_{\mu\nu} \mathrm{I}_{\mu\nu} \psi.$$

On substituting this expression for the variation of the wave function together with the expression $\delta x_\mu = \varepsilon_{\mu\nu} x_\nu$ into the surface integral (20.14) we obtain

$$\oint \left( \frac{\partial L}{\partial \psi_{,\mu}} \delta\psi + L \delta x_\mu \right) d\sigma_\mu = -\frac{1}{2} \varepsilon_{\mu\nu} \oint M_{\mu\nu,\varrho} d\sigma_\varrho = 0, \qquad (20.20)$$

where

$$M_{\mu\nu,\varrho} = x_\mu T_{\nu\varrho} - x_\nu T_{\mu\varrho} - \frac{\partial L}{\partial \psi_{,\varrho}} \mathrm{I}_{\mu\nu} \psi. \qquad (20.21)$$

Thus, we have

$$\oint M_{\mu\nu,\varrho} d\sigma_\varrho = 0,$$

whence it follows that

$$\frac{\partial}{\partial x_\varrho} M_{\mu\nu,\varrho} = 0. \qquad (20.22)$$

The third rank tensor $M_{\mu\nu,\varrho}$ is called the *angular momentum tensor.*

On assuming that the hypersurface in (20.20) is oriented orthogonal to the time axis we obtain the conservation laws

$$-i \int M_{\mu\nu,4} dr \equiv M_{\mu\nu} = \text{const}. \qquad (20.23)$$

From (20.23) it follows that the quantities $M_{\mu\nu}$ form a second rank tensor. This tensor is the angular momentum tensor for the whole field. We shall show later that the term $x_\mu T_{\nu\varrho} - x_\nu T_{\mu\varrho}$ in (20.21) for $M_{\mu\nu,\varrho}$ can be interpreted as an "orbital" angular momentum, while the term $-(\partial L/\partial \psi_\varrho)\mathrm{I}_{\mu\nu}\psi$ can be interpreted as a spin angular momentum.

We note that if we symmetrize the energy-momentum tensor of the field, i.e., if we find such a tensor $\chi_{\mu\varrho\lambda} = -\chi_{\mu\lambda\varrho}$ for which the tensor

$$\theta_{\mu\nu} = T_{\mu\nu} + \frac{\partial\chi_{\mu\nu\lambda}}{\partial x_\lambda} \tag{20.24}$$

will be symmetric, then $\theta_{\mu\nu} = \theta_{\nu\mu}$, then with the aid of the tensor $\theta_{\mu\nu}$ we can define the angular momentum tensor as

$$M_{\mu\nu,\varrho} = x_\mu \theta_{\nu\varrho} - x_\nu \theta_{\mu\varrho}. \tag{20.25}$$

In virtue of the equality $\theta_{\mu\nu} = \theta_{\nu\mu}$ the divergence of this tensor vanishes and

$$\partial M_{\mu\nu,\varrho}/\partial x_\varrho = 0,$$

and therefore the tensor $M_{\mu\nu,\varrho}$ defined by formula (20.25) leads to the correct value for the angular momentum of the whole field.

We shall refer to the tensor $\theta_{\mu\nu}$ as the *symmetric* energy-momentum tensor in contrast to the tensor $T_{\mu\nu}$ to which we shall refer as the *canonical* energy-momentum tensor.

We now show how $\chi_{\mu\nu\varrho}$ can be obtained. On substituting (20.24) into (20.25) and on utilizing the definition of $M_{\mu\nu,\varrho}$ we obtain

$$x_\mu \partial\chi_{\nu\varrho\lambda}/\partial x_\lambda - x_\nu \partial\chi_{\mu\varrho\lambda}/\partial x_\lambda = -\frac{\partial L}{\partial\psi_{,\varrho}} \mathrm{I}_{\mu\nu}\psi + (\partial/\partial x_\lambda)(x_\mu\chi_{\nu\varrho\lambda} - x_\nu\chi_{\mu\varrho\lambda})$$

(to the right-hand side of this equation we have added the nonessential term $(\partial/\partial x_\lambda)(x_\mu\chi_{\nu\varrho\lambda} - x_\nu\chi_{\mu\varrho\lambda})$, whose divergence with respect to the subscript $\varrho$ vanishes). From this it follows that

$$\chi_{\nu\varrho\mu} - \chi_{\mu\varrho\nu} = \frac{\partial L}{\partial\psi_{,\varrho}} \mathrm{I}_{\mu\nu}\psi$$

and, consequently, that

$$\chi_{\nu\varrho\mu} = \frac{1}{2}\frac{\partial L}{\partial\psi_{,\varrho}} \mathrm{I}_{\mu\nu}\psi + a_{\nu\varrho\mu},$$

where $a_{\nu\varrho\mu}$ is for the time being an arbitrary tensor satisfying $a_{\nu\varrho\mu} = a_{\mu\varrho\nu}$. We choose this tensor in such a way that the tensor $\chi_{\nu\varrho\mu}$ will be antisymmetric with respect to the subscripts $\varrho$ and $\mu$. It can be easily seen that

$$\chi_{\mu\nu\varrho} = \frac{1}{2}\left(\frac{\partial L}{\partial\psi_{,\nu}} \mathrm{I}_{\varrho\mu}\psi - \frac{\partial L}{\partial\psi_{,\varrho}} \mathrm{I}_{\nu\mu}\psi - \frac{\partial L}{\partial\psi_{,\mu}} \mathrm{I}_{\nu\varrho}\psi\right).$$

## 20.4. *Current Density Vector*

The wave functions of the field may be real or complex. In the latter case the Lagrangian contains both the functions $\psi_i$ and the complex-conjugate functions $\psi_i^*$ which must be treated as independent of $\psi_i$. Therefore, in the case of complex fields the energy-momentum tensor and the angular momentum tensor should be written

$$T_{\mu\nu} = L\delta_{\mu\nu} - \psi_{,\mu}\frac{\partial L}{\partial \psi_{,\nu}} - \psi^*_{,\mu}\frac{\partial L}{\partial \psi^*_{,\nu}}, \tag{20.26}$$

and

$$M_{\mu\nu,\varrho} = x_\mu T_{\nu\varrho} - x_\nu T_{\mu\varrho} - \frac{\partial L}{\partial \psi_{,\varrho}} \mathrm{I}_{\mu\nu}\psi - \frac{\partial L}{\partial \psi^*_{,\varrho}}(\mathrm{I}_{\mu\nu}\psi)^*. \tag{20.27}$$

We assume that in the case of complex fields the Lagrangian density is invariant under the transformation (149)

$$\psi \to \psi' = \psi e^{i\alpha}, \qquad \psi^* \to \psi^{*\prime} = \psi^* e^{-i\alpha}, \tag{20.28}$$

where $\alpha$ is an arbitrary real constant.

We consider the variation of the action for actual motion by assuming that the infinitesimal transformation (20.28) $\delta\psi = i\delta\alpha\psi$, $\delta\psi^* = -i\delta\alpha\psi^*$ has been performed. Obviously the variation of the action in this case vanishes, and in accordance with (20.14) we obtain

$$\oint \left(\frac{\partial L}{\partial \psi_{,\mu}}\delta\psi + \frac{\partial L}{\partial \psi^*_{,\mu}}\delta\psi^*\right) d\sigma_\mu = i\delta\alpha \oint \left(\frac{\partial L}{\partial \psi_{,\mu}}\psi - \frac{\partial L}{\partial \psi^*_{,\mu}}\psi^*\right) d\sigma_\mu .$$

From this it follows that

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial \psi_{,\mu}}\psi - \frac{\partial L}{\partial \psi^*_{,\mu}}\psi^*\right) = 0.$$

In other words, if we define the vector

$$j_\mu = -ie\left(\frac{\partial L}{\partial \psi_{,\mu}}\psi - \frac{\partial L}{\partial \psi^*_{,\mu}}\psi^*\right), \tag{20.29}$$

where $e$ is a constant, then its divergence will vanish.

Just as in the case of the electron-positron field we interpret $j_\mu(j, i\varrho)$ as the electric current density four-vector associated with the field; its spatial components $j$ define the spatial current density, while its time component $\varrho$ is the charge density. Later we shall show that $e$ represents the charge of the particles associated with the field.

In the case of real fields the current density vector vanishes. Therefore, real fields which do not admit the transformation (20.28) describe neutral particles. Charged particles are always described by complex fields.

### 20.5. *Relativistically Invariant Field Equations*

Free fields obey the superposition principle and, therefore, satisfy linear differential equations which in virtue of the homogeneity of space and time must have constant coefficients.

Without loss of generality we can assume that the differential equations of the free field are homogeneous equations of the first order, since equations of higher order can be reduced to equations of the first order if we include the derivatives of $\psi$ among the wave functions of the field. Equations of this type can be obtained from a variational principle with a Lagrangian density quadratic in $\psi$:

$$L = -\frac{1}{2}\left(\psi^* \mathrm{A}_\mu \frac{\partial}{\partial x_\mu}\psi - \frac{\partial}{\partial x_\mu}\psi^* \mathrm{A}_\mu \psi\right) - \varkappa\psi^* \mathrm{B}\psi, \tag{20.30}$$

where $\mathrm{A}_\mu$ and B are certain matrices and $\varkappa$ is a real constant different from zero.[1] Since $L$ is a real quantity, the matrices $\mathrm{A}_\mu$ and B must obviously satisfy the following conditions:

$$\mathrm{A}_k = -\mathrm{A}_k^+ \quad (k = 1, 2, 3), \quad \mathrm{A}_4 = \mathrm{A}_4^+, \quad \mathrm{B} = \mathrm{B}^+.$$

On assuming that B has an inverse matrix (singularity of the matrix B corresponds to a zero rest mass of the particle) we can introduce the matrices $\Gamma_\mu = \mathrm{B}^{-1}\mathrm{A}_\mu$ and write $L$ in the following form

$$L = -\frac{1}{2}\left(\bar{\psi}\Gamma_\mu \frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar{\psi}}{\partial x_\mu}\Gamma_\mu\psi\right) - \varkappa\bar{\psi}\psi, \tag{20.30'}$$

where $\bar{\psi} = \psi^*\mathrm{B}$. By varying $\psi$ and $\psi'$ independently we obtain the Lagrangian equations

$$\Gamma_\mu \frac{\partial\psi}{\partial x_\mu} + \varkappa\psi = 0, \qquad -\frac{\partial\bar{\psi}}{\partial x_\mu}\Gamma_\mu + \varkappa\bar{\psi} = 0, \tag{20.31}$$

or

$$(\Gamma_\mu)_{ik}\frac{\partial\psi_k}{\partial x_\mu} + \varkappa\psi_i = 0, \qquad -\frac{\partial\bar{\psi}_i}{\partial x_\mu}(\Gamma_\mu)_{ik} + \varkappa\bar{\psi}_k = 0,$$

where the subscripts $i$ and $k$ label the wave functions. These equations are obviously generalizations of the Dirac equations (73).

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[1] The case $\varkappa = 0$ corresponds to particles of zero mass.

We now investigate the conditions for the relativistic invariance of the Lagrangian density (20.30). Under the Lorentz transformation $x' = \mathrm{l}x$ (i.e., $x'_\mu = l_{\mu\nu}x_\nu$) the wave functions undergo the transformation $\psi'(x') = \mathrm{S}\psi(x)$ and the Lagrangian density assumes the following form:

$$L' = -\frac{1}{2}\left(\psi'^*\mathrm{A}\frac{\partial\psi'}{\partial x'} - \frac{\partial\psi'^*}{\partial x'}\mathrm{A}\psi'\right) - \varkappa\psi'^*\mathrm{B}\psi'$$

$$= -\frac{1}{2}\left(\psi^*\mathrm{S}^+\mathrm{A}\mathrm{l}\frac{\partial}{\partial x}(\mathrm{S}\psi) - \mathrm{l}\frac{\partial}{\partial x}(\psi^*\mathrm{S}^+)\,\mathrm{A}\mathrm{S}\psi\right) - \varkappa\psi^*\mathrm{S}^+\mathrm{B}\mathrm{S}\psi,$$

where $\mathrm{A}(\partial/\partial x) = \mathrm{A}_\mu(\partial/\partial x_\mu)$. In order that this expression should coincide with expression (20.30) it is obviously necessary that certain conditions should be satisfied: $\mathrm{S}^+\mathrm{A}\mathrm{l}\,\mathrm{S} = \mathrm{A}$, $\mathrm{S}^+\mathrm{B}\mathrm{S} = \mathrm{B}$. On substituting $\mathrm{A} = \mathrm{B}\Gamma$ into the first condition we obtain $\Gamma = \mathrm{l}\mathrm{S}\Gamma\mathrm{S}^{-1}$ or $\Gamma_\mu = l_{\mu\nu}\mathrm{S}\Gamma_\nu\mathrm{S}^{-1}$. Thus, in order that $L$ should be relativistically invariant the following conditions must be satisfied:

$$\Gamma_\mu = l_{\mu\nu}\mathrm{S}\Gamma_\nu\mathrm{S}^{-1} \quad \text{and} \quad \mathrm{S}^+\mathrm{B}\mathrm{S} = \mathrm{B}. \tag{20.32}$$

It is obviously sufficient that these conditions should be satisfied in the case of the infinitesimal Lorentz transformation $l_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}$, $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$. In such a case in accordance with (20.4) $\mathrm{S} = 1 + \frac{1}{2}\varepsilon_{\mu\nu}\mathrm{I}_{\mu\nu}$ and the conditions (20.32) assume the form

$$\begin{aligned} &[\Gamma_\mu, \mathrm{I}_{\nu\lambda}] = \delta_{\mu\lambda}\Gamma_\nu - \delta_{\mu\nu}\Gamma_\lambda, \quad \text{where} \quad \mu, \nu, \lambda = 1, 2, 3, 4,\\ &\mathrm{I}^+_{kl}\mathrm{B} + \mathrm{B}\mathrm{I}_{kl} = 0, \quad \mathrm{I}^+_{4k}\mathrm{B} - \mathrm{B}\mathrm{I}_{4k} = 0, \quad \text{where} \quad k, l = 1, 2, 3. \end{aligned} \tag{20.33}$$

In particular, from this it follows that

$$\begin{aligned} &\Gamma_k = [\Gamma_4, \mathrm{I}_{k4}],\\ &\mathrm{I}^+_k\mathrm{B} - \mathrm{B}\mathrm{I}_k = 0, \quad k = 1, 2, 3. \end{aligned} \tag{20.34}$$

The following expressions for the energy-momentum tensor, the angular momentum tensor and the current density vector correspond to the Lagrangian density (20.30′):

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2}\left(\bar\psi\Gamma_\nu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar\psi}{\partial x_\mu}\Gamma_\nu\psi\right),\\ M_{\mu\nu,\varrho} &= x_\mu T_{\nu\varrho} - x_\nu T_{\mu\varrho} + \frac{1}{2}\left(\bar\psi\Gamma_\varrho\mathrm{I}_{\mu\nu}\psi - \bar\psi\mathrm{I}_{\mu\nu}\Gamma_\varrho\psi\right),\\ j_\mu &= ie\bar\psi\Gamma_\mu\psi, \end{aligned} \tag{20.35}$$

where

$$\mathrm{I}_{\mu\nu} = \mathrm{B}^{-1}\mathrm{I}^{+}_{\mu\nu}\mathrm{B}.$$

The components $M_{\mu\nu,\varrho}$ for $\mu, \nu = 1, 2, 3$ can also be written in the form

$$M_{kl,\varrho} = x_k T_{l\varrho} - x_l T_{k\varrho} + \frac{i}{2} e_{klm} \overline{\psi}(\Gamma_\varrho \mathrm{I}_m + \mathrm{I}_m \Gamma_\varrho)\psi, \qquad k, l, m = 1, 2, 3. \tag{20.35'}$$

We now proceed to determine some general properties of equations (20.31). We note, first of all, that the same equation (20.31) can be satisfied by wave functions describing different fields, for example, a scalar and a pseudoscalar field. Therefore the field is characterized not only by the matrices $\Gamma_\mu$, but also by the representation $D_R$, according to which the wave functions $\psi$ transform.

The derivatives $\partial\psi/\partial x_\mu$ evidently transform according to the representation which is the direct product of the representation $D_R$ and the representation $(\frac{1}{2}, \frac{1}{2})$ according to which the four-vector $x$ transforms. On the other hand, in accordance with equation (20.31) the quantities $\Gamma_\mu \partial\psi/\partial x_\mu$ transform according to the same representation as $\psi$, i.e., in accordance with the representation $D_R$. In other words, the representation $D_R$ is contained in the representation $D_R \times (\frac{1}{2}, \frac{1}{2})$. From this it follows that if an irreducible representation $(j, k)$ of the Lorentz group is contained in the representation $D_R$, then $D_R$ also contains at least one of the four irreducible representations

$$(j-\tfrac{1}{2}, k-\tfrac{1}{2}), \quad (j-\tfrac{1}{2}, k+\tfrac{1}{2}), \quad (j+\tfrac{1}{2}, k-\tfrac{1}{2}), \quad (j+\tfrac{1}{2}, k+\tfrac{1}{2}).$$

Thus, $D_R$ cannot be an irreducible representation, except in the case $\varkappa = 0$. For example, in the case of the Dirac equation (with $\varkappa \neq 0$) $D_R$ is decomposed into two irreducible representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

If we decompose the representation $D_R$ into irreducible representations $\tau \sim (j, k)$, we can assert that the sum $j+k$ will be either integral or half-integral in all the representations.

We now determine the masses of the particles described by equation (20.31). In order to do this we consider the solutions in the form of plane waves in the rest system of the particle $\psi(x) = \psi^{(0)} e^{-i\varepsilon_0 t}$, where $\varepsilon_0$ is the rest energy of the particle and $\psi^{(0)}$ is a constant; the substitution of $\psi(x)$ into (20.31) yields

$$\varepsilon_0 \Gamma_4 \psi^{(0)} - \varkappa\psi^{(0)} = 0, \tag{20.36}$$

whence it follows that $\varepsilon_0 = (\varkappa/\lambda_i)$, where $\lambda_i$ is a nonvanishing eigenvalue of the matrix $\Gamma_4$. This formula shows that the possible values for the mass of the particles described by equation (20.31) are given by

$$\mu_i = \frac{\varkappa}{\lambda_i}, \tag{20.37}$$

where $\lambda_i$ can be taken equal to any one of the nonvanishing eigenvalues of the matrix $\Gamma_4$.

Thus, in principle equation (20.31) describes particles of different masses. In order that it should describe particles of only one value of the mass it is necessary that the matrix $\Gamma_4$ should have only one nonvanishing eigenvalue.

The matrix $\Gamma_4$ commutes with the infinitesimal operators $\mathrm{I}_{ik} (i, k = 1, 2, 3)$. Therefore one of these operators, for example, $\mathrm{I}_{12} = i\mathrm{I}_3$ and $\Gamma_4$ have common eigenfunctions. We denote them by $\varphi_{\lambda_i, m_i}$ where $m$ is an eigenvalue of $\mathrm{I}_3$. As we already know, this quantity may assume the values $m_i = -l_i, -l_i+1, \ldots, l_i$, where $l_i$ determines the square of the "total angular momentum" $\mathrm{I}^2 = \mathrm{I}_1^2+\mathrm{I}_2^2+\mathrm{I}_3^2$. Thus, the eigenvalues $\lambda_i$ are, generally speaking, degenerate with respect to the "magnetic" quantum numbers $m_i$, with the degree of degeneracy of the eigenvalue $\lambda_i$ being equal to $2l_i+1$.

As has been pointed out previously, spatial rotations correspond to linear transformations of the eigenfunctions of the operator $\mathrm{I}_3$, i.e., of the $2l_i+1$ functions $\varphi_{\lambda_i, m_i}$ (with different values of $m_i$ and the same value of $\lambda_i$). These transformations form irreducible representations of the rotation group $D_{l_i}$. We can therefore assert that particles of mass $\mu_i = \varkappa/\lambda_i$ have spin $l_i$. Thus, the spin is determined by the degree of degeneracy of the nonvanishing eigenvalues of the matrix $\Gamma_4$, with each value of the mass in principle corresponding to its own value of the spin. We can also say this differently, viz., that the spin is determined by the number of independent solutions of the system of equations (20.36) corresponding to a definite value of $\varepsilon_0$, i.e., to the definite eigenvalue $\lambda_i$. The number of these solutions is evidently equal to $2l_i+1$. (For example, in the case of the Dirac equation the number of independent solutions is equal to two in accordance with the fact that the electron spin is equal to $\frac{1}{2}$.)

Since the representation $\tau \sim (j, k)$, regarded as a representation of the rotation group, decomposes into representations $D_l$ with $l$ lying between $|j-k|$ and $|j+k|$, the spin will be integral or half-integral depending on whether the number $j+k$ is integral or half-integral.

We now show that the particle spin is determined by the term $-(\partial L/\partial\psi_{,\varrho})\mathrm{I}_{\mu\nu}\psi$ in the expression (20.21) for the angular momentum tensor or by the term

$$\tfrac{1}{2}(\bar{\psi}\Gamma_{\varrho}\mathrm{I}_{\mu\nu}\psi-\bar{\psi}\bar{\mathrm{I}}_{\mu\nu}\Gamma_{\varrho}\psi)$$

in the expression (20.35). In order to do this we determine the energy and the angular momentum of the field in the case when the field contains only one particle at rest.

In accordance with (20.35) the energy of the field is given by

$$H=-\int T_{44}\,dr=-\tfrac{1}{2}\int dr(\bar{\psi}\Gamma_4(\partial\psi/\partial x_4)-(\partial\bar{\psi}/\partial x_4)\Gamma_4\psi).$$

On substituting into this expression $\psi=\psi^{(0)}\exp(-i\varepsilon_0 t)$, and on assuming that the volume of the field is equal to unity, we obtain

$$H=\varepsilon_0\int\bar{\psi}\Gamma_4\psi\,dr=\varepsilon_0\bar{\psi}^{(0)}\Gamma_4\psi^{(0)}.$$

Since the field contains only one particle the function $\psi^{(0)}$ must be normalized in accordance with the condition

$$|\bar{\psi}^{(0)}\Gamma_4\psi^{(0)}|=1.$$

We note that under this condition we obtain for the charge of the field the value $Q=\pm e$. Indeed, the charge of the field $Q$ is given in accordance with (20.35) by

$$Q=\frac{1}{i}\int j_4\,dr=e\int\bar{\psi}\Gamma_4\psi\,dr=\pm e.$$

Thus, as has been stated earlier, the constant $e$ in the expression (20.29) for the current density vector can be interpreted as the charge of the particles.

We now obtain the component of the angular momentum of the field along the $x$-axis. In accordance with (20.35′) it can be written

$$M_{12}=-i\int M_{12,4}\,dr=\tfrac{1}{2}\int\bar{\psi}(\Gamma_4\mathrm{I}_3+\mathrm{I}_3\Gamma_4)\psi\,dr,$$

since the first term in (20.35′) vanishes for a particle at rest. But the matrices $\Gamma_4$ and $\mathrm{I}_3$ commute, and therefore

$$M_{12}=\int\bar{\psi}\Gamma_4\mathrm{I}_3\psi\,dr.$$

Since $\psi$ is a common eigenfunction of $\Gamma_4$ and $I_3$, we have

$$M_{12} = m \int \bar{\psi} \Gamma_4 \psi \, dr = m,$$

where $m$ is an eigenvalue of $I_3$, or the component of the spin along the $x_3$-axis.

We note that in addition to fields transforming according to the finite-dimensional representations of the Lorentz group we cannot, in principle, exclude the possibility of the existence of fields transforming according to infinite-dimensional representations of the Lorentz group. Particles described by such fields can exist in states with different values of the mass, and, as a rule (73), the masses tend to zero if the spin $l \to \infty$.

### 20.6. *Wave Equations for Particles of Spin Zero and Unity*

The simplest example of relativistically invariant field equations is given by

$$\left(\frac{\partial^2}{\partial x_\mu^2} - m^2\right)\varphi = \left(\Delta - \frac{\partial^2}{\partial t^2} - m^2\right)\varphi = 0, \qquad (20.38)$$

where $\varphi$ is a scalar function of the coordinates and the time. Such an equation in virtue of the scalar nature of $\varphi$ describes particles of spin zero; the mass of the particles is evidently equal to $m$.

We now show how this equation may be brought into the general form (20.31). In order to do this we introduce in addition to $\varphi$ the four-vector $\psi_\mu = (1/\sqrt{m})(\partial\varphi/\partial x_\mu)$. Then equation (20.38) is replaced by

$$-\frac{\partial \psi_\mu}{\partial x_\mu} + m\psi_0 = 0, \qquad -\frac{\partial \psi_0}{\partial x_\mu} + m\psi_\mu = 0, \qquad (20.39)$$

where $\psi_0 = \sqrt{m}\varphi$. This system can be written in the general form (20.31)

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi = 0, \qquad (20.40)$$

where $\psi$ is a five-component wave function consisting of $\psi_0$ and $\psi_\mu$, while $\beta_\mu (\mu = 1, 2, 3, 4)$ are four five-rowed matrices which, as can be easily seen, satisfy relations (51), (104):

$$\beta_\mu \beta_\nu \beta_\varrho + \beta_\varrho \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\varrho + \delta_{\varrho\nu} \beta_\mu . \qquad (20.41)$$

In this case the representation $D_B$ consists of the representation (0,0) according to which transforms the scalar $\psi_0$, and the representation $(\frac{1}{2}, \frac{1}{2})$ according to which transforms the vector $\psi_\mu$.

The wave equation (20.38) can be obtained from a variational principle if for the Lagrangian density we take

$$L = -\left(\frac{\partial \varphi^*}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\mu} + m^2 \varphi^* \varphi\right). \tag{20.42}$$

On taking

$$L = -\left(\frac{\partial \psi_0^*}{\partial x_\mu} \psi_\mu + \psi_\mu^* \frac{\partial \psi_0}{\partial x_\mu} - m\psi_\mu^* \psi_\mu + m\psi_0^* \psi_0\right) \tag{20.43}$$

and on assuming that in carrying out the variation the functions $\psi_0, \psi_\mu, \psi_0^*, \psi_\mu^*$, are independent, we obtain equations (20.39).

The Lagrangian (20.42) corresponds to the energy-momentum tensor

$$T_{\mu\nu} = \frac{\partial \varphi^*}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\nu} + \frac{\partial \varphi^*}{\partial x_\nu} \frac{\partial \varphi}{\partial x_\mu} + \delta_{\mu\nu} \tag{20.44}$$

and to the current density vector

$$j_\mu = ie\left(\frac{\partial \varphi^*}{\partial x_\mu} \varphi - \varphi^* \frac{\partial \varphi}{\partial x_\mu}\right). \tag{20.45}$$

We note that a pseudoscalar field can be treated in a manner similar to the scalar field. In such a case we simply replace the scalar $\varphi$ by the pseudoscalar $\varphi_{\mu\nu\lambda\sigma}$ antisymmetric in any pair of subscripts.

We now consider the wave field describing particles of spin unity (158). Such a field is characterized by the four-vector $\varphi_\mu$ whose components satisfy the equations

$$\left(\frac{\partial^2}{\partial x^2} - m^2\right) \varphi_\nu = 0 \quad \text{and} \quad \frac{\partial \varphi_\nu}{\partial x_\nu} = 0. \tag{20.46}$$

Because of the subsidiary condition $\partial \varphi_\nu / \partial x_\nu = 0$ imposed on $\varphi_\nu$ this field describes only particles of spin 1. Indeed, we consider the free field in the form of the plane wave $\varphi_\mu = \varphi_\mu^{(0)} \exp(ipx)$. It follows from (20.46) that $p^2 + m^2 = 0$ and $p_\mu \varphi_\mu^{(0)} = 0$. If we go over to the coordinate system in which the particle is at rest, i.e., if we set $p = 0$, we obtain $\varphi_4^{(0)} = 0$. Thus, in this system the wave field is described by a three-vector. Under spatial rotations it transforms like a vector (according to the representation $D_1$) and consequently describes particles of spin 1.

If we introduce the antisymmetric tensor

$$\psi_{\mu\nu} = (1/\sqrt{m})\left(\frac{\partial\varphi_\mu}{\partial x_\nu} - \frac{\partial\varphi_\nu}{\partial x_\mu}\right)$$

then the equations (20.46) are replaced by the system

$$-\frac{\partial\psi_{\mu\nu}}{\partial x_\nu} + m\psi_\mu = 0, \qquad -\frac{\partial\psi_\nu}{\partial x_\mu} + \frac{\partial\psi_\mu}{\partial x_\nu} + m\psi_{\mu\nu} = 0, \tag{20.47}$$

where $\psi_\mu = \sqrt{m}\varphi_\mu$. This system can, in turn, be written in the general form (20.31) if we interpret $\psi$ as a ten-component wave function consisting of the vector and the antisymmetric tensor $\psi_{\mu\nu}$:

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi = 0. \tag{20.48}$$

Here $\beta_\mu$ are four ten-rowed matrices which, as may be easily shown, satisfy the same relations (20.41) as the five-rowed matrices $\beta_\mu$ appearing in the equations for a particle of spin zero.

Equations (20.47) can be obtained from a variational principle if for the Lagrangian density we take

$$L = -\frac{1}{2}\psi^*_{\mu\nu}\left(\frac{\partial\psi_\mu}{\partial x_\nu} - \frac{\partial\psi_\nu}{\partial x_\mu}\right) - \frac{1}{2}\left(\frac{\partial\psi^*_\mu}{\partial x_\nu} - \frac{\partial\psi^*_\nu}{\partial x_\mu}\right)\psi_{\mu\nu} + \frac{m}{2}\psi^*_{\mu\nu}\psi_{\mu\nu} - m\psi^*_\mu\psi_\mu \tag{20.49}$$

and in carrying out the variation regard the quantities $\psi_\mu, \psi^*_\mu, \psi_{\mu\nu}, \psi^*_{\mu\nu}$ as independent.

The Lagrangian density (20.49) corresponds to the energy-momentum tensor

$$T_{\mu\nu} = L\delta_{\mu\nu} + \frac{\partial\psi_\varrho}{\partial x_\mu}\psi^*_{\varrho\nu} + \frac{\partial\psi^*_\varrho}{\partial x_\mu}\psi_{\varrho\nu} \tag{20.50}$$

and to the current density vector

$$j_\mu = ie(\psi^*_{\varrho\mu}\psi_\varrho - \psi^*_\varrho\psi_{\varrho\mu}). \tag{20.51}$$

Equations (20.40) for particles of spin zero and equations (20.48) for particles of spin unity can be obtained with the aid of the following Lagrangian density which is of the same form for both kinds of particles:

$$L = -\frac{1}{2}\left(\bar{\psi}\beta_\mu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar{\psi}}{\partial x_\mu}\beta_\mu\psi\right) - m\bar{\psi}\psi. \tag{20.52}$$

This Lagrangian density corresponds to the energy-momentum tensor

$$T_{\mu\nu}=\frac{1}{2}\left(\overline{\psi}\beta_\nu\frac{\partial\psi}{\partial x_\mu}-\frac{\partial\overline{\psi}}{\partial x_\mu}\beta_\nu\psi\right) \tag{20.53}$$

and to the current density vector

$$j_\mu=ie\overline{\psi}\beta_\mu\psi, \tag{20.54}$$

where

$$\overline{\psi}=\psi^*(2\beta_4^2-1)$$

and the matrices $\beta_\mu$ are defined by relations (20.41).

## § 21. Quantization of Fields. Connection between Spin and Statistics

### 21.1. *Nondefiniteness of the Charge in the Case of Integral Spin and of the Energy in the Case of Half-Integral Spin*

We now show that in the case of a wave field describing particles of integral spin it is impossible to introduce a positive definite charge density, while in the case of a wave field describing particles of half-integral spin it is impossible to introduce a positive definite energy density (149).

We first of all consider fields describing particles of integral spin. In this case $\psi$ transforms according to the representation $D_R$ which decomposes into representations $\tau\sim(j, k)$ with integral $j+k$. Therefore $\psi$ contains functions only of classes $+1$ and $-1$ which we respectively denote by $\psi^{(+1)}$ and $\psi^{(-1)}$. The basic field equation satisfied by $\psi$ can be written in terms of the Fourier components of $\psi$ in the form

$$(i\Gamma_\mu k_\mu+\varkappa)\psi=0,$$

where $k_\mu$ is the propagation vector (we assume that $\psi\sim\exp(ikx)$. This equation can be decomposed into two equations:

$$i(\Gamma_\mu)_{+1,+1}k_\mu\psi^{(+1)}+i(\Gamma_\mu)_{+1,-1}k_\mu\psi^{(-1)}+\varkappa\psi^{(+1)}=0,$$
$$i(\Gamma_\mu)_{-1,-1}k_\mu\psi^{(-1)}+i(\Gamma_\mu)_{-1,+1}k_\mu\psi^{(+1)}+\varkappa\psi^{(-1)}=0,$$

where $(\Gamma_\mu)_{\pm1,\mp1}$ are the elements of the matrix $\Gamma_\mu$ operating on the functions $\psi^{(\pm1)}$. It can be easily shown that

$$(\Gamma_\mu)_{+1,+1}=(\Gamma_\mu)_{-1,-1}=0.$$

Indeed, since the propagation vector belongs to the class $-1$, then in the case of nonvanishing $(\Gamma_\mu)_{+1,+1}$ and $(\Gamma_\mu)_{-1,-1}$ the first and the third terms in each equation would belong to different classes, but this is impossible.

Thus, in the case of integral spin the equations have the form

$$\begin{aligned} i(\Gamma_\mu)_{+1,-1} k_\mu \psi^{(-1)} + \varkappa \psi^{(+1)} &= 0, \\ i(\Gamma_\mu)_{-1,+1} k_\mu \psi^{(+1)} + \varkappa \psi^{(-1)} &= 0. \end{aligned} \tag{21.1}$$

We see that the role of the matrices $\Gamma_\mu$ consists of relating functions of different classes.

Equations (21.1) are invariant under the transformation

$$k_\mu \to -k_\mu, \quad \psi^{(+1)} \to a\psi^{(+1)} \quad \text{and} \quad \psi^{(-1)} \to -a\psi^{(-1)}, \tag{21.2}$$

where $a$ is an arbitrary constant. It can be easily shown that $a$ may be taken real. Indeed, the functions $\psi^{(+1)*}$ and $\psi^{(-1)*}$ transform in the same way as the functions $\psi^{(+1)}$ and $\psi^{(-1)}$; therefore from the point of view of relativistic invariance conditions of reality for $\psi^{(+1)}$ and $\psi^{(-1)}$ can have the form

$$\psi^{(+1)*} = \psi^{(+1)} \quad \text{and} \quad \psi^{(-1)*} = \psi^{(-1)}.$$

In order that these relations should remain valid under the transformation (21.2) it is clear that $a$ must be real. Without loss of generality we can take $a = 1$ and rewrite the transformation (21.2) in the form

$$k_\mu \to -k_\mu, \quad \psi^{(+1)} \to \psi^{(+1)}, \quad \psi(^{-1)} \to -\psi^{(-1)}. \tag{21.3}$$

We can construct different tensor quantities with the aid of the functions $\psi^{(+1)}$, $\psi^{(-1)}$ and their derivatives. For subsequent development we shall be interested in the structure of such tensors quadratic in $\psi$. If we go over to the Fourier representation, then the problem reduces to the construction of tensors with the aid of $\psi^{(+1)}$ and $k_\mu$, with these tensors being required to be quadratic or bilinear forms in $\psi$.

We start by constructing the even rank tensor $T_{(2n)}$. Since such a tensor belongs to the class $+1$, it can have only the structure

$$T_{(2n)} \sim \sum k^{2N} \psi^{(+1)} \psi^{(+1)} + \sum k^{2N} \psi^{(-1)} \psi^{(-1)} + \sum k^{2N+1} \psi^{(+1)} \psi^{(-1)}, \tag{21.4}$$

where $N$ are integers (we do not include terms containing $\psi^{(+1)*}$ and $\psi^{(-1)*}$ since they also belong to the classes $+1$ and $-1$).

The odd rank tensor $T_{(2n+1)}$ belongs to the class $-1$ and, consequently, has the following structure:

$$T_{(2n+1)} \sim \sum k^{2N+1}\psi^{(+1)}\psi^{(+1)}+\sum k^{2N+1}\psi^{(-1)}\psi^{(-1)}+\sum k^{2N}\psi^{(+1)}\psi^{(-1)}. \tag{21.5}$$

We now investigate the behavior of these tensors under the transformation (21.3). It is clear that this transformation does not alter $T_{(2n)}$, but reverses the sign of $T_{(2n+1)}$.

In particular, the transformation (21.3) reverses the sign of the current density vector $j_\mu$. In other words, to each solution of the field equation there corresponds another solution with the opposite sign of the current density components. Therefore, in the case of particles of integral spin we cannot introduce a positive definite particle density which transforms like the fourth component of a four-vector.

Thus, we have proved the nondefinite nature of the charge density, and consequently of the total charge, in the case of particles of integral spin.

We now consider fields deseribing particles of half-integral spin. In this case $j+k$ is a half-integer and the functions $\psi$ belong to the classes $+\varepsilon$ and $-\varepsilon$. We denote them respectively by $\psi^{(+\varepsilon)}$ and $\psi^{(-\varepsilon)}$.

In place of equations (21.1) we now obtain the following equations (in terms of their Fourier components):

$$i(\Gamma_\mu)_{+\varepsilon,-\varepsilon}k_\mu\psi^{(-\varepsilon)}+\varkappa\psi^{(+\varepsilon)}=0 \quad \text{and} \quad i(\Gamma_\mu)_{-\varepsilon,+\varepsilon}k_\mu\psi^{(+\varepsilon)}+\varkappa\psi^{(-\varepsilon)}=0, \tag{21.6}$$

where $(\Gamma_\mu)_{\pm\varepsilon,\mp\varepsilon}$ are the elements of the matrix $\Gamma_\mu$ operating on the functions $\psi^{(\pm\varepsilon)}$. These equations, like equations (21.1), are invariant under transformations of the form

$$k_\mu \rightarrow -k_\mu, \quad \psi^{(+\varepsilon)} \rightarrow a\psi^{(+\varepsilon)}, \quad \psi^{(-\varepsilon)} \rightarrow -a\psi^{(-\varepsilon)} \tag{21.3'}$$

but in contrast to the case of integral spin, when $a$ was real, we must now regard $a$ as a purely imaginary quantity. Indeed, from the point of view of relativistic invariance the conditions of reality in the case of half-integral spin can have the following form:

$$\psi^{(+\varepsilon)*}=\psi^{(-\varepsilon)}, \quad \psi^{(-\varepsilon)*}=\psi^{(+\varepsilon)}, \tag{21.7}$$

while the following conditions are impossible:

$$\psi^{(+\varepsilon)*}=\psi^{(+\varepsilon)}, \quad \psi^{(-\varepsilon)*}=\psi^{(-\varepsilon)},$$

since $\psi^{(+\varepsilon)*}$ belongs to the class $-\varepsilon$, while $\psi^{(-\varepsilon)*}$ belongs to the class $+\varepsilon$.

On substituting into (21.7) the transformation (21.3′) we obtain $a = -a^*$.

Without loss of generality we can assume that $a = i$, and rewrite (21.3′) in the form

$$k_\mu \to -k_\mu, \quad \psi^{(+\varepsilon)} \to i\psi^{(+\varepsilon)}, \quad \psi^{(-\varepsilon)} \to -i\psi^{(-\varepsilon)}. \tag{21.8}$$

We now investigate the structure of the tensors that can be constructed with the aid of $\psi^{(\pm\varepsilon)}$ and $k_\mu$, if they are either quadratic or bilinear in $\psi$. By utilizing the multiplication table (20.12) we can easily show that these tensors must have the structure

$$\begin{aligned} T_{(2n)} &\sim \sum k^{2N}\psi^{(+\varepsilon)}\psi^{(+\varepsilon)} + \sum k^{2N}\psi^{(-\varepsilon)}\psi^{(-\varepsilon)} + \sum k^{2N+1}\psi^{(+\varepsilon)}\psi^{(-\varepsilon)}, \\ T_{(2n+1)} &\sim \sum k^{2N+1}\psi^{(+\varepsilon)}\psi^{(+\varepsilon)} + \sum k^{2N+1}\psi^{(-\varepsilon)}\psi^{(-\varepsilon)} + \sum k^{2N}\psi^{(+\varepsilon)}\psi^{(-\varepsilon)} \end{aligned} \tag{21.9}$$

(we have not included terms with $\psi^{(+\varepsilon)*}$ and $\psi^{(-\varepsilon)*}$, which belong to the classes $-\varepsilon$ and $+\varepsilon$).

We now investigate the behavior of these tensors under the transformation (21.8). It is clear that this transformation does not alter $T_{2n+1}$ and reverses the sign of $T_{(2n)}$ in contrast to the transformation (21.3) which does not alter the even rank tensors, and reverses the sign of the odd rank tensors. We, therefore, conclude that in the case of particles of half-integral spin it is impossible to introduce a positive definite energy density, and consequently a positive definite total energy.

From the assertions which we have just proved it does not necessarily follow that in the case of integral spin there always exists a positive definite energy density, and that in the case of half-integral spin there always exists a positive definite charge density. On the contrary, it can be shown that only in the case of spin $\frac{1}{2}$ is it possible to introduce a positive definite charge density, and only in the case of spin 0 and 1 is it possible to introduce a positive definite energy density (it is assumed here that the matrix $\Gamma_0$ is brought into diagonal form).

## 21.2. *Quantization of Fields for Integral and Half-Integral Spin. Pauli's Theorem*

From the impossibility of introducing a positive definite particle density in the case of integral spin, and a positive definite energy density in the case of half-integral spin, there follows a very important physical conclusion, which is that within the framework of the one body problem it is impossible to formulate a theory based on relativistically invariant field equations. In other words, a wave field must be associated not with a single particle, but with a whole set of particles. Moreover, the wave functions must be regarded not only as functions of the coordinates and the time, but also as operators operating in particle number space. We shall refer to them as the operators of the quantized field.

If we start by considering single particle states then, as is well known, we must distinguish two possible cases, viz., when each such state may contain an arbitrary number of particles, and when the number of particles may not exceed unity. In the former case we say that the particles obey Bose-Einstein statistics, and in the latter case that they obey Fermi-Dirac statistics, and we call the particles respectively *bosons* and *fermions*.

In order that the field equations should describe either bosons or fermions the operators of the quantized field must satisfy definite commutation relations. By generalizing the results of §§ 16 and 18 we can say that the commutators of the quantized wave field leading to bosons, and the anticommutators of the quantized wave field leading to fermions, must be *c*-numbers.

We shall now prove the following fundamental theorem due to Pauli: wave fields describing particles of integral spin must be quantized in accordance with Bose-Einstein statistics, while wave fields describing particles of half-integral spin must be quantized according to Fermi-Dirac statistics.

We consider some physical quantity $f(x)$ (for example, current density or energy density) at the two points $x_1(r_1, it_1)$ and $x_2(r_2, it_2)$ separated by a space-like interval $s_{12}^2 = (t_1-t_2)^2-(r_1-r_2)^2 < 0$. Since during the time $|t_1-t_2|$ a disturbance propagated with the speed of light from the first point will have travelled a distance $|t_1-t_2|$ which is smaller than $(r_1-r_2)^2$, the second point cannot experience an effect due to the first point. Therefore measurements at two points separated by a space-like interval cannot influence each other, while the operators corresponding to the quantities $f(x_1)$ and $f(x_2)$ must commute.

In the case of fermions the wave functions $\psi$ do not themselves have a direct physical meaning, since spinors, in general, in view of their transformation properties under rotations can be defined only up to their sign (they change sign as a result of a rotation through an angle $2\pi$, cf. § 9). Therefore, in the case of fermions, only quantities quadratic in $\psi$ have physical meaning, and it is these quantities that must commute at points separated by space-like intervals. This commutativity will exist if the corresponding commutators or anticommutators of the field operators vanish.

A similar conclusion can also be drawn in the case of fields describing bosons. However, in the case of bosons, in contrast to fermions, the wave functions can have a direct physical meaning. In this case the commutators of the boson field operators must vanish if the points to which these operators refer are separated by a space-like interval.

We assume that the commutators or the anticommutators of the fields $\psi_r(x_1)$ and $\psi_s^+(x_2)$ are $c$-numbers which vanish when $(x_1-x_2)^2 > 0$.

From the homogeneity of space-time it follows that the commutators $[\psi_r, \psi_s^+]_-$ and the anticommutators $[\psi_r, \psi_s^+]_-$ can be functions only of the difference of the coordinates $x_1-x_2$, i.e.,

$$[\psi_r(x_1), \psi_s^+(x_2)]_\pm = G_{rs}(x_1-x_2), \tag{21.10}$$

where $G_{rs}(x)$ is a $c$-function. Our problem consists of showing that in the case of integral spin this relation contains a commutator, while in the case of half-integral spin it contains an anticommutator.

We assume that each operator $\psi_r$ contains quantities belonging to only one of the classes $+1$, $-1$, $+\varepsilon$, $-\varepsilon$. In the case of integral spin these will be quantities belonging to the classes $+1$ and $-1$, while in the case of half-integral spin they will be quantities belonging to the classes $+\varepsilon$ and $-\varepsilon$. Since complex conjugation leaves quantities belonging to the classes $+1$ and $-1$ in the original class, and transforms quantities of the class $+\varepsilon$ into those of the class $-\varepsilon$, and *vice versa*, then by using the multiplication table (20.12) we can conclude that in the case of integral spin the quantities $\psi_r\psi_r^+$ must belong to the class $+1$, while in the case of half-integral spin they must belong in the class $-1$.

From this we can conclude that independently of how the fields must be quantized, i.e., independently of whether the commutator or the anticommutator appears in (21.10), the quantities $G_{rs}(x)$ must be tensors of even rank in the case of integral spin, and tensors of odd rank in the case of half-integral spin.

The structure of these tensors can be easily determined. Indeed, we assume that the field $\psi$ describes particles of a single value of the mass $m$. Then we have available only the one scalar function $\Delta(x)$ (cf. § 17), which vanishes for $x^2 > 0$:

$$\Delta(x) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{pr}}\frac{\sin\varepsilon t}{\varepsilon}\,d\mathbf{p}, \qquad \varepsilon = \sqrt{m^2+\mathbf{p}^2}.$$

If we differentiate $\Delta(x)$ once with respect to the coordinates $x$ we obtain a vector, if we differentiate it twice we obtain a second rank tensor, etc. Therefore, in the general case the tensor $G_{rs}(x)$ has the structure $G_{rs}(x) = \mathrm{F}_{rs}^{(n)}(\partial/\partial x)\,\Delta(x)$, where $\mathrm{F}_{rs}^{(n)}(\partial/\partial x)$ is a differential operator containing derivatives with respect to $x_\mu$ multiplied by constant coefficients; $n$ is the order of the highest derivative which coincides with the rank of the tensor $G_{rs}(x)$.

We can, thus, rewrite expressions (21.10) for the case $s = r$ in the following form:

$$[\psi_r(x_1), \psi_r^+(x_2)]_\pm = \begin{cases} \mathrm{F}_{rr}^{(2n)}\left(\dfrac{\partial}{\partial x_1}\right)\Delta(x_1-x_2), & 2(j+k) = 2N, \\ \mathrm{F}_{rr}^{(2n+1)}\left(\dfrac{\partial}{\partial x_1}\right)\Delta(x_1-x_2), & 2(j+k) = 2N+1, \end{cases} \tag{21.11}$$

where the upper line refers to the case of integral, and the lower line to the case of half-integral spin ($N$ is an integer), and where, as can be easily shown, $\mathrm{F}_{rr}^{(2n)}(\partial/\partial x)$ and $\mathrm{F}_{rr}^{(2n+1)}(\partial/\partial x)$ respectively contain derivatives of only even or odd orders.

We now consider the expression

$$K(x_1-x_2) = [\psi_r(x_1), \psi_r^+(x_2)]_\pm + [\psi_r(x_2), \psi_r^+(x_1)]_\pm,$$

symmetric in $x_1$ and $x_2$. Since $\Delta(x)$ is an even function of the spatial coordinates and an odd functions of the time

$$\Delta(\mathbf{r}, t) = \Delta(-\mathbf{r}, t) = -\Delta(\mathbf{r}, -t),$$

the symmetric quantity $K(x_1-x_2)$ must contain an even number of derivatives of $\Delta(x_1-x_2)$ with respect to spatial coordinates and an odd number of derivatives with respect to the time, i.e., an odd total number of derivatives. This is possible, as can be seen from (21.11) in the case of half-integral spin, and is impossible in the case of integral spin if only $\mathrm{F}_{rr}(\partial/\partial x)$ differs from zero. We therefore conclude that in the case of integral spin $K(x_1-x_2)$ must vanish and

$$[\psi_r(x_1), \psi_r^+(x_2)]_\pm + [\psi_r(x_2), \psi_r^+(x_1)]_\pm = 0, \qquad 2(j+k) = 2N. \tag{21.12}$$

But this equality evidently cannot be satisfied if it contains anticommutators, since in this case the left hand side of the equaiton is essentially positive for $x_1 = x_2$.

Thus, we have proved that in the case of integral spin the commutation relations for the operators of the quantized field must contain commutators, i.e., they must be of the form

$$[\psi_r(x_1), \psi_s^+(x_2)]_- = \mathrm{F}_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2) \quad \text{and} \quad 2(j+k) = 2N. \tag{21.13}$$

By utilizing the results of quantizing the electromagnetic field we can conclude from this that the number of particles associated with the field under consideration and existing in a definite state can be arbitrary. This means that particles of integral spin obey Bose-Einstein statistics.

From the arguments just presented we cannot conclude that in the case of half-integral spin the anticommutator of the operators of the quantized field must be a *c*-number, i.e., we cannot conclude that particles of half-integral spin must obey Fermi-Dirac statistics. Formally the commutator can be a *c*-number also in this case, i.e., formally we can carry out the quantization of the field in the case of half-integral spin in accordance with Bose-Einstein statistics. However, we must keep in mind that the energy of the field in the case of half-integral spin is not positive definite, and since the energy of free particles must be positive, it follows from this that quantization in the case of half-integral spin must be carried out in accordance with Fermi-Dirac statistics together with the requirement that all, or practically all, the states of negative energy (we have in mind the single particle states) must be occupied, or, in other words, we must from the outset introduce the concept of the field vacuum in which all the single particle states of negative energy are occupied.

Thus, in the case of half-integral spin the commutation relations for the operators of the quantized field must contain anticommutators, i.e., they must be of the form

$$[\psi_r(x_1), \psi_s^+(x_2)]_+ = \mathrm{F}_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2) \quad \text{and} \quad 2(j+k) = 2N+1. \tag{21.14}$$

### 21.3. *Inversion of Coordinates and Time Reversal*

The transformation of the propagation vector $k_\mu \to -k_\mu$, which we utilized in a purely formal manner in proving the nondefinite nature of the energy in the case of half-integral spin and of the charge in the case of integral spin, acquires an easily visualizable meaning if we go over from the momentum to the coordinate representation, viz., such a transformation corresponds to an inversion of the coordinates and time reversal

$$x_\mu \to -x_\mu. \tag{21.15}$$

This transformation, together with the corresponding transformation of the components of the wave function, is permissible in the case of the equations for the free fields, but the theory as a whole (including the commutation relations and the interaction between the fields) turns out to be noninvariant under the transformation (21.15) alone. Such invariance can be achieved, as will be shown later, if the transformation of the field operators associated with the inversion transformation $x_\mu \to -x_\mu$ is supplemented by a transition to the transposed operators. This may be clearly seen on the example of the electromagnetic field with which we begin our discussion.

We assume that the transformation (21.15) corresponds to the transformation of the operators for the potentials of the electromagnetic field

$$\mathrm{A}_\mu(x) \to \mathrm{A}'_\mu(x') = -\mathrm{A}_\mu(x), \quad \text{where} \quad x' = -x. \tag{21.16}$$

The equations for the free electromagnetic field

$$\frac{\partial^2}{\partial x_\nu^2}\mathrm{A}_\mu(x) = 0 \tag{21.17}$$

are invariant under the transformation

$$\frac{\partial^2}{\partial x_\nu'^2}\mathrm{A}'_\mu(x') = 0,$$

while the commutation relations

$$[\mathrm{A}_\mu(x_1), \mathrm{A}_\nu(x_2)] = -i\delta_{\mu\nu}D(x_1-x_2) \tag{21.17'}$$

are altered. Indeed, on replacing in expression (21.17') $x$ by $-x$, and on substituting into it $\mathrm{A}_\mu(-x) = -\mathrm{A}'_\mu(x)$ we obtain

$$[\mathrm{A}'_\mu(x_1), \mathrm{A}'_\nu(x_2)] = -i\delta_{\mu\nu}D(-x_1+x_2),$$

and since we have $D(-x) = -D(x)$, then

$$[A'_\mu(x_1), A'_\nu(x_2)] = i\delta_{\mu\nu} D(x_1 - x_2).$$

However, it can be easily shown that if we go over from $A'_\mu(x)$ to the transposed operators $A^R_\mu(x) \equiv A'^T_\mu(x)$ (the index T denotes the transposed operator), then the latter will satisfy not only equation (21.17), but also the commutation relations (21.17′). Indeed, since $(AB)^T = B^T A^T$, then

$$[A^R_\mu(x_1), A^R_\nu(x_2)] = -i\delta_{\mu\nu} D(x_1 - x_2).$$

We thus see that the operators $A_\mu(x)$ and $A^R_\mu(x)$ satisfy the same equations of motion and the same commutation relations. From this we can conclude that they are connected by a unitary transformation

$$A^R_\mu(x) = V A_\mu(x) V^{-1},$$

where V is a unitary operator $V^+ = V^{-1}$, operating in particle number space. This relation can also be rewritten in the following form:

$$A'_\mu(x) = U A^T_\mu(x) U^{-1}, \quad U = V^{T-1}, \tag{21.18}$$

where $A'_\mu(x) = -A_\mu(-x)$.

The form of the operator U can be easily obtained. On utilizing the expansion of $A_\mu(x)$ into plane waves (16.13), and on noting that in accordance with (16.18) $c^T_{k\lambda} = c^+_{k\lambda}$, we obtain $A^T_\mu(x) = A_\mu(-x)$, i.e., $-A_\mu(x) = U A_\mu(x) U^{-1}$. Therefore the operator U satisfies the relations

$$U c_{k\lambda} U^{-1} = -c_{k\lambda}. \tag{21.19}$$

Since the operator U obviously commutes with the operators $N_{k\lambda} = c^+_{k\lambda} c_{k\lambda}$ we can regard it as diagonal in the representation in which the $N_{k\lambda}$ are diagonal. Therefore it follows from (21.19) that the values of U corresponding to two consecutive values of $N_{k\lambda}$ are related by the expression $(U)_{N_{k\lambda}+1} = -(U)_{N_{k\lambda}}$; and from this we can conclude that $U = (-1)^{\Sigma N_{k\lambda}}$. (We note that this operator coincides with the operator of charge conjugation for the electromagnetic field, cf. § 23, formula (23.29).)

We now investigate the manner in which the operators of the free electron-positron field transform under an inversion of coordinates and a reflection of the time.

On writing this transformation in the form

$$\psi'(x) = O\psi(-x), \tag{21.20}$$

where O is a matrix which we call the inversion operator, and on using the Dirac equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi(x) = 0,$$

we obtain

$$\left(-\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)O^{-1}\psi'(x) = 0.$$

In order that this equation reduce to

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi'(x) = 0,$$

the operator O must satisfy

$$\gamma_\mu O + O\gamma_\mu = 0, \quad \mu = 1, 2, 3, 4.$$

Since the operation of inversion applied twice in succession must not alter the spinor, i.e., must reproduce it up to its sign, we must have $O^2 = \pm 1$. Finally, the matrix O must commute with all the transformations S of the wave function $\psi$ associated with the proper Lorentz transformations, or, in other words, with all the infinitesimal operators of the Lorentz group $I_{\mu\nu} = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$$[O, I_{\mu\nu}] = 0.$$

From these conditions it follows that O has the form

$$O = \xi\gamma_5$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and $\xi = \pm 1, \pm i$.

The inversion operator O can be represented in the form of the product of the operator of spatial inversion and the operator of time reversal which we respectively denote by $O_r$ and $O_t$:

$$O = O_r O_t. \tag{21.21}$$

We saw in § 9 that $O_r \equiv I = \xi\gamma_4$. Therefore we can say that

$$O_t = \gamma_1\gamma_2\gamma_3. \tag{21.22}$$

We note that $\bar{\psi}(x) = \psi^+(x)\gamma_4$ transforms under the inversion of the coordinates (21.20) like

$$\bar{\psi}'(x) = \overline{\psi'(x)} = \bar{\psi}(-x)\,\bar{O}, \tag{21.20'}$$

where

$$\bar{O} = \gamma_4 O^{+} \gamma_4 = -\xi^* \gamma_5 .$$

We can easily show that

$$O\bar{O} = -1; \quad O\gamma_\mu \bar{O} = \gamma_\mu, \tag{21.23}$$

and, therefore,

$$\begin{aligned} \bar{\psi}'(x)\psi'(x) &= -\bar{\psi}(-x)\psi(-x), \\ \bar{\psi}'(x)\gamma_\mu\psi'(x) &= \bar{\psi}(-x)\gamma_\mu\psi(-x). \end{aligned} \tag{21.23'}$$

Thus, under the transformation $x_\mu \to -x_\mu$, $\bar{\psi}\psi$ changes sign, i.e., behaves not as a scalar, but as a pseudoscalar, while $\bar{\psi}\gamma_\mu\psi$ does not reverse sign, i.e., behaves as a pseudovector.

It can be easily shown that the operators $\psi'(x)$ and $\bar{\psi}'(x)$ satisfy the same commutation relations as the operators $\psi(x)$ and $\bar{\psi}(x)$. Indeed, on replacing $x$ by $-x$, in the expression

$$\{\psi(x_1), \bar{\psi}(x_2)\} = i\left(\gamma_\mu \frac{\partial}{\partial x_{1_\mu}} - m\right)\Delta(x_1 - x_2)$$

and on substituting into it

$$\psi(-x) = O^{-1}\psi'(x), \quad \bar{\psi}(-x) = \bar{\psi}'(x)\bar{O}^{-1},$$

we obtain

$$\{O^{-1}\psi'(x_1), \bar{\psi}'(x_2)\,\bar{O}^{-1}\} = i\left(-\gamma_\mu \frac{\partial}{\partial x_{1_\mu}} - m\right)\Delta(-x_1 + x_2),$$

whence we have

$$\{\psi'(x_1), \bar{\psi}'(x_2)\} = iO\left(-\gamma_\mu \frac{\partial}{\partial x_{1_\mu}} - m\right)\bar{O}\Delta(-x_1 + x_2).$$

On utilizing (21.23) and the fact that the function $\Delta(x)$ is odd, we obtain

$$\{\psi'(x_1), \bar{\psi}'(x_2)\} = i\left(\gamma_\mu \frac{\partial}{\partial x_{1_\mu}} - m\right)\Delta(x_1 - x_2),$$

which is what we set out to prove.

Thus, in contrast to the free electromagnetic field for which the transformation of inversion leaves the field equations invariant, but does not leave the commutation relations invariant, in the case of the free electron-positron field the transformation of inversion leaves both the field equations and the commutation relations invariant.

On the other hand, for the free electron-positron field the transposed operators $\psi^{T}(x)$ satisfy the same equations of motion and the same commutation relations as the operators $\psi(x)$. From this we can conclude that the following relations hold between the operators $\psi'(x)$, $\psi(x)$ and $\psi^{T}(x)$ of the free electron-positron field:

$$\psi'(x) = U_1 \psi(x) U_1^{-1} \quad \text{and} \quad \psi'(x) = U_2 \psi^{T}(x) U_2^{-1}, \tag{21.24}$$

where $U_1$, $U_2$ are unitary operators. However, these relations are not equivalent, since the equations for interacting fields are not invariant under the inversion transformation $x_\mu \to -x_\mu$ alone. We shall show later (cf. subsection 22.4) that all the equations connecting the electromagnetic and the electron-positron fields remain invariant under the transformation of inversion aecompanied by a transition to the transposed operators, i.e., under the transformation

$$A_\mu(x) \to A_\mu^{R}(x) = A_\mu'^{T}(x) \quad \text{and} \quad \psi(x) \to \psi^{R}(x) = \psi'^{T}(x). \tag{21.25}$$

This combined transformation we shall refer to as the transformation of *total inversion* or of *strong inversion.* We emphasize that the transformation of total inversion corresponds not only to a reversal of the sign of the argument of the field operators, but is also associated with an interchange of the initial and final states in the matrix elements of the field operators. This interchange is brought about by the transition to the transposed operators.

Thus, of the two expressions (21.24) only the second expression has a physical meaning. It can be written simultaneously for the operators $A_\mu(x)$ and $\psi(x)$ in the form

$$A'(x) = U A_\mu^{T}(x) U^{-1} \quad \text{and} \quad \psi'(x) = U \psi^{T}(x) U^{-1}, \tag{21.26}$$

where U is a unitary operator operating in particle number space.

We now make an assumption which generalizes expressions (21.26), viz., we assume that in the case of arbitrary wave fields $\psi(x)$ the following relation holds:

$$\psi'(x) = O\psi(-x) = U\psi^{T}(x)\, U^{-1}, \tag{21.27}$$

where O is the transformation matrix for $\psi(x)$ associated with the inversion of coordinates, while U is a unitary operator. Without loss of generality we assume in accordance with (21.3) and (21.8) that the effect of the matrix O reduces to multiplication by $+1$, $-1$, $+i$, $-i$

depending on which of the four classes: $+1$, $-1$, $+\varepsilon$, $-\varepsilon$ is the one to which belongs the component of the field $\psi$:

$$\begin{aligned} O\psi^{(+1)} &= \psi^{(+1)}, & O\psi^{(-1)} &= -\psi^{(-1)}, \\ O\psi^{(+\varepsilon)} &= i\psi^{(+\varepsilon)}, & O\psi^{(-\varepsilon)} &= -i\psi^{(-\varepsilon)}. \end{aligned} \tag{21.28}$$

We now show that assumption (21.27) is equivalent to the theorem on the connection between spin and statistics; in other words we show that if (21.27) holds, then the commutation relations for the field operators must contain commutators in the case of integral spin and anticommutators in the case of half-integral spin (176).

We start from the relations

$$[\psi_r(x_1),\ \psi_s^+(x_2)]_\pm = F_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2), \tag{21.29}$$

where each of the operators $\psi_r, \psi_s$ belongs to one of the classes $+1$, $-1$, $+\varepsilon$, $-\varepsilon$ and $F_{rs}$ is a differential operator containing derivatives multiplied by constant coefficients. Since $[\psi_r(x_1), \psi_s^+(x_2)]_\pm$ is a $c$-number, we can write

$$\begin{aligned} F_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2) &= U[\psi_r(x_1), \psi_s^+(x_2)]_\pm^T U^{-1} \\ &= \pm\ [U\psi_r^T(x_1)U^{-1}, U\psi_s^{+T}(x_2)U^{-1}]_\pm \\ &= \pm\ \left[(O\psi(-x_1))_r, (O\psi(-x_2))_s^+\right]_\pm \\ &= \pm\sum_{r's'} O_{rr'}O_{ss'}^*[\psi_{r'}(-x_1), \psi_{s'}^+(-x_2)]_\pm . \end{aligned}$$

The bracket $[\psi_{r'}(-x_1), \psi_{s'}^+(-x_2)]_\pm$ appearing in this expression is evidently equal to $F_{r's'}\left(-\frac{\partial}{\partial x_1}\right)\Delta(-x_1+x_2)$, and since $\Delta(-x) = -\Delta(x)$, we have

$$[\psi_{r'}(-x_1),\ \psi_{s'}^+(-x_2)]_\pm = -F_{r's'}\left(-\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2).$$

Therefore

$$F_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2) = \mp\sum_{r's'} O_{rr'}O_{ss'}^* F_{r's'}\left(-\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2). \tag{21.29'}$$

In the case of integral spin the quantity $\psi_r\psi_s^+$ belongs to the class $+1$, if $\psi_r$ and $\psi_s$ belong to the same class ($+1$ or $-1$), and to the class $-1$, if $\psi_r$ and $\psi_s$ belong to different classes. In the former case the operator $\mathrm{F}_{rs}\left(\frac{\partial}{\partial x}\right)$ must contain derivatives of only even order, while in the latter case it must contain derivatives of only odd order. From this and from (21.28) we can conclude that in the case of integral spin

$$\sum_{r's'} \mathrm{O}_{rr'}\mathrm{O}^{*}_{ss'}\mathrm{F}_{r's'}\left(-\frac{\partial}{\partial x_1}\right) = \mathrm{F}_{rs}\left(\frac{\partial}{\partial x_1}\right), \quad 2(j+k) = 2N.$$

In a similar manner we can show that in the case of half-integral spin

$$\sum_{r's'} \mathrm{O}_{rr'}\mathrm{O}^{*}_{ss'}\mathrm{F}_{r's'}\left(-\frac{\partial}{\partial x_1}\right) = -F_{rs}\left(\frac{\partial}{\partial x_1}\right), \quad 2(j+k) = 2N+1.$$

Therefore, formula (21.29′) can be rewritten as

$$\mathrm{F}_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2) = \begin{cases} \mp\, \mathrm{F}_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2), & 2(j+k) = 2N, \\ \pm\, \mathrm{F}_{rs}\left(\frac{\partial}{\partial x_1}\right)\Delta(x_1-x_2), & 2(j+k) = 2N+1. \end{cases}$$

From this we see that in the case of integral spin the lower sign must be taken, which corresponds to the commutator in the commutation relations (21.29), while in the case of half-integral spin we must take the upper sign, which corresponds to the anticommutator in the commutation relations (21.29).

Thus, we have shown that the theorem on the connection between spin and statistics is a consequence of

$$\psi'(x) = \mathrm{U}\psi^{\mathrm{T}}(x)\mathrm{U}^{-1}.$$

The inverse theorem can also be proved, which consists of the fact that the invariance of the theory of wave fields under the transformation of total inversion follows from Pauli's theorem on the connection between spin and statistics (151), (129).

# Fundamental Equations of Quantum Electrodynamics

## § 22. Interacting Electromagnetic and Electron-Positron Fields

### 22.1. *System of Equations for Interacting Fields*

In the preceding sections we began by assuming the independent existence of the electromagnetic and the electron-positron fields, and we studied their properties separately, making use of two independent systems of equations—Maxwell's and Dirac's equations. In actual fact, however, as a result of the interaction between them, the electromagnetic and the electron-positron fields form a single dynamical system, which must be described by a coupled system of equations.

In order to set up this system of equations we can utilize, firstly, the Dirac equations for the electron wave function $\psi(x)$ in the presence of the electromagnetic field $A_\mu(x)$ (cf. § 12)

$$\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-ieA_\mu\right)+m\right)\psi=0,\quad \bar{\psi}\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu}+ieA_\mu\right)-m\right)=0 \tag{22.1}$$

and, secondly, the classical equations for the potentials of the electromagnetic field in the presence of currents

$$\Box A_\mu = -j_\mu, \tag{22.2}$$

where $j_\mu$ is the current density four-vector associated with the motion of the electron. In this expression we must take for the electron current density in the state $\psi$ the expression

$$j_\mu = ie\bar{\psi}\gamma_\mu\psi, \tag{22.3}$$

which formally coincides with the expression for the current density in the case of the free motion of the electron. As can be easily shown, this vector in virtue of the Dirac equations satisfies the equation of continuity

$$\frac{\partial}{\partial x_\mu}j_\mu = 0 \tag{22.3'}$$

and can be used for determining the electromagnetic field to which the electron in the state $\psi$ gives rise.

The Dirac equation for a single electron situated in a given electromagnetic field, and the wave equation for the potential of the electromagnetic field produced by a given electron current enable us to solve a number of problems, for example, scattering problems, electron energy eigenvalue problems, etc. However, within the framework of the one-body problem it is impossible to formulate a general theory of the interaction between electrons and the electromagnetic field. In order to formulate such a theory, we must, as in the preceding chapter, utilize the concept of quantized fields, and treat the potential of the electromagnetic field and the electron wave function not as ordinary quantities ($c$-numbers), but as certain operators ($q$-numbers) operating in particle-number space and satisfying, as functions of the coordinates and the time, certain differential equations. We denote these operators by the bold-face letters $\mathbf{A}_\mu$ and $\boldsymbol{\psi}_\lambda$ and we adopt equations (22.1) and (22.2), in which $\boldsymbol{\psi}$, $\bar{\boldsymbol{\psi}}$ and $\mathbf{A}_\mu$ are regarded as field operators, as the fundamental system of equations for the interacting fields. In doing this, as has been already explained in § 19, it is convenient to alter somewhat the definition of the current density vector and to assume that it is given by

$$\mathbf{j}_\mu = \frac{ie}{2}(\bar{\boldsymbol{\psi}}\gamma_\mu\boldsymbol{\psi} - \bar{\boldsymbol{\psi}}^c\gamma_\mu\boldsymbol{\psi}^c) = \frac{ie}{2}[\bar{\boldsymbol{\psi}}, \gamma_\mu\boldsymbol{\psi}], \tag{22.4}$$

where $\boldsymbol{\psi}^c = \mathrm{C}\bar{\boldsymbol{\psi}}$, $\bar{\boldsymbol{\psi}}^c = \mathrm{C}^{-1}\boldsymbol{\psi}$ are the charge-conjugate operators of the electron-positron field. This vector, like the vector (22.3), satisfies the equation of continuity, and can, therefore, be interpreted as the current-density vector. We shall use this definition of the current density because it does not lead to an infinite vacuum charge.

In addition to the system of differential equations for the field operators we must also formulate the commutation relations between these operators. Since the time-dependence of the operators is determined by the field equations, the commutation relations can be specified only at some initial instant of time.

The introduction of definite initial commutation relations is equivalent to prescribing them at identical instants of time $t = t'$. For such commutation relations we adopt the expressions

$$[\mathbf{A}_\mu(x), \mathbf{A}_\nu(x')]_{t=t'} = 0,$$

$$\left[\mathbf{A}_\mu(x), \frac{\partial}{\partial t'}\mathbf{A}_\nu(x')\right]_{t=t'} = i\delta_{\mu\nu}\delta(\boldsymbol{r}-\boldsymbol{r}'),$$
$$\left[\frac{\partial \mathbf{A}_\mu(x)}{\partial t}, \frac{\partial \mathbf{A}_\nu(x')}{\partial t'}\right]_{t=t'} = 0,$$
$$\{\psi_\alpha(x), \psi_\beta^\dagger(x')\}_{t=t'} = \delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{22.5}$$
$$\{\psi_\alpha(x), \psi_\beta(x')\}_{t=t'} = 0,$$
$$[\psi_\alpha(x), \mathbf{A}_\mu(x')]_{t=t'} = 0,$$
$$\left[\psi_\alpha(x), \frac{\partial \mathbf{A}_\mu(x')}{\partial t}\right]_{t=t'} = 0.$$

In the case of free fields these relations are equivalent to the general relations (17.1), (19.4) for arbitrary instants of time $t$ and $t'$. However, in the case of interacting fields the general commutation relations for arbitrary values of $t \neq t'$ cannot be formulated in advance, since in order to do this we would have to obtain the general solution of the equations of motion of the coupled fields.

The state of the system of interacting fields is specified by a unit vector in Hilbert space—the state vector $\Phi^0$. In our method of description this vector does not vary with time, since the variation in the state of the system is taken into account by the variation with time of the field operators (such a method of describing the state of the system is called in quantum mechanics the Heisenberg picture — cf. subsection 23.1).

However, not all state vectors are admissible: just as in the case of the free electromagnetic field, the allowable states are restricted by the subsidiary condition (16.26).

Thus, we adopt the following system of fundamental equations of quantum electrodynamics for the field operators:

$$\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ie\mathbf{A}_\mu\right) + m\right)\psi = 0,$$
$$\bar{\psi}\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + ie\mathbf{A}_\mu\right) - m\right) = 0,$$
$$\frac{\partial^2}{\partial x_\nu^2}\mathbf{A}_\mu = -\frac{ie}{2}[\bar{\psi}, \gamma_\mu\psi],$$
$$\frac{\partial \Phi^0}{\partial t} = 0, \tag{22.6}$$

$$\begin{aligned}
&\left(\Phi^0, \frac{\partial \mathbf{A}_\mu}{\partial x_\mu}\Phi^0\right) = 0,\\
&\left[\mathbf{A}_\mu(x), \frac{\partial}{\partial t'}\mathbf{A}_\nu(x')\right]_{t=t'} = i\delta_{\mu\nu}\delta(\boldsymbol{r}-\boldsymbol{r}'),\\
&\{\boldsymbol{\psi}_\alpha(x), \boldsymbol{\psi}_\beta^\dagger(x')\}_{t=t'} = \delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'),\\
&[\mathbf{A}_\mu(x), \mathbf{A}_\nu(x')]_{t=t'} = 0,\\
&\{\boldsymbol{\psi}_\alpha(x), \boldsymbol{\psi}_\beta(x')\}_{t=t'} = 0, \qquad\qquad (22.6,\ \textit{cont.})\\
&[\boldsymbol{\psi}_\alpha(x), \mathbf{A}_\mu(x')]_{t=t'} = 0,\\
&\left[\frac{\partial \mathbf{A}_\mu(x)}{\partial t}, \frac{\partial \mathbf{A}_\nu(x')}{\partial t'}\right]_{t=t'} = 0,\\
&\left[\boldsymbol{\psi}_\alpha(x), \frac{\partial \mathbf{A}_\mu(x')}{\partial t'}\right]_{t=t'} = 0.
\end{aligned}$$

We note that from the third and the fifth equations of this system we can easily obtain the second pair of Maxwell's equations, which now means that the expectation value of the operator $\frac{\partial}{\partial x_\mu}\mathbf{F}_{\mu\nu}+\mathbf{j}_\nu$ in the state $\Phi^0$ is equal to zero:

$$\left(\Phi^0, \left(\frac{\partial}{\partial x_\mu}\mathbf{F}_{\mu\nu}+\mathbf{j}_\nu\right)\Phi^0\right) = 0, \tag{22.7}$$

where $\mathbf{F}_{\mu\nu}$ is the operator for the field tensor:

$$\mathbf{F}_{\mu\nu} = (\partial/\partial x_\mu)\mathbf{A}_\nu-(\partial/\partial x_\nu)\mathbf{A}_\mu.$$

### 22.2. *Lagrangian. Energy-Momentum Tensor*

The equations of motion of quantum electrodynamics can be obtained from the variational principle $\delta\int L d^4x = 0$, where $L$ is the Lagrangian density.

We first treat the quantities $\psi, \bar{\psi}, A_\mu$ as $c$-numbers. In this case the Dirac equation (22.1) for an electron in the electromagnetic field, and the wave equation for the electromagnetic potential (22.2) can be obtained from the variational principle if for the Lagrangian density we take

$$\begin{aligned}
L \equiv L_0+L_{e\gamma} = &-\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu}-\frac{1}{2}\bar{\psi}\left(\gamma_\nu\frac{\partial}{\partial x_\nu}+m\right)\psi\\
&+\frac{1}{2}\psi\left(\tilde{\gamma}_\nu\frac{\partial}{\partial x_\nu}-m\right)\bar{\psi}+ie\bar{\psi}\gamma_\nu\psi A_\nu,
\end{aligned} \tag{22.8}$$

where $L_0$ is the sum of the corresponding functions for the free fields $L_0 = L_\gamma + L_e$:

$$L_\gamma = -\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu} \tag{22.9}$$

and

$$L_e = -\frac{1}{2}\bar{\psi}\left(\gamma_\nu\frac{\partial}{\partial x_\nu}+m\right)\psi+\frac{1}{2}\psi\left(\tilde{\gamma}_\nu\frac{\partial}{\partial x_\nu}-m\right)\bar{\psi}, \tag{22.10}$$

while $L_{e\gamma}$ is the part of the Lagrangian density which describes the interaction between the fields:

$$L_{e\gamma} = ie\bar{\psi}\gamma_\nu\psi A_\nu. \tag{22.11}$$

Indeed, according to (16.4) the field equations have the following form ($\psi$, $\bar{\psi}$, $A_\mu$ are treated as independent variables):

$$\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial(\partial A_\nu/\partial x_\mu)}-\frac{\partial L}{\partial A_\nu}=0,$$

$$\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial(\partial\psi/\partial x_\mu)}-\frac{\partial L}{\partial\psi}=0, \qquad \frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial(\partial\bar{\psi}/\partial x_\mu)}-\frac{\partial L}{\partial\bar{\psi}}=0.$$

On substituting into these expressions

$$\frac{\partial L}{\partial A_\nu}=ie\bar{\psi}\gamma_\nu\psi, \qquad \frac{\partial L}{\partial(\partial A_\nu/\partial x_\mu)}=-\frac{\partial A_\nu}{\partial x_\mu},$$

$$\frac{\partial L}{\partial\psi}=\frac{1}{2}\bar{\psi}(ie\gamma_\nu A_\nu-m)+\frac{1}{2}\left(\tilde{\gamma}_\nu\left(\frac{\partial}{\partial x_\nu}+ieA_\nu\right)-m\right)\bar{\psi}, \quad \frac{\partial L}{\partial(\partial\psi/\partial x_\nu)}=-\frac{1}{2}\bar{\psi}\gamma_\nu,$$

$$\frac{\partial L}{\partial\bar{\psi}}=\frac{1}{2}\psi(ie\tilde{\gamma}_\nu A_\nu-m)-\frac{1}{2}\left(\gamma_\nu\left(\frac{\partial}{\partial x_\nu}-ieA_\nu\right)+m\right)\psi, \quad \frac{\partial L}{\partial(\partial\bar{\psi}/\partial x_\nu)}=\frac{1}{2}\psi\tilde{\gamma}_\nu,$$

we obtain equations (22.1) and (22.2).

We note that the part of the Lagrangian density describing the interaction between the fields is equal to the scalar product of the potential of the electromagnetic field $A_\nu$ and the current density vector $\mathrm{j}_\nu$:

$$L_{e\gamma} = A_\nu \mathrm{j}_\nu. \tag{22.11'}$$

This vector can be defined with the aid of $L$ in accordance with the general formula (20.29)

$$j_\mu = -ie\left(\frac{\partial L}{\partial(\partial\psi/\partial x_\mu)}\psi-\bar{\psi}\frac{\partial L}{\partial(\partial\bar{\psi}/\partial x_\mu)}\right)=ie\bar{\psi}\gamma_\mu\psi.$$

We note that the function $L_{e\gamma}$ is the only invariant in which the electromagnetic field potentials appear linearly, and the electron functions appear in bilinear form, and which does not contain any derivatives of $\psi$ and $A_\mu$.

In accordance with the general formula (20.17) the Lagrangian density (22.8) corresponds to the following energy-momentum tensor:

$$T_{\mu\nu}=L\delta_{\mu\nu}-\frac{\partial L}{\partial(\partial A_\varrho/\partial x_\nu)}\frac{\partial A_\varrho}{\partial x_\mu}-\frac{\partial L}{\partial(\partial\psi/\partial x_\nu)}\frac{\partial\psi}{\partial x_\mu}-\frac{\partial L}{\partial(\partial\bar{\psi}/\partial x_\nu)}\frac{\partial\bar{\psi}}{\partial x_\mu}$$

$$=\frac{\partial A_\lambda}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\mu}-\frac{1}{2}\delta_{\mu\nu}\frac{\partial A_\lambda}{\partial x_\varrho}\frac{\partial A_\lambda}{\partial x_\varrho}+\frac{1}{2}\bar{\psi}\gamma_\nu\frac{\partial\psi}{\partial x_\mu}-\frac{1}{2}\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\nu\psi. \tag{22.12}$$

It represents a sum of expressions for the energy-momentum tensors of the electromagnetic and the electron-positron fields.

In order to obtain with the aid of the variational principle the equations for the quantized fields we must take for the Lagrangian density the operator

$$\mathbf{L}=-\frac{1}{2}\frac{\partial\mathbf{A}_\mu}{\partial x_\nu}\frac{\partial\mathbf{A}_\mu}{\partial x_\nu}-\frac{1}{2}\bar{\boldsymbol{\psi}}\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-ie\mathbf{A}_\mu\right)+m\right)\boldsymbol{\psi}$$

$$-\frac{1}{2}\bar{\boldsymbol{\psi}}^c\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu}+ie\mathbf{A}_\mu\right)+m\right)\bar{\boldsymbol{\psi}}^c$$

$$=-\frac{1}{2}\frac{\partial\mathbf{A}_\mu}{\partial x_\nu}\frac{\partial\mathbf{A}_\mu}{\partial x_\nu}-\frac{1}{2}\bar{\boldsymbol{\psi}}\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m\right)\boldsymbol{\psi}$$

$$-\frac{1}{2}\bar{\boldsymbol{\psi}}^c\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m\right)\boldsymbol{\psi}^c+\mathbf{j}_\mu\mathbf{A}_\mu, \tag{22.13}$$

and in varying the action treat $\mathbf{A}_\mu,\boldsymbol{\psi},\bar{\boldsymbol{\psi}}$ (or $\mathbf{A}_\mu,\boldsymbol{\psi}^c,\bar{\boldsymbol{\psi}}^c$) as the independent variables. On noting that

$$\bar{\boldsymbol{\psi}}^c\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu}+ie\mathbf{A}_\mu\right)+m\right)\boldsymbol{\psi}^c=\boldsymbol{\psi}\left(\tilde{\gamma}_\mu\left(\frac{\partial}{\partial x_\mu}+ie\mathbf{A}_\mu\right)-m\right)\bar{\boldsymbol{\psi}},$$

on carrying out the variation, and on neglecting the inessential terms having the form of divergences, we obtain the expression

$$\begin{aligned}\delta \mathbf{L} = {} & \frac{1}{2}\,\delta \mathbf{A}_\mu(\Box \mathbf{A}_\mu + \mathbf{j}_\mu) + \frac{1}{2}(\Box \mathbf{A}_\mu + \mathbf{j}_\mu)\,\delta \mathbf{A}_\mu \\ & - \frac{1}{2}\,\delta \bar{\psi}\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ie\mathbf{A}_\mu\right) + m\right)\psi \\ & + \frac{1}{2}\left(\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ie\mathbf{A}_\mu\right) + m\right)\psi\,\delta \bar{\psi} \\ & - \frac{1}{2}\,\delta \psi\left(\tilde{\gamma}_\mu\left(\frac{\partial}{\partial x_\mu} + ie\mathbf{A}_\mu\right) - m\right)\bar{\psi} \\ & + \frac{1}{2}\left(\tilde{\gamma}_\mu\left(\frac{\partial}{\partial x_\mu} + ie\mathbf{A}_\mu\right) - m\right)\bar{\psi}\,\delta\psi = 0,\end{aligned}$$

from which the first three equations (22.6) follow. If in carrying out the variation of $L$ we treat $\psi^c$ and $\bar{\psi}^c$ as the independent variables in place of $\psi$ and $\bar{\psi}$ we obtain in place of the first two equations (22.6) the equations for the charge-conjugate operators $\psi^c$ and $\bar{\psi}^c$.

We note that in the expression (22.13) for $L$ the order of arrangement of the field operators is important; if we treat these quantities as $c$-numbers, expression (22.13) does not go over into (22.8).

In the theory of quantized fields, just as in $c$-number theory, the energy-momentum tensor can be defined by

$$\begin{aligned}\mathbf{T}_{\mu\nu} = {} & \frac{1}{2}\left(\frac{\partial \mathbf{A}_\sigma}{\partial x_\nu}\frac{\partial \mathbf{A}_\sigma}{\partial x_\mu} + \frac{\partial \mathbf{A}_\sigma}{\partial x_\mu}\frac{\partial \mathbf{A}_\sigma}{\partial x_\nu} - \delta_{\mu\nu}\frac{\partial \mathbf{A}_\sigma}{\partial x_\lambda}\frac{\partial \mathbf{A}_\sigma}{\partial x_\lambda}\right) \\ & + \frac{1}{4}\left(\bar{\psi}\gamma_\nu\frac{\partial \psi}{\partial x_\mu} - \frac{\partial \bar{\psi}}{\partial x_\mu}\gamma_\nu\psi\right) + \frac{1}{4}\left(\bar{\psi}^c\gamma_\nu\frac{\partial \psi^c}{\partial x_\mu} - \frac{\partial \bar{\psi}^c}{\partial x_\mu}\gamma_\nu\psi^c\right), \qquad (22.14)\end{aligned}$$

which is equal to the sum of the energy-momentum tensors for the free fields. It can be easily shown that this tensor is a Hermitian operator and satisfies the continuity equation

$$\frac{\partial}{\partial x_\nu}\mathbf{T}_{\mu\nu} = 0.$$

From this follows the conservation of the energy-momentum four-vector for the field

$$\mathbf{P}_\mu = -i\int \mathbf{T}_{\mu 4}\,dr = \text{const.} \qquad (22.15)$$

### 22.3. *Field Equations in Poisson Bracket Form*

We now show that the operators $\mathbf{A}_\mu(x)$ and $\psi_\alpha(x)$ satisfy the following relations:

$$\frac{\partial}{\partial x_\nu}\mathbf{A}_\mu(x) = i[\mathbf{A}_\mu(x), \mathbf{P}_\nu], \qquad \frac{\partial}{\partial x_\nu}\psi_\alpha(x) = i[\psi_\alpha(x), \mathbf{P}_\nu]. \tag{22.16}$$

Since $\mathbf{P}_\nu = \text{const.}$, we can assume that in the integral defining $\mathbf{P}_\nu$ all the operators are taken at the same instant of time as $\mathbf{A}_\mu(x)$. We therefore obtain on utilizing (22.14) and (22.5) the following result:

$$\begin{aligned} i[\mathbf{A}_\mu(x), \mathbf{P}_\nu] = -i\int dr'\Big(&\frac{\partial}{\partial x'_\nu}\mathbf{A}_\sigma(x')\Big[\mathbf{A}_\mu(x), \frac{\partial}{\partial t'}\mathbf{A}_\sigma(x')\Big] \\ &+\Big[\mathbf{A}_\mu(x), \frac{\partial}{\partial x'_\nu}\mathbf{A}_\sigma(x')\Big]\frac{\partial}{\partial t'}\mathbf{A}_\sigma(x') \\ &-i\delta_{4\nu}\frac{\partial}{\partial x'_\varrho}\mathbf{A}_\sigma(x')\Big[\mathbf{A}_\mu(x), \frac{\partial}{\partial x'_\varrho}\mathbf{A}_\sigma(x')\Big)\Big]_{t'=t} \\ = \int dr'\Big(&\frac{\partial}{\partial x'_\nu}\mathbf{A}_\sigma(x')\,\delta_{\mu\sigma}\delta(\boldsymbol{r}-\boldsymbol{r}') \\ &-i\delta_{\mu\sigma}\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta_{\nu4}\frac{\partial}{\partial t'}\mathbf{A}_\sigma(x') \\ &-\delta_{\nu4}\delta_{\mu\sigma}\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta_{\varrho4}\frac{\partial}{\partial x'_\varrho}\mathbf{A}_\sigma(x')\Big)_{t=t'} = \frac{\partial}{\partial x_\nu}\mathbf{A}_\mu(x). \end{aligned}$$

We now prove the second of relations (22.16). It can be easily shown that

$$\begin{aligned} i[\psi_\alpha(x), \boldsymbol{P}_\mu] = \frac{1}{2}\int dr'\Big[\psi_\alpha(x), \Big(&\psi_\beta^+(x')\frac{\partial}{\partial x'_\mu}\psi_\beta(x') \\ &-\frac{\partial}{\partial x'_\mu}\psi_\beta^+(x')\,\psi_\beta(x')\Big)\Big]_{t'=t} \\ = \frac{1}{2}\int dr'\Big(&\{\psi_\alpha(x), \psi_\beta^+(x')\}\frac{\partial}{\partial x'_\mu}\psi_\beta(x') \\ -\psi_\beta^+(x')\Big\{\psi_\alpha(x), \frac{\partial}{\partial x'_\mu}\psi_\beta(x')\Big\} &- \Big\{\psi_\alpha(x), \frac{\partial}{\partial x'_\mu}\psi_\beta^+(x')\Big\}\psi_\beta(x') \\ +\frac{\partial}{\partial x'_\mu}\psi_\beta^+(x')\,\{\psi_\alpha(x), \psi_\beta(x')\}\Big)_{t'=t}&. \end{aligned}$$

From the commutation relations and the equations of motion it follows that

$$\left\{\psi_\alpha(x), \frac{\partial}{\partial x'_\mu}\psi_\beta(x')\right\}_{t'=t} = 0,$$

and therefore

$$i[\psi_\alpha(x), P_\mu] = \int dr' \left( \{\psi_\alpha(x), \psi^+_\beta(x')\} \frac{\partial}{\partial x'_\mu}\psi_\beta(x') \right)$$
$$- \frac{1}{2}\frac{\partial}{\partial x'_\mu}\left( \{\psi_\alpha(x), \psi^+_\beta(x')\}\psi_\beta(x') \right)_{t'=t}.$$

The integral of the second term vanishes: for $\mu = 1, 2, 3$ this is evident, while for $\mu = 4$ it follows from the law of conservation of charge:

$$\frac{d}{dt}\int \psi^+(x)\psi(x)\, dr = \frac{d}{dt}\int \bar{\psi}\gamma_4\psi\, dr = 0.$$

Therefore we finally obtain

$$i[\psi_\alpha(x), \mathbf{P}_\mu] = \int dr' \{\psi_\alpha(x), \psi^+_\beta(x')\} \frac{\partial}{\partial x'_\mu}\psi_\beta(x') = \frac{\partial}{\partial x_\mu}\psi_\alpha(x).$$

We note that by utilizing (22.16) we can prove the validity of the general relation

$$\frac{\partial}{\partial x_\mu}\mathbf{F}(x) = i[\mathbf{F}(x), \mathbf{P}_\mu], \tag{22.17}$$

where $\mathbf{F}(x)$ is an arbitrary function of the field variables at the point $x$. This relation enables us to determine the dependence of the matrix elements of the operator $\mathbf{F}(x)$ on the coordinates. Indeed, on going over to the representation in which the operators $\mathbf{P}_\mu$ are diagonal, we obtain, from (22.17),

$$\frac{\partial}{\partial x_\mu}\left(\mathbf{F}(x)\right)_{12} = i\left(\mathbf{F}(x)\right)_{12}(P^{(2)}_\mu - P^{(1)}_\mu),$$

where $\left(\mathbf{F}(x)\right)_{12}$ is the matrix element of the operator $\mathbf{F}(x)$ between the states 1 and 2 of the system for which the energy-momentum vector of the field is equal to $P^{(1)}_\mu$ and $P^{(2)}_\mu$. From this it follows that

$$\left(\mathbf{F}(x)\right)_{12} = F_{12}e^{-i\left(P^{(1)}_\mu - P^{(2)}_\mu\right)x_\mu}, \tag{22.18}$$

where $F_{12}$ no longer depends on the coordinates. This formula is a natural relativistic generalization of the well-known quantum mechanical formula which determines the dependence of the matrix elements on the time.

### 22.4. *Invariance Properties of the Equations of Quantum Electrodynamics*

We now show that the fundamental equations of quantum electrodynamics are invariant under Lorentz and gauge transformations, and under the transformations of space inversion, charge conjugation and strong inversion.

Invariance under Lorentz transformations is immediately evident from the fact that the Lagrangian density is a scalar.

By a gauge transformation we mean a transformation of the following form:

$$\begin{aligned} \mathbf{A}_\mu &\to \mathbf{A}_\mu - \frac{\partial \eta}{\partial x_\mu}, \\ \psi &\to e^{-ie\eta}\psi, \\ \bar{\psi} &\to e^{ie\eta}\bar{\psi}, \end{aligned} \tag{22.19}$$

where $\eta$ is a scalar function satisfying

$$\Box\, \eta = 0, \tag{22.20}$$

but otherwise arbitrary.

The invariance of the equations of quantum electrodynamics under these transformations follows from the fact that as a result of transformation (22.19) the Lagrangian density acquires an inessential additional term which can be written in the form

$$\frac{\partial}{\partial x_\mu}\left(\mathbf{A}_\nu \frac{\partial^2 \eta}{\partial x_\mu \partial x_\nu}\right) - \mathbf{A}_\nu \frac{\partial}{\partial x_\nu}\frac{\partial^2 \eta}{\partial x_\mu^2} + \frac{\partial}{\partial x_\nu}\left(\frac{\partial \eta}{\partial x_\mu}\frac{\partial^2 \eta}{\partial x_\mu \partial x_\nu}\right) - \frac{\partial \eta}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\partial^2 \eta}{\partial x_\mu \partial x_\nu} + j_\mu \frac{\partial \eta}{\partial x_\mu}.$$

Here the first and third terms are of the form of four-dimensional divergences and therefore may be neglected, while the second and fourth terms vanish in virtue of equation (22.20). Finally, the last term, which is equal to

$$(\partial/\partial x_\mu)(\eta j_\mu) - \eta(\partial j_\mu/\partial x_\mu),$$

may be omitted, since $\partial j_\mu/\partial x_\mu = 0$.

We note that equation (22.20) must hold in order that the potentials $A_\mu$ and $A'_\mu = A_\mu - (\partial\eta/\partial x_\mu)$ should satisfy $\Box A_\mu = -j_\mu$, while the Dirac equations retain their form for any function $\eta$ not even necessarily satisfying equation (22.20).

We now consider the transformation of charge conjugation

$$\begin{aligned} \psi(x) &\to \psi^c(x) = C\bar{\psi}(x), \\ \bar{\psi}(x) &\to \bar{\psi}^c(x) = C^{-1}\psi(x), \\ A_\mu(x) &\to A^c_\mu(x) = -A_\mu(x). \end{aligned} \tag{22.21}$$

Under this transformation the sign of the current reverses in accordance with (22.4): $j_\mu \to j^c_\mu = -j_\mu$, and therefore the Lagrangian density remains unchanged.

We note that the reversal of the sign of $j_\mu$ agrees with the simple obvious interpretation of charge conjugation as the reversal of the sign of the charge.

The invariance of the field equations under the transformation of charge conjugation is a mathematical formulation of the symmetry of the theory with respect to both signs of the charge, i.e., with respect to the electron and the positron.

Finally, we prove the invariance of the equations of quantum electrodynamics under the transformation of strong inversion.

In the expression (22.13) for L we substitute $x \to -x$:

$$\begin{aligned} L(-x) = L\{A_\mu(-x), \psi(-x)\} = &-\frac{1}{2}\frac{\partial A_\mu(-x)}{\partial x_\nu}\frac{\partial A_\mu(-x)}{\partial x_\nu} \\ &-\frac{1}{2}\bar{\psi}(-x)\left[\gamma_\mu\left(-\frac{\partial}{\partial x_\mu} - ieA_\mu(-x)\right) + m\right]\psi(-x) \\ &-\frac{1}{2}\psi(-x)\left[\gamma^T_\mu\left(-\frac{\partial}{\partial x_\mu} + ieA_\mu(-x)\right) - m\right]\bar{\psi}(-x). \end{aligned}$$

Further, on substituting into this expression

$$\begin{aligned} \psi(-x) &= O^{-1}\psi'(x), \\ \bar{\psi}(-x) &= \bar{\psi}'(x)O^{-1}, \\ A_\mu(-x) &= -A'_\mu(x), \end{aligned} \tag{22.22}$$

where O is the inversion operator introduced in subsection 21.3, and on utilizing expressions (21.23)

$$\mathrm{O}\bar{\mathrm{O}} = -1, \qquad \mathrm{O}\gamma_\mu\bar{\mathrm{O}} = \gamma_\mu,$$
$$\bar{\mathrm{O}}^{\mathrm{T}}\mathrm{O}^{\mathrm{T}} = -1, \qquad \bar{\mathrm{O}}^{\mathrm{T}}\gamma_\mu^{\mathrm{T}}\mathrm{O}^{\mathrm{T}} = \gamma_\mu^{\mathrm{T}},$$

we obtain

$$\begin{aligned} \mathbf{L}(-x) = & -\frac{1}{2}\frac{\partial \mathbf{A}'_\mu(x)}{\partial x_\nu}\frac{\partial \mathbf{A}'_\mu(x)}{\partial x_\nu} \\ & -\frac{1}{2}\bar{\psi}'(x)\left[\gamma_\mu\left(-\frac{\partial}{\partial x_\mu}+ie\mathbf{A}'_\mu(x)\right)-m\right]\psi'(x) \\ & -\frac{1}{2}\bar{\psi}'(x)\left[\gamma_\mu^{\mathrm{T}}\left(-\frac{\partial}{\partial x_\mu}-ie\mathbf{A}'_\mu(x)\right)+m\right]\psi'(x). \end{aligned} \tag{22.23}$$

We now go over from **L** to the transposed operator $\mathbf{L}^{\mathrm{T}}$:

$$\begin{aligned} \mathbf{L}^{\mathrm{T}}(-x) = & -\frac{1}{2}\frac{\partial \mathbf{A}^R_\mu(x)}{\partial x_\nu}\frac{\partial \mathbf{A}^R_\mu(x)}{\partial x_\nu} \\ & -\frac{1}{2}\psi^R(x)\left[\gamma_\mu^T\left(-\frac{\overleftarrow{\partial}}{\partial x_\mu}+ie\mathbf{A}^R_\mu(x)\right)-m\right]\bar{\psi}^R(x) \\ & -\frac{1}{2}\bar{\psi}^R(x)\left[\gamma_\mu\left(-\frac{\overleftarrow{\partial}}{\partial x_\mu}-ie\mathbf{A}^R_\mu(x)\right)+m\right]\psi^R(x), \end{aligned}$$

where in accordance with (21.25) we have

$$\mathbf{A}^R_\mu(x) = \mathbf{A}'^{\mathrm{T}}_\mu(x); \qquad \psi^R(x) = \psi'^{\mathrm{T}}(x) \tag{22.24}$$

and the arrow over $\partial/\partial x_\mu$ denotes that the derivative is taken of the expression standing to the left, and not to the right of $\partial/\partial x_\mu$. Since the Lagrangian density appears under the integral sign we may change the "direction" of differentiation in $\mathbf{L}^{\mathrm{T}}$ if we simultaneously reverse the sign of $\partial/\partial x_\mu$, i.e.,

$$\begin{aligned} \mathbf{L}^{\mathrm{T}}\{\mathbf{A}_\mu(-x),\psi(-x)\} = & -\frac{1}{2}\frac{\partial \mathbf{A}^R_\mu(x)}{\partial x_\nu}\frac{\partial \mathbf{A}^R_\mu(x)}{\partial x_\nu} \\ & -\frac{1}{2}\psi^R(x)\left[\gamma_\mu^{\mathrm{T}}\left(\frac{\partial}{\partial x_\mu}+ie\mathbf{A}^R_\mu(x)\right)-m\right]\bar{\psi}^R(x) \\ & -\frac{1}{2}\bar{\psi}^R(x)\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-ie\mathbf{A}^R_\mu(x)\right)+m\right]\psi^R(x). \end{aligned}$$

A comparison of this expression with (22.13) shows that $\mathbf{L}^{\mathrm{T}}(-x)$ can be obtained from $\mathbf{L}\{\mathbf{A}(x),\psi(x)\}$ if the following substitution is made in $\mathbf{L}(x)$:

$$\begin{aligned} \psi(x) &\to \psi^{R}(x) = \psi'^{\mathrm{T}}(x), \\ \bar{\psi}(x) &\to \bar{\psi}^{R}(x) = \bar{\psi}'^{\mathrm{T}}(x), \\ \mathbf{A}_\mu(x) &\to \mathbf{A}_\mu^{R}(x) = \mathbf{A}_\mu'^{\mathrm{T}}(x), \end{aligned} \tag{22.25}$$

i.e.,

$$\left(\mathbf{L}\{\mathbf{A}_\mu(-x),\psi(-x)\}\right)^{\mathrm{T}} = \mathbf{L}\{\mathbf{A}_\mu^{R}(x),\psi^{R}(x)\}.$$

From this it follows that the operators $\psi^{R}(x),\bar{\psi}^{R}(x)$ and $\mathbf{A}_\mu^{R}(x)$ satisfy the same equations as the operators $\psi(x),\bar{\psi}(x)$, $\mathbf{A}_\mu(x)$. In other words, the equations of quantum electrodynamics are invariant under the transformation of total inversion.

It can be easily shown that the commutation relations (22.5) are also invariant under the transformation of strong inversion.

The necessity of supplementing the transformation of inversion by the transition to transposed operators is clearly seen using as an example the equation

$$\Box\mathbf{A}_\mu(x) = -\mathbf{j}_\mu(x).$$

The left-hand side of this equation reverses its sign under the transformation of inversion, while in accordance with (21.3) the current density does not change its sign:

$$\Box\mathbf{A}'_\mu(x) = \mathbf{j}'_\mu(x).$$

On the other hand,

$$[\psi_\alpha,\psi_\beta]^{\mathrm{T}} = -\,[\psi_\alpha^{\mathrm{T}},\psi_\beta^{\mathrm{T}}].$$

Therefore, we have

$$\Box\mathbf{A}_\mu^{R}(x) = -\mathbf{j}_\mu^{R}(x),$$

where

$$\mathbf{j}_\mu^{R}(x) = -\mathbf{j}_\mu^{\mathrm{T}}(-x).$$

Thus, in order to preserve the invariance of the equation connecting $\mathbf{A}_\mu$ with $\mathbf{j}_\mu$, simultaneously with carrying out the transformation of inversion we must also go over to the transposed operators.

Thus, the equations of quantum electrodynamics are invariant not only under Lorentz transformations (which also include the transformation of inversion in ordinary space), but also under transformations of total inversion $R$ and of charge conjugation C. They are therefore also invariant under the product of these fundamental transformations. In particular, the equations of quantum electrodynamics are invariant under the transformation J = RC, which is called *weak* inversion. In accordance with formulas (22.21) and (22.25) this transformation has the following form:

$$\begin{aligned} \mathbf{A}_\mu(x) &\to \mathbf{A}^J_\mu(x) = \mathbf{A}^{\mathrm{T}}_\mu(-x), \\ \psi(x) &\to \psi^J(x) = \left(\Omega^{-1}\bar{\psi}(-x)\right)^{\mathrm{T}}, \\ \bar{\psi}(x) &\to \bar{\psi}^J(x) = (\psi(-x)\,\Omega)^{\mathrm{T}}, \end{aligned} \qquad (22.26)$$

where $\Omega^{-1} = \mathrm{OC} = \xi\gamma_5\mathrm{C}$. In virtue of (10.2) this matrix satisfies the conditions

$$\Omega\gamma_\mu\Omega^{-1} = \gamma^{\mathrm{T}}_\mu \quad (\mu = 1, 2, 3, 4, 5,);$$

$$\Omega\Omega^{+} = 1, \quad \Omega^{\mathrm{T}} = -\Omega, \quad (\gamma_5\gamma_\mu)^{\mathrm{T}} = -\Omega\gamma_5\gamma_\mu\Omega^{-1}, \quad \gamma^{\mathrm{T}}_{[\mu\nu]} = -\Omega\gamma_{[\mu\nu]}\Omega^{-1},$$

where

$$\gamma_{[\mu\nu]} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).$$

It can be easily shown that in contrast to the transformation of total inversion which reverses the sign of the current density

$$\mathbf{j}^R_\mu(x) = -\mathbf{j}^{\mathrm{T}}_\mu(-x)$$

the transformation of weak inversion does not reverse the sign of the current density

$$\mathbf{j}^J_\mu(x) = \mathbf{j}^{\mathrm{T}}_\mu(-x).$$

This is associated with the that fact that the current density reverses sign both under the transformation R and under the transformation C.

From this we can conclude that the transformation of weak inversion is analogous to the space-time inversion in classical electrodynamics. Indeed, since under the transformation $x_\mu \to -x_\mu$ the velocity of the particles is not changed, the current density also does not change sign, and from this it follows in turn that under the transformation $x_\mu \to -x_\mu$ the potential of the field $\mathrm{A}_\mu$ should also remain unchanged, since the Lagrangian for the particle must be invariant

$$L = -mc^2\sqrt{1-(v^2/c^2)} + \frac{e}{c}Av - e\varphi$$

($\varphi$ is the scalar potential, $m$ is the particle mass).

From the form of the energy-momentum tensor (22.14) it follows that both the transformation of total inversion and the transformation of weak inversion leave $\mathrm{T}_{\mu\nu}$ invariant

$$\mathrm{T}^{R}_{\mu\nu}(x) = \mathrm{T}^{\mathrm{T}}_{\mu\nu}(-x) \quad \text{and} \quad \mathrm{T}^{J}_{\mu\nu}(x) = \mathrm{T}^{\mathrm{T}}_{\mu\nu}(-x).$$

Thus, the principal physical difference between the transformations R and J consists of the fact that the former changes the current density while the latter leaves it unchanged.

The operator J, as well as the operator O, can be represented in the form of a product of the operators of space and time inversion which we denote by P and T:

$$\mathrm{J} \equiv \mathrm{PT} = \mathrm{RC}. \tag{22.27}$$

In this case the operator P is defined in the same way as in (21.21), i.e.,

$$\mathrm{P}\psi(\boldsymbol{r}, t) = \mathrm{O}_r\psi(-\boldsymbol{r}, t).$$

As regards the operator T, it is defined by formula (22.27). We note that $\mathrm{T} \neq \mathrm{O}_t$.

Since the square of each of these operators differs from the identity operator at most by a phase factor, it follows from (22.27) that R = CPT, T = RCP.

The product of the operators of space inversion and charge conjugation is called the operator of *combined* inversion (114),

$$\mathrm{K} = \mathrm{PC}.$$

The equations of quantum electrodynamics are invariant not only under proper Lorentz transformations, but also under the transformation of space inversion P and charge conjugation C, and therefore they are also invariant under the transformations T and K. However, in the general case of arbitrary interacting fields we may require only the invariance of the theory under the proper Lorentz group (without the transformation of space inversion), and also under the transformation R.

In particular, it is well known that in the case of $\beta$-decay parity is not conserved, i.e., the theory is not invariant under the transformation P (123). However, it may be assumed that in this case the invariance under the operation of combined inversion is preserved (114).

## § 23. Equations of Quantum Electrodynamics in the Interaction Picture. Invariant Perturbation Theory

### 23.1. *Heisenberg and Schrödinger Pictures. Interaction Picture*

It is well known that in quantum mechanics the state of a system is described by a unit vector in Hilbert space—the state vector. In this description the variation of a state with time consists of the variation of the components of the state vector along the coordinate axes. This variation can be described in two different ways (49). In the first method of description the state vector is stationary while the coordinate system rotates. Such a description is referred to as the *Heisenberg picture.* In the second method of description the coordinate system in Hilbert space is fixed, while the state vector rotates. Such a description is referred to as the *Schrödinger picture.*

In the Heisenberg picture the operators corresponding to different physical quantities vary with time, while in the Schrödinger picture the operators do not depend on the time, but the state vector or the wave function $\Psi\ (t)$ varies with time. This variation is determined by the Schrödinger equation:

$$i\frac{\partial\Psi(t)}{\partial t} = \check{\mathrm{H}}\Psi(t)\,, \tag{23.1}$$

where $\check{\mathrm{H}}$ is the energy operator for the system.

The transition from the Heisenberg to the Schrödinger picture is given by the canonical transformation

$$\Psi(t) = e^{-i\check{\mathrm{H}}t}\Phi^0, \qquad \check{\mathrm{F}} = e^{-i\check{\mathrm{H}}t}\mathrm{F}e^{i\check{\mathrm{H}}t}, \tag{23.2}$$

where $\Phi^0$ is the constant wave function in the Heisenberg picture, while $\check{\mathrm{F}}$ and $\mathrm{F}$ are operators corresponding to some physical quantity in the Schrödinger and in the Heisenberg pictures. The determination of the explicit form of this canonical transformation is equivalent to a complete solution of the corresponding quantum mechanical problem.

It follows from (23.2) that

$$\mathrm{F} = e^{i\check{\mathrm{H}}t}\check{\mathrm{F}}e^{-i\check{\mathrm{H}}t}, \qquad \check{\mathrm{H}} = \mathrm{H}. \tag{23.2'}$$

On differentiating this relation with respect to time we obtain the following equation:

$$-i\frac{\partial}{\partial t}\mathbf{F} = [\mathbf{H}, \mathbf{F}], \tag{23.3}$$

which determines the variation with time of the Heisenberg operators.

We denote by $\Phi_n^0$ the wave function of the stationary state of energy $E_n$ in the Heisenberg picture.

It satisfies the equation

$$\mathbf{H}\Phi_n^0 = E_n\Phi_n^0.$$

The wave function $\Psi_n(t)$ in the Schrödinger picture which corresponds to $\Phi_n^0$ has in accordance with (23.2) the form

$$\Psi_n(t) = e^{-iE_n t}\Phi_n^0.$$

The matrix elements corresponding to some physical quantity $F$ are the same in both pictures:

$$(\Phi_n^0, \mathbf{F}\Phi_{n'}^0) = (\Psi_n, \check{\mathrm{F}}\Psi_{n'}). \tag{23.4}$$

On substituting into this expression $\Psi_n(t) = e^{-iE_n t}\Phi_n^0$, we obtain the dependence of the matrix elements on the time

$$(\Phi_n^0, \mathbf{F}\Phi_{n'}^0) = (\Phi_n^0, \check{\mathrm{F}}\Phi_{n'}^0)e^{i(E_n - E_{n'})t}. \tag{23.4'}$$

In addition to the Heisenberg and the Schrödinger pictures another intermediate picture is possible, viz., a description of the system in which both the coordinate axes of Hilbert space and the state vector rotate, with the coordinate axes rotating in a prescribed manner. Such a picture is frequently used in quantum mechanics for the approximate solution of problems by means of perturbation theory. It was introduced into quantum electrodynamics in a relativistically invariant manner by Tomonaga (197) and by Schwinger (172) and has received the name of the *interaction picture*.

In order to explain the introduction of this picture we consider a quantum mechanical system with the Hamiltonian $\check{\mathrm{H}}$ and decompose it into the two terms $\check{\mathrm{H}}_0$ and $\check{\mathrm{H}}_1$ which we call the free Hamiltonian and the interaction Hamiltonian

$$\check{\mathrm{H}} = \check{\mathrm{H}}_0 + \check{\mathrm{H}}_1.$$

We carry out the canonical transformation

$$\Phi(t) = e^{i\check{\mathrm{H}}_0 t}\Psi(t), \qquad \text{and} \qquad \mathrm{F} = e^{i\check{\mathrm{H}}_0 t}\check{\mathrm{F}}e^{-i\check{\mathrm{H}}_0 t}, \tag{23.5}$$

which, in contrast to the transformation (23.2′), involves not the whole Hamiltonian of the system $\check{\mathrm{H}}$ but only a part of it, $\check{\mathrm{H}}_0$. We now determine the variation with time of the function $\Phi(t)$. On differentiating $\Phi(t)$ with respect to time and on utilizing (23.1) we obtain

$$i\frac{\partial}{\partial t}\Phi = -\check{\mathrm{H}}_0\Phi + e^{i\check{\mathrm{H}}_0 t}(\check{\mathrm{H}}_0 + \check{\mathrm{H}}_1)e^{-i\check{\mathrm{H}}_0 t}\Phi = e^{i\check{\mathrm{H}}_0 t}\check{\mathrm{H}}_1 e^{-i\check{\mathrm{H}}_0 t}\Phi,$$

i.e.,

$$i\frac{\partial}{\partial t}\Phi = \mathrm{H}_1\Phi, \tag{23.6}$$

where

$$\mathrm{H}_1 = e^{i\check{\mathrm{H}}_0 t}\check{\mathrm{H}}_1 e^{-i\check{\mathrm{H}}_0 t}.$$

We see that the function $\Phi(t)$ satisfies the Schrödinger equation with the Hamiltonian $\mathrm{H}_1$. We refer to this function as the wave function of the system in the interaction picture, and we regard the canonical transformation (23.5) as the transition from the Schrödinger picture to the new interaction picture.

We now investigate the way in which the operators vary with time in the interaction picture. By differentiating the second relation (23.5) with respect to time we obtain

$$-i\frac{\partial}{\partial t}\mathrm{F} = \check{\mathrm{H}}_0\mathrm{F} - \mathrm{F}\check{\mathrm{H}}_0,$$

and since in accordance with (23.5)

$$\mathrm{H}_0 = \check{\mathrm{H}}_0,$$

we have

$$-i\frac{\partial}{\partial t}\mathrm{F} = [\mathrm{H}_0, \mathrm{F}]. \tag{23.7}$$

In contrast to (23.3) these relations involve not the whole Hamiltonian H, but only a part of it, $\mathrm{H}_0$.

Thus the fundamental equations of quantum mechanics in the three pictures—the Heisenberg, the Schrödinger and the interaction picture—have the form

$$
\begin{aligned}
&i\frac{\partial \Phi^0}{\partial t} = 0, \quad -i\frac{\partial \mathbf{F}}{\partial t} = [\mathbf{H}, \mathbf{F}]\text{—Heisenberg picture,}\\
&i\frac{\partial \psi}{\partial t} = \check{\mathrm{H}}\Psi, \quad -i\frac{\partial \check{\mathrm{F}}}{\partial t} = 0\text{—Schrödinger picture,}\\
&i\frac{\partial \Phi}{\partial t} = \mathrm{H}_1\Phi, \quad -i\frac{\partial \mathrm{F}}{\partial t} = [\mathrm{H}_0, \mathrm{F}]\text{—interaction picture.}
\end{aligned}
\tag{23.8}
$$

The functions $\Phi^0$, $\Psi(t)$, $\Phi(t)$ and the operators $\mathbf{F}$, $\hat{\mathrm{F}}$, $\mathrm{F}$ in the three pictures are related among themselves by the following canonical transformations:

$$
\begin{aligned}
&\Psi(t) = e^{-i\check{\mathrm{H}}t}\Phi^0, \quad \Phi(t) = e^{i\mathrm{H}_0 t}\Psi(t),\\
&\check{\mathrm{F}} = e^{-i\check{\mathrm{H}}t}\mathrm{F}e^{i\check{\mathrm{H}}t}, \quad \mathrm{F} = e^{i\check{\mathrm{H}}_0 t}\check{\mathrm{F}}e^{i\check{\mathrm{H}}_0 t}.
\end{aligned}
\tag{23.8'}
$$

## 23.2. *Transition to the Interaction Picture in Quantum Electrodynamics*

Quantum electrodynamics is concerned with the investigation of the dynamical system involving both the electromagnetic and the electron-positron fields. This system, like any other quantum mechanical system, is characterized by a Hamiltonian function H which can be written in the form $\mathrm{H} = \mathrm{H}_0 + \mathrm{V}$, where $\mathrm{H}_0 = \mathrm{H}_\gamma + \mathrm{H}_e$ is the sum of the Hamiltonians of the free fields, while V is the Hamiltonian of the interaction between the fields. Such a decomposition of the Hamiltonian can be utilized for introducing the interaction picture into quantum electrodynamics. In this picture the field operators satisfy equations of the form (23.7), i.e., the equations for the Heisenberg operators of the free field (16.3) and (18.3). Because of this the explicit form of the field operators is known in the interaction picture, and the commutation relations between them at any time are also known. This makes the use of the interaction picture particularly convenient for the solution of specific problems.

We write the fundamental system of equations of quantum electrodynamics in the interaction picture

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi = 0, \quad \overline{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right) = 0,$$

$$\Box A_\mu = 0, \quad i\frac{\partial}{\partial t}\Phi(t) = V(t)\Phi(t), \tag{23.9}$$

$$[A_\mu(x)A_\nu(x')] = -i\delta_{\mu\nu}D(x-x'), \quad \{\psi_\alpha(x)\overline{\psi}_\beta(x')\} = -iS_{\alpha\beta}(x-x').$$

For the investigation of this system we must know the explicit form of the operator $V(t)$. We now show that

$$V(t) = -\int j_\mu(x)A_\mu(x)\,dr, \tag{23.10}$$

where $j_\mu$ is the current density vector written in terms of the operators of the electron-positron field $\psi$ and $\overline{\psi}$ in the interaction picture in accordance with the general formula (22.4), and $A_\mu$ is the operator for the potential of the electromagnetic field also in the interaction picture. To achieve this we go over to the interaction picture by starting from the fundamental equations of quantum electrodynamics in the Heisenberg picture, i.e., from equations (22.6) for the time-dependent field operators. As has been pointed out already, this transition can be regarded as a canonical transformation. If $\mathbf{F}$ and F are operators corresponding to some physical quantity in the Heisenberg picture and in the interaction picture, then such a transformation is of the form

$$F = S(t)\,\mathbf{F}S^+(t), \tag{23.11}$$

where $S(t)$ is a unitary operator

$$S^+(t)\,S(t) = S(t)S^+(t) = 1,$$

which we shall refer to as the transformation operator. The transformation operator also relates the state vector $\Phi(t)$ in the interaction picture to the state vector $\Phi^0$ in the Heisenberg picture

$$\Phi(t) = S(t)\Phi^0. \tag{23.12}$$

If we assume that at time $t = -\infty$ there is no interaction between the fields, then we can impose the following condition on the operator $S(t)$: $S(-\infty) = 1$. It follows from (23.12) that

$$i\frac{\partial\Phi}{\partial t} = i\frac{\partial S(t)}{\partial t}\Phi^0 = i\frac{\partial S(t)}{\partial t}S^+(t)\Phi.$$

By comparing this expression with the fourth equation (23.9), we see that the operator V$(t)$ is related to S$(t)$ by

$$\mathrm{V}(t)=i\frac{\partial \mathrm{S}(t)}{\partial t}\mathrm{S}^{+}(t). \tag{23.13}$$

It can be easily shown that V$(t)$ is a Hermitian operator. Indeed, from the unitary nature of S$(t)$ it follows that

$$\frac{\partial}{\partial t}\left(\mathrm{S}(t)\mathrm{S}^{+}(t)\right)=\frac{\partial \mathrm{S}(t)}{\partial t}\mathrm{S}^{+}(t)+\mathrm{S}(t)\frac{\partial \mathrm{S}^{+}(t)}{\partial t}=0,$$

and therefore

$$\mathrm{V}(t)=i\frac{\partial \mathrm{S}(t)}{\partial t}\mathrm{S}^{+}(t)=-i\mathrm{S}(t)\frac{\partial \mathrm{S}^{+}(t)}{\partial t},$$

i.e.,

$$\mathrm{V}(t)=\mathrm{V}^{+}(t). \tag{23.14}$$

In order to obtain the form of the operator V$(t)$ we determine the derivatives with respect to time of the field operators in the interaction picture and compare them with the corresponding derivatives in the Heisenberg picture. By differentiating (23.11) with respect to time we obtain

$$\frac{\partial \mathrm{F}}{\partial t}=\mathrm{S}\frac{\partial \mathrm{F}}{\partial t}\mathrm{S}^{+}+\frac{\partial \mathrm{S}}{\partial t}\mathrm{S}^{+}\mathrm{SFS}^{+}+\mathrm{SFS}^{+}\mathrm{S}\frac{\partial \mathrm{S}^{+}}{\partial t}, \qquad \mathrm{S}\equiv\mathrm{S}(t).$$

By utilizing (23.13) we can put this expression in the form

$$\frac{\partial \mathrm{F}}{\partial t}=\mathrm{S}(t)\frac{\partial \mathbf{F}}{\partial t}\mathrm{S}^{+}(t)+i[\mathrm{F},\mathrm{V}(t)]. \tag{23.15}$$

In a similar manner we can derive the following relationship:

$$\frac{\partial^2 \mathrm{F}}{\partial t^2}=\mathrm{S}(t)\frac{\partial^2 \mathbf{F}}{\partial t^2}\mathrm{S}^{+}(t)+i\left[\mathrm{S}(t)\frac{\partial \mathbf{F}}{\partial t}\mathrm{S}^{+}(t),\mathrm{V}(t)\right]+i\frac{\partial}{\partial t}[\mathrm{F},\mathrm{V}(t)]. \tag{23.15'}$$

It follows from (23.15) and (23.15′) that

$$\begin{aligned}\frac{\partial \psi}{\partial t}&=\mathrm{S}(t)\frac{\partial \boldsymbol{\psi}}{\partial t}\mathrm{S}^{+}(t)+i[\psi,\mathrm{V}(t)],\\ \frac{\partial^2 \mathrm{A}_\mu}{\partial t^2}&=\mathrm{S}(t)\frac{\partial^2 \mathbf{A}_\mu}{\partial t^2}\mathrm{S}^{+}(t)+i\left[\mathrm{S}(t)\frac{\partial \mathbf{A}_\mu}{\partial t}\mathrm{S}^{+}(t),\mathrm{V}(t)\right]+i\frac{\partial}{\partial t}[\mathrm{A}_\mu,\mathrm{V}(t)].\end{aligned} \tag{23.15''}$$

With the aid of these formulas we evaluate the expressions $\left(\gamma_\mu(\partial/\partial x_\mu)+m\right)\psi$ and $\Box \mathrm{A}_\mu$, which must vanish in virtue of (23.9). We first of all obtain $\left(\gamma_\mu(\partial/\partial x_\mu)+m\right)\psi$:

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m\right)\psi = \mathrm{S}(t)\left(\gamma_k\frac{\partial}{\partial x_k}+m\right)\boldsymbol{\psi}\mathrm{S}^+(t)-i\gamma_4\frac{\partial\psi}{\partial t}$$

$$= \mathrm{S}(t)\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m\right)\boldsymbol{\psi}\mathrm{S}^+(t)+\gamma_4[\psi, \mathrm{V}(t)] = 0, \qquad k = 1, 2, 3.$$

Since $\left(\gamma_\mu(\partial/\partial x_\mu)+m\right)\boldsymbol{\psi} = ie\hat{\mathrm{A}}\boldsymbol{\psi}$, we have

$$\mathrm{S}(t)\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+m\right)\boldsymbol{\psi}\mathrm{S}^+(t) = ie\mathrm{S}(t)\hat{\mathrm{A}}\boldsymbol{\psi}\mathrm{S}^+(t) = ie\hat{\mathrm{A}}\psi,$$

and we finally obtain

$$[\mathrm{V}(t), \psi] = ie\gamma_4\hat{\mathrm{A}}\psi. \tag{23.16}$$

The operators $\mathrm{A}_\mu$ and $\psi$ commute, and therefore it follows from (23.16) that the operator $\mathrm{V}(t)$ must be linear in $\mathrm{A}_\mu$. On the other hand, the commutator of the operators $\mathrm{A}_\mu$ and $\mathrm{A}_\nu$ taken at the same instant of time vanishes. Therefore, we have

$$[\mathrm{A}_\mu(x), \mathrm{V}(t)] = 0. \tag{23.17}$$

We now evaluate $\Box \mathrm{A}_\mu$ by utilizing the second formula (23.15''):

$$\Delta \mathrm{A}_\mu-\frac{\partial^2\mathrm{A}_\mu}{\partial t^2} = \mathrm{S}(t)\Box \mathrm{A}_\mu\mathrm{S}^+(t)-i\left[\mathrm{S}(t)\frac{\partial\mathrm{A}_\mu}{\partial t}\mathrm{S}^+(t), \mathrm{V}(t)\right]$$

$$-i\frac{\partial}{\partial t}[\mathrm{A}_\mu, \mathrm{V}(t)] = 0.$$

Here the last term vanishes in accordance with (23.17), while the first term is equal to $-\mathrm{j}_\mu$, since

$$\mathrm{S}(t)\Box \mathrm{A}_\mu\mathrm{S}^+(t) = -\mathrm{S}(t)\mathbf{j}_\mu\mathrm{S}^+(t) = -\mathbf{j}_\mu;$$

further, on noting that in accordance with (23.15) and (23.17)

$$\frac{\partial\mathrm{A}_\mu}{\partial t} = \mathrm{S}(t)\frac{\partial\mathrm{A}_\mu}{\partial t}\mathrm{S}^+(t), \tag{23.17'}$$

we obtain finally

$$i\left[\mathrm{V}(t), \frac{\partial\mathrm{A}_\mu}{\partial t}\right] = \mathrm{j}_\mu. \tag{23.18}$$

We can conclude on the basis of this relation that the operator $V(t)$ must be linear in $j_\mu$. Thus, $V(t)$ involves both operators $A_\mu$ and $j_\mu$ linearly.

We now show that both relations (23.16) and (23.18) are satisfied if we assume that

$$V(t) = -\int j_\mu(x) A_\mu(x) d\boldsymbol{r}.$$

In order to do this we use the commutation relations for the field operators in the interaction picture at the same instant of time, which can be easily obtained from the corresponding commutation relations in the Heisenberg picture with the aid of formulas (23.11) and (23.17′):

$$\begin{gathered}
[A_\mu(x), A_\nu(x')]_{t'=t} = 0, \\
\left[A_\mu(x), \frac{\partial}{\partial t'} A_\nu(x')\right]_{t'=t} = i\delta_{\mu\nu}\,\delta(\boldsymbol{r}-\boldsymbol{r}'), \\
\{\psi_\alpha(x), \psi_\beta^+(x')\}_{t'=t} = \delta_{\alpha\beta}\,\delta(\boldsymbol{r}-\boldsymbol{r}').
\end{gathered} \tag{23.19}$$

By utilizing the third of these equations we obtain

$$\begin{aligned}
i[\psi, V(t)] &= -i\int A_\nu(\boldsymbol{r}', t)[\psi(\boldsymbol{r}, t), j_\nu(\boldsymbol{r}', t)]d\boldsymbol{r}' \\
&= e\int A_\nu(\boldsymbol{r}', t)(\gamma_\nu)_{\alpha\beta}\{\psi(\boldsymbol{r}, t), \psi(\boldsymbol{r}', t)\}\psi_\beta(\boldsymbol{r}', t)d\boldsymbol{r}' = e\gamma_4 \hat{A}\psi,
\end{aligned}$$

as is required by relation (23.16). Further, by utilizing the second relation (23.19) we obtain

$$i\left[\frac{\partial A_\mu}{\partial t}, V(t)\right] = -i\int j_\nu(\boldsymbol{r}', t)\left[\frac{\partial A_\mu(\boldsymbol{r}, t)}{\partial t}, A_\nu(\boldsymbol{r}', t)\right]d\boldsymbol{r}' = -j_\mu(x),$$

i.e., the relation (23.18).

We must still prove the equivalence of the commutation relations (23.19) in the interaction picture involving different instants of time, and the commutation relations in the Heisenberg picture for the same time. In order to do this we note that the solution of the equation $\Box A_\mu = 0$ is uniquely determined by specifying $A_\mu(x)$ and $\partial A_\mu(x)/\partial t$ over a certain hypersurface $t = t_0$:

$$A_\mu(x) = \int\limits_{t'=t_0} \left(D(x-x')\frac{\partial A_\mu(x')}{\partial t'} - A_\mu(x')\frac{\partial D(x-x')}{\partial t'}\right) d\boldsymbol{r}'. \tag{23.20}$$

We can easily show this, if we recall that

$$D(\boldsymbol{r}, 0) = \left.\frac{\partial^2 D(x)}{\partial t^2}\right|_{t=0} = 0, \qquad \left.\frac{\partial D(x)}{\partial t}\right|_{t=0} = \delta(\boldsymbol{r}). \tag{23.21}$$

In a similar manner the solution of the equation $(\gamma_\mu(\partial/\partial x_\mu)+m)\psi = 0$ is uniquely determined by specifying $\psi(x)$ over the hypersurface $t = t_0$:

$$\psi_a(x) = -i \int\limits_{t'=t_0} S_{a\beta}(x-x')(\gamma_4)_{\beta\varrho}\psi_\varrho(x')d\boldsymbol{r}', \tag{23.22}$$

where

$$S_{a\beta}(x) = -\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{a\beta} \Delta(x).$$

Formula (23.22) immediately follows from the fact that

$$S(\boldsymbol{r}, 0) = i\gamma_4\,\delta(\boldsymbol{r}). \tag{23.22$'$}$$

We now derive the commutation relations (23.9) from the commutation relations (22.5). We evaluate the bracket $[\mathrm{A}_\mu(x), \mathrm{A}_\nu(x')]$. By utilizing (23.20) we obtain

$$[\mathrm{A}_\mu(x), \mathrm{A}_\nu(x')] = \int\limits_{t''=t} D(x'-x'')\left[\mathrm{A}_\mu(x), \frac{\partial \mathrm{A}_\nu(x'')}{\partial t''}\right] d\boldsymbol{r}''$$
$$- \int\limits_{t''=t} [\mathrm{A}_\mu(x), \mathrm{A}_\nu(x'')] \frac{\partial D(x'-x'')}{\partial t''}\, d\boldsymbol{r}''.$$

In accordance with (23.19) this expression is equal to

$$[\mathrm{A}_\mu(x), \mathrm{A}_\nu(x')] = \int\limits_{t''=t} D(x'-x'')\left[\mathrm{A}_\mu(x), \frac{\partial \mathrm{A}_\nu(x'')}{\partial t''}\right] d\boldsymbol{r}'' = -i\delta_{\mu\nu} D(x-x'),$$

which is what we set out to prove.

Further, we evaluate the bracket $\{\psi_a(x), \psi_\beta^+(x')\}$. On utilizing (23.19) and (23.22) we obtain

$$\{\psi_a(x), \psi_\beta^+(x')\} = -i\left\{ \int\limits_{t''=t'} S_{a\varrho}(x-x'')(\gamma_4)_{\varrho\lambda}\psi_\lambda(x'')d\boldsymbol{r}'', \psi_\beta^+(x')\right\}$$
$$= -i \int\limits_{t''=t'} S_{a\varrho}(x-x'')(\gamma_4)_{\varrho\lambda}\{\psi_\lambda(x''), \psi_\beta^+(x')\}\, d\boldsymbol{r}'' = -iS_{a\varrho}(x-x')(\gamma_4)_{\varrho\beta},$$

from which it follows that

$$\{\psi_a(x), \overline{\psi}_\beta^+(x')\} = -iS_{a\beta}(x-x').$$

Thus, we have obtained the commutation relations in the interaction picture from the commutation relations in the Heisenberg picture.

In concluding this subsection we note that, as can be seen from a comparison of (23.10) with (22.11′), the operator $\mathrm{V}(t)$ can also be represented in the form

$$\mathrm{V}(t) = -\int \mathrm{L}_{e\gamma}(x)d\boldsymbol{r}, \tag{23.23}$$

where $L_{e\gamma}(x)$ is the part of the Lagrangian density which describes the interaction between the fields. This result can be obtained by means of a direct derivation of the Hamiltonian from the Lagrangian. Relation (23.23) holds in virtue of the fact that $L_{e\gamma}$ contains only the "generalized coordinates" $A_\mu$, $\psi$, $\bar{\psi}$, and does not contain their time derivatives.

The transition to the interaction picture can also be carried out by starting with the fundamental equation of quantum electrodynamics in the Schrödinger picture

$$i\frac{\partial}{\partial t}\Psi(t) = \check{H}\Psi(t)$$

where $\Psi(t)$ is the time dependent state vector of the system of fields and $\check{H}$ is the time-independent Hamiltonian of the system. The latter coincides with the Hamiltonian of the system of fields in the Heisenberg picture $\mathbf{H}$ and is defined by the volume integral of the time component $\mathbf{T}_{44}$ of the energy-momentum tensor

$$\check{H} = \mathbf{H} = -\int \mathbf{T}_{44} d\mathbf{r},$$

$$\mathbf{T}_{44} = -\frac{1}{2}\frac{\partial \mathbf{A}_\sigma}{\partial \mathbf{r}}\frac{\partial \mathbf{A}_\sigma}{\partial \mathbf{r}} + \frac{1}{2}\frac{\partial \mathbf{A}_\sigma}{\partial t}\frac{\partial \mathbf{A}_\sigma}{\partial t} + \frac{1}{4}\left(\bar{\psi}\gamma_4\frac{\partial \psi}{\partial x_4} - \frac{\partial \bar{\psi}}{\partial x_4}\gamma_4\psi\right)$$

$$+\frac{1}{4}\left(\bar{\psi}^c\gamma_4\frac{\partial \psi^c}{\partial x_4} - \frac{\partial \bar{\psi}^c}{\partial x_4}\gamma_4\psi^c\right),$$

where $\mathbf{A}_\sigma$ and $\psi$ are the field operators in the Heisenberg picture. Since the field equations (22.6) contain the first derivative with respect to time of the operators $\psi$ and the second derivative with respect to time of the operators $\mathbf{A}_\sigma$, the independent variables in these expressions are $\psi$, $\mathbf{A}_\sigma$ and $\partial \mathbf{A}_\sigma/\partial t$. Insofar as the quantities $\partial\psi/\partial x_4$ (and also $\partial\bar{\psi}/\partial x_4$ and $\partial\psi^c/\partial x_4$) are concerned, they are not independent and must be eliminated with the aid of the first two equations (22.6):

$$\gamma_4\frac{\partial \psi}{\partial x_4} = -\boldsymbol{\gamma}\frac{\partial \psi}{\partial \mathbf{r}} - m\psi + ie\hat{\mathbf{A}}\psi,$$

$$\frac{\partial \bar{\psi}}{\partial x_4}\gamma_4 = -\frac{\partial \bar{\psi}}{\partial \mathbf{r}}\boldsymbol{\gamma} + m\bar{\psi} - ie\hat{\mathbf{A}}\bar{\psi}$$

and analogous equations for the charge-conjugate operators $\psi^c$ and $\bar{\psi}^c$. As a result we obtain the following expression for $T_{44}$:

$$T_{44} = -\frac{1}{2}\left(\frac{\partial A_\sigma}{\partial r}\frac{\partial A_\sigma}{\partial r} - \frac{\partial A_\sigma}{\partial t}\frac{\partial A_\sigma}{\partial t}\right)$$
$$+\frac{1}{4}\left\{-\bar{\psi}\left(\gamma\frac{\overrightarrow{\partial}}{\partial r}+m\right)\psi+\bar{\psi}\left(\gamma\frac{\overleftarrow{\partial}}{\partial r}-m\right)\psi\right\}$$
$$+\frac{1}{4}\left\{-\bar{\psi}^c\left(\gamma\frac{\overrightarrow{\partial}}{\partial r}+m\right)\psi^c+\bar{\psi}^c\left(\gamma\frac{\overleftarrow{\partial}}{\partial r}-m\right)\psi^c\right\}$$
$$+A_\mu j_\mu,$$

where

$$j_\mu = \frac{ie}{2}(\bar{\psi}\gamma_\mu\psi - \bar{\psi}^c\gamma_\mu\psi^c)\cdot$$

On going over from the field operators F in the Heisenberg picture to the operators in the Schrödinger picture $\check{F}$ in accordance with the general formula

$$\check{F} = e^{-iHt}Fe^{iHt}$$

we obtain the following expression for the Hamiltonian of the fields in the Schrödinger picture

$$\check{H} = \check{H}_0 + \check{H}_1$$

where

$$\check{H}_0 = \int dr\left\{\frac{1}{2}\left(\frac{\partial\check{A}_\sigma}{\partial r}\frac{\partial\check{A}_\sigma}{\partial r} - \dot{\check{A}}_\sigma\dot{\check{A}}_\sigma\right)\right.$$
$$+\frac{1}{4}\left[-\check{\psi}\left(\gamma\frac{\overrightarrow{\partial}}{\partial r}+m\right)\check{\psi}+\check{\psi}\left(\gamma\frac{\overleftarrow{\partial}}{\partial r}-m\right)\check{\psi}\right]$$
$$\left.+\frac{1}{4}\left[-\check{\psi}^c\left(\gamma\frac{\overleftarrow{\partial}}{\partial r}+m\right)\check{\psi}^c+\check{\psi}^c\left(\gamma\frac{\overleftarrow{\partial}}{\partial r}-m\right)\check{\psi}^c\right]\right\},$$
$$\bar{H}_1 = -\int \check{j}_\mu(r)\check{A}_\mu(r)dr.$$

The operator $\check{H}_0$ is evidently the Hamiltonian for the system of noninteracting fields, while $\check{H}_1$ is the operator for the energy of interaction between the fields in the Schrödinger picture. Moreover, the field operators in the Schrödinger picture satisfy the same commutation relations as the field operators in the Heisenberg picture at the same instant of time

$$\{\check{\psi}(\boldsymbol{r}), \check{\bar{\psi}}(\boldsymbol{r}')\} = \gamma_4 \delta(\boldsymbol{r}-\boldsymbol{r}'),$$

$$\{\check{\psi}(\boldsymbol{r}), \check{\psi}(\boldsymbol{r}')\} = 0,$$

$$[\check{\mathrm{A}}_\mu(\boldsymbol{r}), \check{\dot{\mathrm{A}}}_\nu(\boldsymbol{r}')] = i\delta_{\mu\nu}\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

$$[\check{\mathrm{A}}_\mu(\boldsymbol{r}), \check{\mathrm{A}}_\nu(\boldsymbol{r}')] = 0,$$

$$[\check{\dot{\mathrm{A}}}_\mu(\boldsymbol{r}), \check{\dot{\mathrm{A}}}_\nu(\boldsymbol{r}')] = 0.$$

We now go over to the interaction picture in accordance with the general formulas (23.8″). The state vector for the system of fields $\Phi(t)$ defined by

$$\Phi(t) = e^{i\check{\mathrm{H}}_0 t}\Psi(t),$$

will then satisfy the equation

$$i\frac{\partial}{\partial t}\Phi(t) = \mathrm{V}(t)\Phi(t)$$

where

$$\mathrm{V}(t) = e^{i\check{\mathrm{H}}_0 t}\mathrm{H}_1 e^{-i\check{\mathrm{H}}_0 t} = -\int \mathrm{j}_\mu(x)\mathrm{A}_\mu(x)\, d\boldsymbol{r}$$

and the field operators $\mathrm{A}_\mu(x)$, $\psi_a(x)$, $\mathrm{j}_\mu(x)$ are defined in the interaction picture by the relation

$$\mathrm{F} = e^{i\check{\mathrm{H}}_0 t}\check{\mathrm{F}}e^{-i\check{\mathrm{H}}_0 t}.$$

In virtue of the cannonical nature of this transformation, the commutation relations between the field operators at the same instant of time in the interaction picture coincide with the commutation relations in the Schrödinger picture. Therefore, the equations of motion for the fields in the interaction picture $-i(\partial\mathrm{F}/\partial t) = [\check{\mathrm{H}}_0, \mathrm{F}]$ are, as can be easily seen, of the same form as the equations of motion for the free fields (23.9). Indeed, we have

$$-i\frac{\partial\psi}{\partial t} = [\check{\mathrm{H}}_0, \psi] = -e^{i\check{\mathrm{H}}_0 t}[\check{\psi}(\boldsymbol{r}), \check{\mathrm{H}}_0]e^{-i\check{\mathrm{H}}_0 t}$$

$$= e^{i\check{\mathrm{H}}_0 t}\gamma_4\left(\boldsymbol{\gamma}\frac{\partial}{\partial \boldsymbol{r}}\check{\psi}(\boldsymbol{r}) + m\check{\psi}(\boldsymbol{r})\right)e^{-i\check{\mathrm{H}}_0 t}$$

$$= \gamma_4\boldsymbol{\gamma}\frac{\partial}{\partial \boldsymbol{r}}\psi(x) + \gamma_4 m\psi(x),$$

i.e.,

$$\gamma_\mu \frac{\partial}{\partial x_\mu} \psi(x) + m\,\psi(x) = 0.$$

Similarly we have

$$\begin{aligned}\frac{\partial}{\partial t} \dot{\mathrm{A}}_\mu(x) &= i e^{i\check{\mathrm{H}}_0 t} [\check{\mathrm{H}}_0, \check{\dot{\mathrm{A}}}_\mu(\boldsymbol{r})] e^{i\check{\mathrm{H}}_0 t} \\ &= e^{i\check{\mathrm{H}}_0 t} \frac{\partial^2}{\partial x_j^2} \check{\mathrm{A}}_\mu(\boldsymbol{r})\, e^{-i\check{\mathrm{H}}_0 t} \\ &= \frac{\partial^2}{\partial x_j^2} \mathrm{A}_\mu(x), \quad \text{where} \quad j = 1, 2, 3,\end{aligned}$$

i.e.,

$$\square \mathrm{A}_\mu(x) = 0.$$

Since the field operators in the interaction picture satisfy the equations for the free fields, and the commutation relations between them at the same instant of time have the same form as the commutation relations between the free fields, it follows that the commutation relations between the field operators in the interaction picture coincide with the commutation relations for the free fields at different instants of time.

### 23.3. *Charge Conjugation Operator*

The charge conjugation transformation (22.21) corresponds to a transformation of the state vector $\Phi^0$ of the system of fields, i.e., of the system of electrons and photons. If we write this transformation in the form

$$\Phi^{0c} = \Lambda \Phi^0, \tag{23.24}$$

where $\Lambda$ is the transformation operator which we shall call the charge conjugation operator, then in accordance with the transformation law for the operators, corresponding to the transformation of the wave function, the following relations must hold:

$$\begin{aligned}\psi \to \psi^c &= \Lambda \psi \Lambda^{-1}, \\ \mathrm{A}_\mu \to \mathrm{A}_\mu^c &= -\mathrm{A}_\mu = \Lambda \mathrm{A}_\mu \Lambda^{-1},\end{aligned} \tag{23.25}$$

or

$$\Lambda \psi \Lambda^{-1} = \mathrm{C}\bar{\psi}, \quad \Lambda \bar{\psi} \Lambda^{-1} = \mathrm{C}^{-1} \psi \quad \text{and} \quad \Lambda \mathrm{A}_\mu \Lambda^{-1} = -\mathrm{A}_\mu.$$

We now demonstrate the manner in which the explicit form of the charge conjugation operator $\Lambda$ can be obtained (214). This can be done in the interaction picture, since in this picture the explicit form of the field operators is known.

We write the operator $\Lambda$ in the form of a product of operators acting respectively on the photon and the electron variables:

$$\Lambda = \Lambda_\gamma \Lambda_e, \tag{23.26}$$

and we use the expressions for the field operators in the form of expansions (16.13), (18.7). In this way we obtain the expressions

$$\begin{aligned} \Lambda_\gamma c_\lambda &= -c_\lambda \Lambda_\gamma, & \Lambda_\gamma c_\lambda^+ &= -c_\lambda^+ \Lambda_\gamma, \\ \Lambda_e a_r \Lambda_e^{-1} &= b_r, & \Lambda_e a_r^+ \Lambda_e^{-1} &= b_r^+, \\ \Lambda_e b_r \Lambda_e^{-1} &= a_r, & \Lambda_e b_r^+ \Lambda_e^{-1} &= a_r^+, \end{aligned} \tag{23.27}$$

relating the operators $\Lambda_\gamma$ and $\Lambda_e$ with the creation and annihilation operators for photons, electrons and positrons.

We first of all determine the form of the operator $\Lambda_\gamma$. It can be written in the form of the product

$$\Lambda_\gamma = \prod_\lambda \Lambda_{\gamma\lambda},$$

where the operator $\Lambda_{\gamma\lambda}$ acts only on the photon variables $\lambda \equiv (\boldsymbol{k}, \lambda)$ and satisfies the following relations:

$$\begin{aligned} \Lambda_{\gamma\lambda} c_\lambda &= -c_\lambda \Lambda_{\gamma\lambda}, \\ \Lambda_{\gamma\lambda} c_\lambda^+ &= -c_\lambda^+ \Lambda_{\gamma\lambda}, \\ \Lambda_{\gamma\lambda} c_{\lambda'} &= c_{\lambda'} \Lambda_{\gamma\lambda}, \\ \Lambda_{\gamma\lambda} c_{\lambda'}^+ &= c_{\lambda'}^+ \Lambda_{\gamma\lambda}, \quad \text{where} \quad \lambda' \neq \lambda. \end{aligned} \tag{23.28}$$

Since the operator $\Lambda_{\gamma\lambda}$ commutes with the operator $N_\lambda = c_\lambda^+ c_\lambda$, it can be regarded as diagonal in the representation in which $N_\lambda$ is diagonal. From (23.28) it follows that the values of $\Lambda_{\gamma\lambda}$ corresponding to two consecutive values of $N_\lambda$, are related by

$$(\Lambda_{\gamma\lambda})_{N_\lambda+1} = -(\Lambda_{\gamma\lambda})_{N_\lambda}.$$

From this we can conclude that

$$\Lambda_{\gamma\lambda} = (-1)^{N_\lambda}. \tag{23.29}$$

Therefore, the operator $\Lambda_\gamma$ has the form

$$\Lambda_\gamma = (-1)^N, \tag{23.30}$$

where $N = \Sigma N_\lambda$ is the total number of photons.

We now obtain the form of the operator $\Lambda_e$. Just as in the case of the operator $\Lambda_\gamma$ we write it in the form of the product

$$\Lambda_e = \prod_r \Lambda_{er},$$

where the operator $\Lambda_{er}$ refers to the electron variables $r$ and satisfies the relations

$$\Lambda_{er} \mathrm{a}_r \Lambda_{er}^{-1} = \mathrm{b}_r, \quad \Lambda_{er} \mathrm{b}_r \Lambda_{er}^{-1} = \mathrm{a}_r. \tag{23.31}$$

In place of $\mathrm{a}_r$, $\mathrm{b}_r$, $\mathrm{a}_r^+$, $\mathrm{b}_r^+$ we introduce the operators $\alpha_r$, $\beta_r$, $\alpha_r^+$, $\beta_r^+$:

$$\alpha_r = \frac{1}{\sqrt{2}} (\mathrm{a}_r + \mathrm{b}_r), \quad \alpha_r^+ = \frac{1}{\sqrt{2}} (\mathrm{a}_r^+ + \mathrm{b}_r^+),$$

$$\beta_r = \frac{1}{\sqrt{2}} (\mathrm{a}_r - \mathrm{b}_r), \quad \beta_r^+ = \frac{1}{\sqrt{2}} (\mathrm{a}_r^+ - \mathrm{b}_r^+),$$

which, as can be easily verified, satisfy the same commutation relations as the operators $\mathrm{a}_r$, $\mathrm{b}_r$.

Expressions (23.31) defining the operator $\Lambda_{er}$ can be rewritten in terms of $\alpha_r$ and $\beta_r$ in the form

$$\Lambda_{er} \alpha_r \Lambda_{er}^{-1} = \alpha_r, \quad \Lambda_{er} \beta_r \Lambda_{er}^{-1} = -\beta_r. \tag{23.31'}$$

We see that the relation between the operators $\Lambda_{er}$ and $\beta_r$ is the same as between the operators $\Lambda_{\gamma\lambda}$ and $\mathrm{c}_\lambda$. Therefore, on assuming that the operator $\Lambda_{er}$ is diagonal in the representation in which $\beta_r^+ \beta_r$ is diagonal we immediately obtain in analogy with (23.29)

$$\Lambda_{er} = (-1)^{\beta_r^+ \beta_r}$$

and, consequently,

$$\Lambda_e = (-1)^{\sum_r \beta_r^+ \beta_r} \tag{23.32}$$

Since the eigenvalues of the operator $\beta_r^+ \beta_r$ are equal to unity or zero, we have $(-1)^{\beta_r^+ \beta_r} = 1 - 2\beta_r^+ \beta_r$ and the operator $\Lambda_e$ can be written in the form

$$\Lambda = \prod_r (1 - 2\beta_r^+ \beta_r) = \prod_r (1 - \mathrm{a}_r^+ \mathrm{a}_r - \mathrm{b}_r^+ \mathrm{b}_r + \mathrm{a}_r^+ \mathrm{b}_r + \mathrm{b}_r^+ \mathrm{a}_r). \tag{23.33}$$

This last expression shows that the result of operating with $\Lambda_e$ on the state vector $\Phi$ for the system reduces to the following: it transforms each state occupied by an electron into a state occupied by a positron (of the same momentum), and *vice versa*, and does not change the unoccupied electron-positron states.

It can be easily shown that the operator $\Lambda$ commutes with the energy, momentum and angular momentum operators for the system, and anticommutes with its total charge operator $Q$. Therefore, the states of the system can be eigenstates of the operator $\Lambda$ only in the case when $Q = 0$.

Since two consecutive applications of $\Lambda$ are equivalent to the identity transformation, i.e., $\Lambda^2 = 1$, the eigenvalues of $\Lambda$ are equal to

$$\Lambda = \pm 1. \tag{23.34}$$

Therefore, for given values of the energy and the momentum of a neutral system we can distinguish two states corresponding to different values of "charge parity" $\Lambda = \pm 1$. The state is said to be *charge-even,* if $\Lambda = +1$, and *charge-odd* if $\Lambda = -1$.

For a neutral system the law of conservation of charge parity holds.

Relation (23.30) shows that a system consisting of an even number of photons is charge-even, while a system consisting of an odd number of photons is charge-odd. Therefore, an arbitrary neutral system, if it has a definite charge parity, can decay either only into an even or only into an odd number of photons. For example, a $\pi^0$-meson decays into two photons, and therefore the $\pi^0$-meson is charge-even.

We consider a system consisting of an electron and a positron. The state vector for such a system can be represented in momentum space in the form of a superposition of three states:

$$\Phi = \sum_{p_1,\sigma_1,\sigma_2} A(p, \sigma_1, \sigma_2) \mathrm{a}^+_{\sigma_1}(p) \mathrm{b}^+_{\sigma_2}(-p) \Phi_0,$$

where $\Phi_0$ is the vacuum state vector; $p$, $\sigma_1$ and $-p, \sigma_2$ are the momenta and the spin variables of the two particles, whose total momentum is assumed to be equal to zero.

The result of operating with $\Lambda_e$ on $\Phi$ is in accordance with (23.27)

$$\begin{aligned}\Lambda_e \Phi &= \sum_{p,\sigma_1,\sigma_2} \mathrm{A}(p, \sigma_1, \sigma_2) \Lambda_e \mathrm{a}^+_{\sigma_1}(p) \Lambda_e^{-1} \cdot \Lambda_e \mathrm{b}^+_{\sigma_2}(-p) \Lambda_e^{-1} \Lambda_e \Phi_0 \\ &= \sum_{p,\sigma_1,\sigma_2} \mathrm{A}(p, \sigma_1, \sigma_2) \mathrm{b}^+_{\sigma_1}(p) \mathrm{a}^+_{\sigma_2}(-p) \Phi_0,\end{aligned}$$

and, since a+ and b+ anticommute, we have

$$\Lambda_e \Phi = - \sum_{p,\sigma_1,\sigma_2} A(p, \sigma_1, \sigma_2) a^+_{\sigma_2}(-p) b^+_{\sigma_1}(p) \Phi_0$$

$$= - \sum_{p,\sigma_1,\sigma_2} A(-p, \sigma_2, \sigma_1) a^+_{\sigma_1}(p) b^+_{\sigma_2}(-p) \Phi_0.$$

Thus, the result of operating with $\Lambda_e$ reduces to the reversal of the sign of $A$ and the exchange of the momenta and the spins of the two particles:

$$\Lambda_e A(p, \sigma_1, \sigma_2) = -A(-p, \sigma_2, \sigma_1).$$

This expression can be rewritten as

$$\Lambda_e \psi(1, 2) = -\psi(2, 1), \tag{23.35}$$

where $\psi$ is the wave function of the system, while 1 and 2 denote the complete sets of variables of the two particles in momentum or configuration space.

It follows from (23.35) that the symmetric states of the electron-positron system are charge-odd, while the antisymmetric states are charge-even.

If $l$ is the relative orbital angular momentum of the electron and the positron, while $s$ is the total spin of the system ($s = 0,1$), then

$$\Lambda = (-1)^{l+s}. \tag{23.36}$$

In particular, the electron-positron system in the state ${}^1S(l = 0, s = 0)$ can decay only into an even number of photons, while the same system in the state ${}^3S(l = 0, s = 1)$ can decay only into an odd number of photons.

We note that not every neutral system can be characterized by definite charge parity. For example, we consider the hydrogen atom. We denote its wave function by $\psi_{\mathrm{H}}$. Then $\Lambda\psi_{\mathrm{H}} = \psi_{\bar{\mathrm{H}}}$, where $\psi_{\bar{\mathrm{H}}}$ is the wave function for an atom of "antihydrogen" consisting of an antiproton and a positron. The functions $\psi_{\bar{\mathrm{H}}}$ and $\psi_{\mathrm{H}}$ are not connected by any simple relation, and the eigenfunctions of the operator $\Lambda$ are given not by $\psi_{\mathrm{H}}$ and $\psi_{\bar{\mathrm{H}}}$, but by the combinations $\psi_{\mathrm{H}} \pm \psi_{\bar{\mathrm{H}}}$.

## 23.4. *Perturbation Theory*

In quantum electrodynamics the interaction operator in accordance with (23.10) is proportional to the electron charge $e$ which can be treated as a small parameter. Therefore, the solution of the fundamental equation of quantum electrodynamics in the interaction picture

$$i\frac{\partial \Phi(t)}{\partial t} = \mathrm{V}(t)\Phi(t) \tag{23.37}$$

can be sought by using the method of successive approximations in the form of an expansion into a series in powers of the parameter $e$.

We write $\Phi(t)$ in the form

$$\Phi(t) = \mathrm{S}(t, t_0)\Phi(t_0), \tag{23.37'}$$

where $\Phi(t_0)$ is the value of $\Phi(t)$ at some initial instant of time $t = t_0$, while $\mathrm{S}(t, t_0)$ is an unknown operator which obviously satisfies the equation

$$i\frac{\partial \mathrm{S}(t, t_0)}{\partial t} = \mathrm{V}(t)\mathrm{S}(t, t_0) \tag{23.38}$$

and the initial condition

$$\mathrm{S}(t_0, t_0) = 1. \tag{23.38'}$$

We note that if at $t = -\infty$ the wave function $\Phi(t)$ coincides with $\Phi^0$, $\Phi(-\infty) = \Phi^0$, then the operator $\mathrm{S}(t, -\infty)$ goes over into the transformation operator $\mathrm{S}(t)$ introduced earlier (cf. (23.12)).

We seek the solution of equation (23.38) in the form of a series in powers of the charge $e$:

$$\mathrm{S}(t, t_0) = \sum_{k=0}^{\infty} \mathrm{S}_k(t, t_0), \tag{23.39}$$

where $\mathrm{S}_k(t, t_0)$ is proportional to $e^k$. On substituting this expansion into (23.38), and on equating terms of the same order, we obtain, on taking into account the initial condition (23.38'):

$$\begin{aligned} \mathrm{S}_k(t, t_0) &= -i\int_{t_0}^{t} \mathrm{V}(t')\mathrm{S}_{k-1}(t', t_0)\, dt', \\ \mathrm{S}_0(t, t_0) &= 1. \end{aligned} \tag{23.40}$$

On applying this formula $k$ times we obtain

$$\mathrm{S}_k(t, t_0) = (-i)^k \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{k-1}} dt_k \mathrm{V}(t_1)\,\mathrm{V}(t_2)\ldots \mathrm{V}(t_k). \tag{23.41}$$

The $k$-fold integral appearing in this expression can be conveniently represented in a somewhat different form. For the sake of simplicity we first discuss the term $S_2(t, t_0)$:

$$S_2(t, t_0) = -\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 V(t_1) V(t_2). \tag{23.41'}$$

In the $(t_1, t_2)$ plane ($t_1$ is the abscissa, $t_2$ is the ordinate of a point) the region of integration in (23.41′) will obviously be given by the triangle situated below the bisector of the angle between the coordinate axes (Fig. 11).

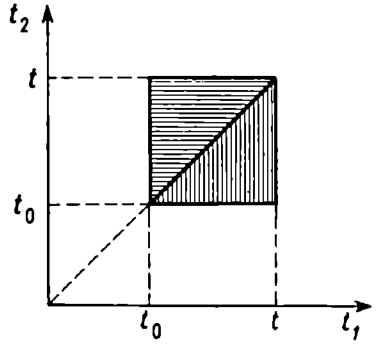

Fig. 11.

We now interchange the variables $t_1$ and $t_2$:

$$S_2(t, t_0) = -\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 V(t_2) V(t_1).$$

If in the $(t_1, t_2)$ plane we, as before, regard $t_1$ as the abscissa and $t_2$ as the ordinate, the region of integration will now be given by the triangle situated above the bisector of the angle between the coordinate axes. The integrand will now differ from the previous integrand of expression (23.41′) by a change in the order of the factors. If the operators $V(t_1)$ and $V(t_2)$ commuted, then both integrands would coincide and $S_2(t, t_0)$ would be given by half of the integral over the whole square. We can do this also in the general case of noncommuting operators $V(t_1)$ and $V(t_2)$ if we introduce the chronological operator $T$(cf. subsection 17.2)

$$T\big(V(t_1) V(t_2)\big) = \begin{cases} V(t_1) V(t_2), & t_2 < t_1, \\ V(t_2) V(t_1), & t_1 < t_2. \end{cases} \tag{23.42}$$

Then we have

$$S_2(t, t_0) = -\tfrac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\left(\mathrm{V}(t_1)\, \mathrm{V}\, t_2)\right).$$

In a similar manner it can be shown that in the general case $S_k(t, t_0)$ has the form (52)

$$S_k(t, t_0) = \frac{(-i)^k}{k!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \dots \int_{t_0}^{t} dt_k\, T\left(\mathrm{V}(t_1)\, \mathrm{V}(t_2) \dots \mathrm{V}(t_k)\right), \quad (23.43)$$

where $T$ is the chronological operator which arranges the factors in such a way that the time arguments of the operators $\mathrm{V}(t)$ decrease from left towards right.

This solution can be interpreted as follows. We write the formal solution of equation (23.37) in the form

$$\Phi(t) = \left(\exp\left(-i \int_{t_0}^{t} \mathrm{V}(t)\, dt\right)\right) \Phi(t_0). \quad (23.44)$$

This expression is a formally correct solution of the equation for $\Phi(t)$ if the values of the operator $\mathrm{V}(t)$ at different times commute. However, if two operators A and B do not commute, then $\exp(\mathrm{A}+\mathrm{B}) \neq \exp \mathrm{A} \cdot \exp \mathrm{B}$. Therefore formula (23.44) cannot be used directly in the case when the operators $\mathrm{V}(t_1), \mathrm{V}(t_2), \dots$ do not commute. In order to find out how to proceed in this case, we note that the solution of equation (23.37) can always be written in the form

$$\Phi(t) = \lim_{\tau_n \to \tau_{n+1}} \prod_{n=0}^{\infty} \left(1 - i \int_{\tau_{n+1}}^{\tau_n} \mathrm{V}(t)\, dt\right) \Phi(t_0),$$

where $\tau_n$ are times decreasing with increasing $n$ and tending towards $t_0$:

$$\tau_0 = t > \tau_1 > \tau_2 > \dots > \tau_\infty = t_0.$$

This expression shows that the function $\Phi(t_0)$ is first of all acted upon by the operator $\mathrm{V}(t)$ at the time $t = t_0$, and is only then acted upon by the values of the same operator at later times. Therefore, expression (23.44) can be used, but we must preserve the correct order of operating with the values of the operator $\mathrm{V}(t)$ at different times: the values of the operator referring to earlier times must be applied before those corresponding to later times. In order to express this we introduce at the beginning of the right hand side of formula (23.44) the chronological operator $T$:

$$\Phi(t) = T\left(\exp\left(-i\int_{t_0}^{t} \mathrm{V}(t')\,dt'\right)\right)\Phi(t_0) \tag{23.45}$$

or

$$\Phi(t) = \mathrm{S}(t, t_0)\,\Phi(t_0),$$

where

$$\mathrm{S}(t, t_0) = T\left(\exp\left(-i\int_{t_0}^{t} \mathrm{V}(t')\,dt'\right)\right). \tag{23.45'}$$

If the exponential is expanded into a series, we obtain

$$S(t, t_0) = \sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_k\, T\big(\mathrm{V}(t_1)\ldots \mathrm{V}(t_k)\big). \tag{23.46}$$

Perturbation theory can also be used to obtain the solutions of the fundamental equations of quantum electrodynamics in the Heisenberg picture. On expanding the field operators $\mathrm{A}_\mu$ and $\psi$ into series in powers of the electron charge

$$\begin{aligned} \mathrm{A}_\mu &= \mathrm{A}_\mu^{(0)} + e\mathrm{A}_\mu^{(1)} + e^2\mathrm{A}_\mu^{(2)} + \ldots, \\ \psi &= \psi^{(0)} + e\psi^{(1)} + e^2\psi^{(2)} + \ldots, \\ \bar{\psi} &= \bar{\psi}^{(0)} + e\bar{\psi}^{(1)} + e^2\bar{\psi}^{(2)} + \ldots \end{aligned} \tag{23.47}$$

and on substituting these expansions into (22.6) we obtain, equating the coefficients of like powers of the charge,

$$\begin{gathered} \left(\gamma_\mu\frac{\partial}{\partial x_\mu} + m\right)\psi^{(0)} = 0, \\ \Box\,\mathrm{A}_\mu^{(0)} = 0, \\ \left(\gamma_\mu\frac{\partial}{\partial x_\mu} + m\right)\psi^{(n+1)} = \frac{i}{2}\sum_{n'=0}^{n}\{\mathrm{A}_\mu^{(n')}, \gamma_\mu\psi^{(n-n')}\}, \\ \Box\,\mathrm{A}_\mu^{(n+1)} = -\frac{i}{2}\sum_{n'=0}^{n}[\bar{\psi}^{(n')}, \gamma_\mu\psi^{(n-n')}], \quad \text{where } n = 0, 1, 2, \ldots \end{gathered} \tag{23.48}$$

These equations enable us to express $\psi^{(1)}$, $A_\mu^{(1)}$ in terms of $\psi^{(0)}$, $A_\mu^{(0)}$; $\psi^{(2)}$, $A_\mu^{(2)}$ in terms of $\psi^{(1)}$, $A_\mu^{(1)}$, etc., i.e., we can finally express all the $\psi^{(n)}$, $A_\mu^{(n)}$ in terms of $\psi^{(0)}$, $A_\mu^{(0)}$. Therefore, in order that the problem of obtaining the field operators should be completely unique, we must also specify the commutation relations which must be satisfied by the operators $\psi^{(0)}$ and $A_\mu^{(0)}$. (Due to the existence of recurrence relations it is not possible to specify general commutation relations between the field operators, as we have seen earlier.) But $\psi^{(0)}$ and $A_\mu^{(0)}$ are the operators for the free fields, and the quantization rules satisfied by them are known.

We now demonstrate the method of solving the recurrence equations (23.48). We note, first of all, that the particular solution of the inhomogeneous Dirac equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) \psi(x) = -\varphi(x), \qquad (23.49)$$

where $\varphi(x)$ is a spinor which vanishes at $t = -\infty$, can be written in the form of the following integral over four-dimensional space:

$$\psi_a(x) = \int S^R_{a\lambda}(x-x')\varphi_\lambda(x')\, d^4x', \qquad (23.50)$$

where the Green's function, or more accurately the Green's matrix $S^R$, is defined by

$$S^R(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right) \Delta_R(x). \qquad (23.50')$$

Indeed, on making use of (23.50′), we obtain

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) \psi = \int (\Box - m^2)\Delta_R(x-x')\varphi(x')\, d^4x',$$

but according to (17.49′)

$$(\Box - m^2)\Delta_R(x) = -\delta(x),$$

and therefore

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) \psi(x) = -\int \delta(x-x')\varphi(x')\, d^4x' = -\varphi(x).$$

Since the function $\Delta_R(x)$ vanishes for $t < 0$, the solution (23.50) vanishes at $t = -\infty$.

In a similar manner the particular solution of the inhomogeneous equation

$$\Box A_\mu(x) = -a_\mu(x), \tag{23.51}$$

which vanishes at $t = -\infty$, can be written as

$$A_\mu(x) = \int D_R(x-x')\, a_\mu(x')\, d^4x'. \tag{23.52}$$

Formulas (23.50) and (23.52) enable us to obtain in turn solutions of equations (23.48) vanishing at $t = -\infty$. It can be shown that these solutions are of the following form (171), (172):

$$\begin{aligned} \psi^{(n)}(x) &= i^n \int_{-\infty}^{t} d^4x_1 \int_{-\infty}^{t_1} d^4x_2 \ldots \\ &\ldots \int_{-\infty}^{t_{n-1}} d^4x_n \Big[ \mathrm{U}^{(0)}(x_n), [\ldots [\mathrm{U}^{(0)}(x_1), \psi^{(0)}(x)] \ldots ] \Big], \\ \mathrm{A}_\mu^{(n)}(x) &= i^n \int_{-\infty}^{t} d^4x_1 \int_{-\infty}^{t_1} d^4x_2 \ldots \\ &\ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}^4x_n \Big[ \mathrm{U}^{(0)}(x_n), [\ldots [\mathrm{U}^{(0)}(x_1), \mathrm{A}_\mu^{(0)}(x)] \ldots ] \Big], \end{aligned} \tag{23.53}$$

where

$$\mathrm{U}^{(0)}(x) = -\mathrm{j}_\mu^{(0)}(x)\, \mathrm{A}_\mu^{(0)}(x),$$

$$\mathrm{j}_\mu^{(0)}(x) = \frac{i}{2} [\bar{\psi}^{(0)}(x), \gamma_\mu \psi^{(0)}(x)].$$

## § 24. The Scattering Matrix

### 24.1. *The Scattering Problem and the Definition of the Scattering Matrix*

In quantum electrodynamics, as in quantum mechanics, there are two fundamental types of problems that have to be solved. Problems of the first type consist of determining the energy levels of a system, while problems of the second type are collision problems. We shall in future concern ourselves with problems of the second type. They can be formulated in the following way. Given the state of the electromagnetic and the electron-positron fields at the time $t = -\infty$, it is required to obtain the state of the fields at $t = \infty$. In the interaction picture which we shall use these states are described by the wave functions $\Phi(-\infty)$ and $\Phi(\infty)$. In accordance with (23.45) they are related by

$$\Phi(\infty) = \mathrm{S}\Phi(-\infty), \tag{24.1}$$

where the operator S is of the form

$$\mathrm{S} = \mathrm{S}(\infty, -\infty) = T\left(\exp\left(-i \int_{-\infty}^{\infty} \mathrm{V}(t)\, dt\right)\right). \tag{24.2}$$

This operator which transforms the wave function of the initial state $\Phi(-\infty)$ into the wave function of the final state $\Phi(\infty)$ is called the *scattering matrix* or the S-*matrix*.

In quantum mechanical collision theory it is usually assumed that the initial and the final states of the system are free, i.e., it is assumed that asymptotically at $t = \mp\infty$ the interaction operator $\mathrm{V}(t)$ vanishes and $\mathrm{V}(\mp\infty) = 0$. Such an assumption is justified if the collision does not lead to the appearance or destruction of bound states, since, if we exclude such cases, then at $t = \mp\infty$ the particles are far from each other, and therefore $\mathrm{V}(\mp\infty) = 0$. Without taking into account the appearance and destruction of bound states we assume that in quantum electrodynamics we have in the initial and final states of the system only free electrons and photons. In doing this we use here the term "free electron" only in the sense that there is no interaction between the electron and the quantized electromagnetic field; insofar as an external electromagnetic field is concerned, it may be present, and bound states due to the external field can be taken into account (cf. the expansion of the operator for the electron-positron field in terms of stationary states in the presence of an external field (18.7)).

We note that, strictly speaking, the electron cannot be regarded as free even asymptotically at $t = \mp\infty$ since the interaction of the electron with its own electromagnetic field leads to a change in its mass. We shall discuss this problem in greater detail in Chapter VII.

Since for $\mathrm{V} \to 0$ the state vector in the interaction picture coincides with the state vector in the Heisenberg picture, we can assume that

$$\Phi(-\infty) = \Phi^0,$$

and rewrite (24.1) as

$$\Phi(\infty) = \mathrm{S}\Phi^0. \tag{24.1'}$$

We characterize the different states of the free fields by the set of state vectors $\Phi_r^0$, where $r$ is the set of indices specifying the numbers of electrons and of photons having different values of energy, momentum and other physical quantities. If at the initial time $t=-\infty$ the system was in the state $\Phi_i^0$, then at $t=\infty$ it will be in the state

$$\Phi(\infty)=\mathrm{S}\Phi_i^0.$$

Therefore, the probability amplitude for the transition of the system from the state $\Phi_i^0$ at $t=-\infty$ into the state $\Phi_f^0$ at $t=\infty$ will be equal to

$$\big(\Phi_f^0, \Phi(\infty)\big)=(\Phi_f^0, \mathrm{S}\Phi_i^0)\equiv(f|\mathrm{S}|i). \tag{24.3}$$

We see that the probabilities of the different processes are determined by the elements of the scattering matrix S connecting the corresponding initial and final states.

### 24.2. *Matrix Elements of Field Operators*

In accordance with (23.10) and (23.46) the scattering matrix can be represented in the form

$$\mathrm{S}=T\Big(\exp\Big(-i\int \mathrm{U}(x)\,\mathrm{d}^4x\Big)\Big) \tag{24.4}$$

or

$$S=\sum_{n=0}^{\infty}\mathrm{S}^{(n)}=\sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int d^4x_1\ldots\int d^4x_n\,T\big(\mathrm{U}(x_1)\ldots\mathrm{U}(x_n)\big), \tag{24.5}$$

where

$$\mathrm{U}(x)=-\mathrm{L}_{e\gamma}=-\mathrm{j}_\mu(x)\,\mathrm{A}_\mu(x), \tag{24.6}$$

and the integration with respect to each variable is taken over the whole four-dimensional space.

Since both the chronological operator $T$ and the interaction operator $\mathrm{U}(x)$ are invariant under Lorentz transformations, the scattering matrix is relativistically invariant.

If the electron is acted upon by an external electromagnetic field then we must take for the operators $\psi$, $\overline{\psi}$ appearing in the expression for the current density $\mathrm{j}_\mu$ the expansions (18.7) in terms of the eigenfunctions of the Dirac equation for the electron in this field.

In many cases the external field is sufficiently weak to be taken into account by means of perturbation theory. In order to do this we must replace $A_\mu(x)$ in the expression for $U(x)$ by the sum $A_\mu(x) + A^e_\mu(x)$ where $A_\mu(x)$ is the potential of the quantized electromagnetic field, while $A^e_\mu(x)$ is the potential of the external field which, in contrast to $A_\mu(x)$, is a *c*-number. Insofar as the current density vector $j_\mu$ is concerned, it must be constructed with the aid of the expansions of $\psi$ and $\bar{\psi}$ in terms of free particle eigenfunctions.

An intermediate method of taking the external field into account is also possible when one of its parts has to be taken into account exactly by means of the Dirac equation, while its other part is taken into account approximately by adding it to the expression for the quantized potential. In this case the operators $\psi$ and $\bar{\psi}$ are defined with the aid of the expansions in terms of the eigenfunctions of the Dirac equation containing only the first part of thc external field.

We shall say that a scattering process, i.e., the process of interaction between the fields, is an effect of the *n*th order if the element of the S-matrix corresponding to this process is proportional to $e^n$. It is obvious that all the *n*th order processes are described by the matrix $S^{(n)}$ which is the *n*th term in the expansion of the S-matrix into a series in powers of the electron charge.

Since the operators $A_\mu(x)$ and $j_\mu(x)$ commute, $S^{(n)}$ has the structure

$$S^{(n)} = \int K_{\mu\nu} \dots (x_1, x_2, \dots) \, T\big(A_\mu(x_1) \, A_\nu(x_2) \dots\big) \, d^4x_1 d^4x_2 \dots,$$

where $K_{\mu\nu} \dots (x_1, x_2, \dots)$ contains a chronological product of the current density operators. The latter satisfy the equation of continuity, and therefore

$$(\partial K_{\mu\nu} \dots / \partial x_{1_\mu}) = (\partial K_{\mu\nu} \dots / \partial x_{2_\nu}) = \dots = 0.$$

From this we can easily conclude that the addition to $T\big(A_\mu(x_1) \, A_\nu(x_2)\big)$ of the expression $\big(\partial^2 \varphi(x_1 - x_2)\big)/(\partial x_{1_\mu} \, \partial x_{2_\mu})$, containing an arbitrary function $\varphi(x)$ (cf. formula (17.14)) does not alter the value of $S^{(n)}$, in accordance with the gauge invariance of the theory.

We now proceed to establish the rules for evaluating the elements of the scattering matrix connecting any given states.

The individual terms of the expansion (24.5) are integrals containing products of operators $\psi(x)$, $\bar{\psi}(x)$, $A_\mu(x)$. As has been shown in §§ 16,18, $\psi(x)$ is an electron annihilation operator and a positron creation operator, $\bar{\psi}(x)$ is an electron creation operator and a positron annihilation operator, and $A_\mu(x)$ is a photon creation and annihilation operator.

In order to obtain the matrix element of the operator $\psi(x)$ we use the general expansion (18.7) of the operator $\psi(x)$ in terms of the eigenfunctions of the electron in an external field, and also formulas (18.17) defining the matrix elements of the electron and positron creation and annihilation operators. From these formulas it follows that the matrix elements of the operators $\psi(x)$ and $\bar{\psi}(x)$, corresponding to the annihilation and creation of an electron in the state $r$, are given by

$$\begin{aligned} \left(\Phi_{0_r^+}(\infty), \psi(x)\,\Phi_{1_r^+}(-\infty)\right) &\equiv \left(0_r^+|\psi(x)|1_r^+\right) = \psi_r^{(+)}(x), \\ \left(\Phi_{1_r^+}(\infty), \bar{\psi}(x)\,\Phi_{0_r^+}(-\infty)\right) &\equiv \left(1_r^+|\bar{\psi}(x)|0_r^+\right) = \bar{\psi}_r^{(+)}(x), \end{aligned} \qquad (24.7)$$

where $\psi_r^{(+)}(x)$ is the normalized wave function of positive frequency for an electron in the $r$-state, while $\Phi_{1_r^+}(\pm\infty)$ and $\Phi_{0_r^+}(\pm\infty)$ are the wave functions for the system of fields at the time $t = \pm\infty$ (the subscript $1_r^+$ or $0_r^+$ denotes the number of electrons in the corresponding state).

The matrix elements for the creation or annihilation of a positron in the state $r$ are given by

$$\begin{aligned} (1_r^-|\psi(x)|0_r^-) &= \psi_r^{(-)}(x), \\ (0_r^-|\bar{\psi}(x)|1_r^-) &= \bar{\psi}_r^{(-)}(x), \end{aligned} \qquad (24.8)$$

where $\psi_r^{(-)}$ is the normalized wave function of negative frequency for an electron in the $r$-state.

If the states of the electron can be described by plane waves, then the matrix elements for the annihilation and creation of an electron of momentum $p$ and polarization $r$ will be given by

$$\begin{aligned} \left(0_{pr}^+|\psi(x)|1_{pr}^+\right) &= \frac{1}{\sqrt{\Omega}}\, u^r(p)\, e^{ipx}, \\ \left(1_{pr}^+|\bar{\psi}(x)|0_{pr}^+\right) &= \frac{1}{\sqrt{\Omega}}\, \bar{u}^r(p)\, e^{-ipx}, \end{aligned} \qquad (24.9)$$

where $\Omega$ is the normalization volume.

Finally, the matrix elements for the creation and annihilation of a positron of momentum $p$ and polarization $r$ are given by

$$\begin{aligned} \left(1_{pr}^-|\psi(x)|0_{pr}^-\right) &= \frac{1}{\sqrt{\Omega}}\, v^r(-p)\, e^{-ipx}, \\ \left(0_{pr}^-|\bar{\psi}(x)|1_{pr}^-\right) &= \frac{1}{\sqrt{\Omega}}\, \bar{v}^r(-p)\, e^{ipx}. \end{aligned} \qquad (24.10)$$

These matrix elements have the form of normalized Dirac plane waves of momentum $-\boldsymbol{p}$ corresponding to negative frequency.

In order to determine the matrix elements of the operator $\mathrm{A}_\mu(x)$ we use the expansion of $\mathrm{A}_\mu(x)$ in terms of plane waves and formulas (16.38) defining the photon annihilation and creation matrix elements $\mathbf{c}_{k\lambda}$ and $\mathbf{c}^{+}_{k\lambda}$.

From these formulas it follows that the matrix elements of the operator $\mathrm{A}_\mu(x)$ corresponding to the annihilation and creation of a photon of momentum $\boldsymbol{k}$ and polarization $\lambda$ are respectively given by

$$\begin{aligned}(0_{k\lambda}|\mathrm{A}_\mu(x)|1_{k\lambda}) &= \frac{1}{\sqrt{2\omega\Omega}}e_\mu^{(\lambda)}e^{ikx},\\ (1_{k\lambda}|\mathrm{A}_\mu(x)|0_{k\lambda}) &= \frac{1}{\sqrt{2\omega\Omega}}e_\mu^{(\lambda)}e^{-ikx},\end{aligned} \tag{24.11}$$

where $e_\mu^{(\lambda)}$ is the four-dimensional unit polarization vector

$$e_\mu^{(\lambda)}e_\mu^{(\lambda)} = e^{(\lambda)2}-e_0^{(\lambda)2} = 1$$

$\left(e^{(\lambda)}\right.$ is the spatial and $e_0^{(\lambda)}(= e_4^{(\lambda)}/i)$ is the time component of the vector $\left.e_\mu^{(\lambda)}\right)$.

If for the potentials of the electromagnetic field we utilize not plane but spherical waves, then in place of (24.11) we obtain

$$\begin{aligned}(0_{jM\lambda}|\mathrm{A}_\mu(x)|1_{jM\lambda}) &= (\mathrm{A}_{jM\lambda})_\mu,\\ (1_{jM\lambda}|\mathrm{A}_\mu(x)|0_{jM\lambda}) &= (\mathrm{A}_{jM\lambda})^*_\mu,\end{aligned} \tag{24.12}$$

where $\mathrm{A}_{jM\lambda}$ are the normalized potentials of photon states of definite momentum and parity (cf. § 4).

## 24.3. *Representation of the Scattering Matrix as a Sum of Normal Products*

We return to the general expansion (24.5) of the scattering matrix. Since the operators $\psi$, $\bar{\psi}$, $\mathrm{A}_\mu$ are expressed as a sum of creation and annihilation operators for the individual particles, each term in the expansion (24.5) may be written as a sum of products of creation and annihilation operators for electrons, positrons and photons in different states. We wish to determine the conditions under which such products have nonvanishing matrix elements corresponding to some process $i \rightarrow f$ which is of interest to us. If, for example, in the state *i* there exists a single electron and no positrons or photons, while in the state *f* there exists an electron and a photon, then, obviously, one of the annihilation operators must "annihilate" an electron in the state *i*, two creation operators must "create" an electron and a photon in the state *f*, while all the other operators must break up into pairs with the operators belonging to each pair "creating" and "annihilating" the same particle.

Such virtual processes of creation and subsequent annihilation of the same particle introduce considerable complications into the evaluation of the elements of the scattering matrix. We shall therefore attempt to transform the scattering matrix in such a way that there should be no need at all to discuss such virtual processes. Evidently the problem is reduced to representing the scattering matrix in the form of a sum of normal or ordered products of particle creation and annihilation operators in which the creation operators appear to the left of the annihilation operators. In the evaluation of matrix elements of such products the annihilation operators will "annihilate" only those particles which are present in the initial state, while the creation operators will "create" only those particles which must occur in the final state. Insofar as the virtual processes of particle creation and annihilation are concerned they will clearly not appear in our discussion.

In order to write the scattering matrix in such a form we note that in accordance with (19.29)

$$\mathrm{j}_\mu(x) = ie\,N(\bar{\psi}(x)\gamma_\mu\psi(x))$$

and therefore

$$\begin{aligned} \mathrm{U}(x) &= -\mathrm{j}_\mu(x)\mathrm{A}_\mu(x) = -ieN\big(\bar{\psi}(x)\hat{\mathrm{A}}(x)\psi\,(x)\big), \\ \hat{\mathrm{A}}(x) &= \gamma_\mu \mathrm{A}_\mu(x). \end{aligned} \tag{24.13}$$

From this it follows that the *n*th term in the expansion of the *S*-matrix may be written in the following form:

$$\mathrm{S}^{(n)} = \frac{(-e)^n}{n!}\int d^4x_1 \dots \int d^4x_n\, T\big(N(\bar{\psi}(x_1)\hat{\mathrm{A}}(x_1)\,\psi(x_1)) \dots N(\psi(x_n)\hat{\mathrm{A}}(x_n)\psi(x_n))\big). \tag{24.14}$$

The individual factors of the *T*-product appearing in this formula represent *N*-products of field operators relating to the same instant of time. We shall refer to a *T*-product of this type as a mixed *T*-product.

Thus, our problem consists of writing a mixed $T$-product in the form of a sum of $N$-products. This may be accomplished with the aid of the following two theorems (210).

I. A $T$-product of field operators is equal to the sum of their $N$-products in which the operators are paired in all possible ways.

In other words, if $\chi_1, \chi_2, \ldots, \chi_n$ stand for the field operators $\psi, \overline{\psi}$, $A_\mu$ then the following relation holds:

$$\begin{aligned} T(\chi_1\chi_2 \cdots \chi_n) &\equiv \delta_P \chi_{i_1}\chi_{i_2} \cdots \chi_{i_n} \\ &= N(\chi_1\chi_2 \cdots \chi_n) + N(\chi_1^a\chi_2^a\chi_3 \cdots \chi_n) + N(\chi_1^a\chi_2\chi_3^a\chi_4 \cdots \chi_n) \\ &+ \ldots + N(\chi_1^a\chi_2^a\chi_3^b\chi_4^b \cdots \chi_n) + \ldots + N(\chi_1^a\chi_2^b \cdots \chi_{n-1}^b\chi_n^a) + \ldots \qquad (24.15) \end{aligned}$$

Here the operators $\chi_{i_1}, \chi_{i_2}, \cdots \chi_{i_n}$ in the right hand side of the first equation, which is a definition of the $T$-product, are arranged in chronological order, i.e., in such a way that the time increases from right to left, and $\delta_p$ is the parity of the permutation of the electron-positron operators required to achieve this.

The different letters serving as superscripts of operators in the right hand side of the equation denote different pairings defined for adjacent operators by means of

$$\chi_i^a\chi_k^a = T(\chi_i\chi_k) - N(\chi_i\chi_k).$$

Pairings between non-neighboring operators are defined in the following manner: if the paired operators are photon operators, then they may simply be written next to each other; however, if they are electron-positron operators, then they may be placed next to each other provided we first multiply the $N$-product by the parity of the permutation of the electron-positron operators required to achieve this order. For example, if $\chi_1, \chi_2, \ldots$ are electron-positron operators, then

$$N(\chi_1^a\chi_2\chi_3^b\chi_4^b\chi_5^a\chi_6) = -(\chi_1^a\chi_5^a)(\chi_3^b\chi_4^b)N(\chi_2\chi_6)$$

(since $\chi_1^a\chi_5^a$ and $\chi_3^b\chi_4^b$ are $c$-numbers, they have been taken outside the $N$-product).

II. A mixed product of field operators is equal to the sum of their $N$-products of the form (24.15), but this sum does not contain pairings between operators appearing in the same inner $N$-product. For example,

$$\begin{aligned} T(\chi_1 N(\chi_2\chi_3\chi_4)) &= N(\chi_1\chi_2\chi_3\chi_4) \\ &+ N(\chi_1^a\chi_2^a\chi_3\chi_4) + N(\chi_1^a\chi_2\chi_3^a\chi_4) + N(\chi_1^a\chi_2\chi_3\chi_4^a). \qquad (24.15') \end{aligned}$$

This sum does not contain the pairings $\chi_2^a\chi_3^a, \chi_2^a\chi_4^a, \chi_3^a\chi_4^a$.

In order to prove theorem I we make the preliminary remark that by a simultaneous identical rearrangement of factors within *T*- and *N*-products in (24.15) we do not violate this equation, so that without loss of generality we may assume that the operators in (24.15) are already arranged in chronological order.

We now arrange the factors in (24.15) in such a way that all the creation operators stand to the left of the annihilation operators.[1] We shall refer to such an arrangement of the operators as an *N*-ordered one. In order to achieve this we take the *N*-unordered creation operator standing at the left and interchange it in turn with all the annihilation operators standing to the left of this operator. In the course of this additional terms with pairings between the operators being interchanged appear in accordance with the following formula:

$$\chi_1\chi_2 = T(\chi_1\chi_2) = N(\chi_1\chi_2)+\chi_1^a\chi_2^a = \pm\, \chi_2\chi_1+\chi_1^a\chi_2^a.$$

We perform the ordering operation also with respect to the other unordered creation operators. As a result we express the initial *T*-product in terms of a sum of *N*-products. These *N*-products may appear both with positive and with negative signs, but if the factors within the *N*-products are rearranged in such a way that they again become *T*-ordered, i.e., arranged in chronological order, then, obviously, all the *N*-products will appear with positive signs and we shall obtain an expression for the *T*-product in the form of a sum of *N*-products of the form (24.15). This sum will, obviously, not contain all possible pairings, but only pairings between all pairs of *N*-unordered operators; however, since pairings between *N*-ordered operators, which are at the same time *T*-ordered operators, vanish, we can assume that the sum contains all possible pairings. Thus we have proved the validity of theorem I.

Theorem II can be proved in a similar manner. In the course of the proof it is only necessary to keep in mind that operators appearing within the same *N*-product need not be interchanged since such operators are already *N*-ordered; therefore there will be no pairings corresponding to them.

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[1] For the sake of simplicity we assume that each of the operators $\chi_i$ is either a creation or an annihilation operator.

### 24.4. *General Relation between T- and N-orderings*

We now show that not only the individual terms in the expansion of the scattering matrix, but also the matrix as a whole can be represented in the form of an $N$-ordered operator.

First of all we consider the functional of the electromagnetic field operators

$$F\{\mathrm{A}\} = \sum_{n=0}^{\infty} \int K^{(n)}_{\nu_1, \nu_2, \ldots, \nu_n}(x_1, x_2, \ldots, x_n) \mathrm{A}_{\nu_1}(x_1) \mathrm{A}_{\nu_2}(x_2) \ldots$$
$$\ldots \mathrm{A}_{\nu_n}(x_n) d^4x_1 \, d^4x_2 \ldots d^4x_n, \qquad (24.16)$$

where $K^{(n)}_{\nu_1, \nu_2, \ldots, \nu_n}$ are $c$-numbers, $\nu_k = 1, 2, 3, 4$, and the summation is over $n$ and $\nu_k$. We show that (89)

$$TF\{\mathrm{A}\} = N(e^{\Delta} F\{\mathrm{A}\}), \qquad (24.17)$$

where

$$\Delta = \frac{1}{2} \int\int \delta_{\mu\nu} D^c(x-x') \frac{\delta}{\delta \mathrm{A}_\mu(x)} \frac{\delta}{\delta \mathrm{A}_\nu(x')} d^4x \, d^4x'. \qquad (24.17')$$

In this expression the functional derivative with respect to the operator $\mathrm{A}_\mu(x)$ is defined as follows. If $\delta \mathrm{A}_\mu(x)$ is the increment of the operator $\mathrm{A}_\mu(x)$ then the variation of the functional given by

$$\delta F\{\mathrm{A}_\mu(x)\} = F\{\mathrm{A}_\mu(x) + \delta \mathrm{A}_\mu(x)\} - F\{\mathrm{A}_\mu(x)\},$$

is related to the functional derivative $[\delta F\{\mathrm{A}_\mu(x)\}]/[\delta \mathrm{A}_\mu(x')]$ by

$$\delta F\{\mathrm{A}_\mu(x)\} = \int \frac{\delta F\{\mathrm{A}_\mu(x)\}}{\delta \mathrm{A}_\mu(x')} \delta \mathrm{A}_\mu(x') d^4x'. \qquad (24.18)$$

In order to prove formula (24.17) we first show that $\Delta F\{\mathrm{A}\}$ is obtained from $F\{\mathrm{A}\}$ if in each term of the expansion (24.16) we form in all possible ways single pairings between the operators $\mathrm{A}_\mu(x)$ and then sum the expressions so obtained. In order to accomplish this it is sufficient to verify

$$\Delta \big(\mathrm{A}_\mu(x_1) \mathrm{A}_\nu(x_2)\big) = \mathrm{A}^a_\mu(x_1) \mathrm{A}^a_\nu(x_2),$$

which follows immediately from (24.17′) and (17.10):

$$\begin{aligned}
&\Delta\left(\mathrm{A}_\lambda(x_1)\,\mathrm{A}_\sigma(x_2)\right)\\
&=\frac{1}{2}\int\int D^c(x-x')\,d^4x\,d^4x'\,\frac{\delta}{\delta\mathrm{A}_\nu(x)}\,\frac{\delta}{\delta\mathrm{A}_\nu(x')}\,\mathrm{A}_\lambda(x_1)\,\mathrm{A}_\sigma(x_2)\\
&=\frac{1}{2}\int\int D^c(x-x')\,d^4x\,d^4x'\,\frac{\delta}{\delta\mathrm{A}_\nu(x)}\big(\mathrm{A}_\lambda(x_1)\,\delta_{\sigma\nu}\,\delta(x_2-x')\\
&\quad+\mathrm{A}_\sigma(x_2)\,\delta_{\lambda\nu}\,\delta(x_1-x')\big)\\
&=\frac{1}{2}\int\int D^c(x-x')\,\big(\delta_{\lambda\nu}\,\delta_{\sigma\nu}\,\delta(x-x_1)\,\delta(x'-x_2)\\
&\quad+\delta_{\lambda\nu}\,\delta_{\sigma\nu}\,\delta(x'-x_1)\,\delta(x-x_2)\big)\,d^4x\,d^4x'\\
&=\mathrm{A}_\lambda^a(x_1)\,\mathrm{A}_\sigma^a(x_2).
\end{aligned}$$

In a similar manner it is possible to show that the functional $(\Delta^k/k!)F\{\mathrm{A}\}$ is obtained from the functional $F\{\mathrm{A}\}$ if in each term of expansion (24.16) we form in all possible ways $k$ pairings between the operators $\mathrm{A}_\mu$ and then sum the expressions so obtained.

From this it follows that theorem I stated above on the expansion of a $T$-product in terms of $N$-products can be written in the following form applicable to the operators $\mathrm{A}_\mu$:

$$TF\{\mathrm{A}\}=\sum_{k=0}^{\infty}N\left(\frac{\Delta^k}{k!}\,F\{\mathrm{A}\}\right).$$

On noting that $\sum_{k=0}^{\infty}(\Delta^k/k!)=e^{\Delta}$, we obtain formula (24.17).

A formula of the same type also holds for electron-positron operators, viz., if $F\{\psi,\overline{\psi}\}$ is a functional of the operators $\psi$ and $\overline{\psi}$ of the form

$$F\{\psi,\overline{\psi}\}=\sum_{n_1,n_2=0}^{\infty}\int G^{(n_1,n_2)}(x_1,x_2,\ldots,x_{n_1};x_1',x_2',\ldots,x_{n_2}')\,\psi(x_1)\,\overline{\psi}(x_1')\ldots\psi(x_{n_1})\,\overline{\psi}(x_{n_2}')\cdot d^4x_1\ldots d^4x_{n_2}',\tag{24.19}$$

where the summation is over $n_1$ and $n_2$ and also over the spinor indices on which $G^{(n_1,n_2)}$ depends, then

$$TF\{\psi,\overline{\psi}\}=N(e^{\Sigma}F\{\psi,\overline{\psi}\}),\tag{24.20}$$

where

$$\Sigma=\int\int\frac{\delta}{\delta\psi_\alpha(x)}\,S_{\alpha\beta}^c(x-x')\,\frac{\delta}{\delta\overline{\psi}_\beta(x')}\,d^4x\,d^4x',\tag{24.20'}$$

and the functional derivatives with respect to $\psi$ and $\overline{\psi}$ are defined in the following manner. If $\delta\psi(x)$ and $\delta\overline{\psi}(x)$ are increments of the operators $\psi(x)$ and $\overline{\psi}(x)$, then the variation of the functional $F\{\psi, \overline{\psi}\}$ given by

$$\delta F\{\psi, \overline{\psi}\} = F\{\psi+\delta\psi, \overline{\psi}+\delta\overline{\psi}\}-F\{\psi, \overline{\psi}\},$$

is related to the functional derivatives by

$$\delta F\{\psi, \overline{\psi}\} = \int \frac{\delta F}{\delta\psi_\alpha(x)}\, \delta\psi_\alpha(x)\, d^4x + \int \delta\overline{\psi}_\beta(x) \frac{\delta F}{\delta\overline{\psi}_\beta(x)}\, d^4x. \tag{24.21}$$

In order to take into account correctly the change of sign resulting from the interchange of electron-positron operators, the increments of the operators $\psi$ and $\overline{\psi}$ are subjected to the following conditions:

$$\{\delta\psi, \psi\} = \{\delta\psi, \overline{\psi}\} = \{\delta\overline{\psi}, \psi\} = \{\delta\overline{\psi}, \overline{\psi}\} = 0 \tag{24.22}$$

or to the equivalent conditions

$$\left\{\frac{\delta}{\delta\psi_\alpha(x)}, \overline{\psi}_\beta(x')\right\} = 0, \quad \left\{\frac{\delta}{\delta\overline{\psi}_\beta(x)}, \psi_\alpha(x')\right\} = 0,$$
$$\left\{\frac{\delta}{\delta\psi_\alpha(x)}, \psi_\beta(x')\right\} = \delta_{\alpha\beta}\,\delta(x-x'), \quad \left\{\frac{\delta}{\delta\overline{\psi}_\alpha(x)}, \overline{\psi}_\beta(x')\right\} = \delta_{\alpha\beta}\,\delta(x-x'). \tag{24.22'}$$

It follows from formulas (24.17) and (24.20) that if we have a general functional $F\{A_\mu, \psi, \overline{\psi}\}$ of the operators $A_\mu, \psi, \overline{\psi}$, then

$$TF\{A_\mu, \psi, \overline{\psi}\} = N(e^{\Delta+\Sigma}F\{A_\mu, \psi, \overline{\psi}\}). \tag{24.23}$$

We now apply these formulas to the transformation of the S-matrix. Let the functional $F\{A_\mu, \psi, \overline{\psi}\}$ be of the form

$$F\{A_\mu, \psi, \overline{\psi}\} = \exp\left(i\int j_\mu(x)\, A_\mu(x)\, d^4x\right). \tag{24.24}$$

In accordance with (24.4) the scattering matrix is related to $F\{A_\mu, \psi, \overline{\psi}\}$ by

$$S = TF\{A_\mu, \psi, \overline{\psi}\} = T_\psi T_A F\{A_\mu, \psi, \overline{\psi}\}, \tag{24.24'}$$

where the subscripts $\psi$ and $A$ on $T$ denote respectively the $T$-orderings of the operators $\psi$, $\overline{\psi}$ and $A_\mu$. In accordance with the proofs just given,

$$T_A F\{A_\mu, \psi, \overline{\psi}\} = N_A\left(e^{\Delta}\cdot \exp\left(i\int j_\mu(x)\, A_\mu(x)\, d^4x\right)\right),$$

where $N_A$ denotes the $N$-ordering of the operators $\mathrm{A}_\mu$. On noting that

$$N_A \frac{\delta}{\delta \mathrm{A}_\nu(x')} \exp\left(i \int \mathrm{j}_\mu(x)\, \mathrm{A}_\mu(x)\, d^4x\right) = i N_A\left(\mathrm{j}_\nu(x') \exp\left(i \int \mathrm{j}_\mu(x)\, \mathrm{A}_\mu(x)\, d^4x\right)\right)$$

it can be easily shown that

$$N_A\left(\Delta \exp\left(i \int \mathrm{j}_\mu(x)\, \mathrm{A}_\mu(x)\, d^4x\right)\right)$$
$$= -N_A\left(\tfrac{1}{2} \int d^4x' \int d^4x'' D^c(x'-x'')\, \mathrm{j}_\nu(x')\, j_\nu(x'') \exp\left(i \int \mathrm{j}_\mu(x)\, \mathrm{A}_\mu(x)\, d^4x\right)\right).$$

Therefore we have

$$T_A F\{\mathrm{A}, \psi, \bar{\psi}\} = \exp\left(-\tfrac{1}{2} \int d^4x' \int d^4x''\, \mathrm{j}_\nu(x')\, D^c(x'-x'')\, \mathrm{j}_\nu(x'')\right)$$
$$\times N_A\left(\exp\left(i \int \mathrm{j}_\mu(x)\, \mathrm{A}_\mu(x)\, d^4x\right)\right).$$

From this it follows that

$$\mathrm{S} = T_\psi\left(\exp\left(-\tfrac{1}{2} \int d^4x' \int d^4x''\, \mathrm{j}_\mu(x')\, D^c(x'-x'')\, \mathrm{j}_\mu(x'')\right)\right.$$
$$\left. \times N_A\left(\exp i \int \mathrm{j}_\nu(x)\, \mathrm{A}_\nu(x)\, d^4x\right)\right). \quad (24.25)$$

In processes in which no photons participate the last factor in (24.25) is equal to unity, and the scattering matrix assumes the form

$$\mathrm{S} = T_\psi \exp\left(-\tfrac{1}{2} \int d^4x' \int d^4x''\, \mathrm{j}_\nu(x')\, D^c(x'-x'')\, \mathrm{j}_\nu(x'')\right). \quad (24.26)$$

If the motion of the electrons can be regarded as given, i.e., if we treat the current density $\mathrm{j}_\mu(x)$ as a given $c$-function $J_\mu(x)$, then the scattering matrix can be written as

$$\mathrm{S} = \exp\left(-\tfrac{1}{2} \int d^4x' \int d^4x''\, J_\nu(x')\, D^c(x'-x'')\, J_\nu(x'')\right)$$
$$\times N \exp\left(i \int J_\mu(x)\, \mathrm{A}_\mu(x)\, d^4x\right), \quad (24.27)$$

where the only operators are the electromagnetic potentials $\mathrm{A}_\mu(x)$.

### 24.5. *Symmetry of the Scattering Matrix under Time Reversal*

In proving the invariance of the equations of quantum electrodynamics under the transformation of total inversion $R$ we have shown (cf. subsection 22.4) that

$$\left(\mathrm{L}\{\mathrm{A}_\mu(-x), \psi(-x)\}\right)^{\mathrm{T}} = \mathrm{L}\{\mathrm{A}_\mu^R(x), \psi^R(x)\}.$$

From this it follows that

$$\left(\mathrm{j}_\mu(-x)\,\mathrm{A}_\mu(-x)\right)^{\mathrm{T}} = \mathrm{j}_\mu^R(x)\,\mathrm{A}_\mu^R(x).$$

On the other hand, $R$ is a transformation which does not alter the equations or the commutation relations of quantum electrodynamics, and can be regarded as a unitary transformation; therefore

$$\mathrm{j}_\mu^R(x)\,\mathrm{A}_\mu^R(x) = \mathrm{U}^{\mathrm{T}}\mathrm{j}_\mu(x)\,\mathrm{A}_\mu(x)\,\mathrm{U}^{\mathrm{T}-1},$$

where U is a unitary operator and, consequently,

$$\mathrm{j}_\mu(-x)\,\mathrm{A}_\mu(-x) = \mathrm{U}^{-1}\left(\mathrm{j}_\mu(x)\,\mathrm{A}_\mu(x)\right)^{\mathrm{T}}\mathrm{U}. \tag{24.28}$$

We now use this relation to investigate the symmetry properties of the elements of the scattering matrix under time reversal.

We write the elements of the scattering matrix in the form

$$S_{i\to f} = \left(\Phi_f,\ T\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}(t)\,dt\right]\right)\Phi_i\right)$$

where

$$\mathrm{V}(t) = -\int \mathrm{j}_\mu(x)\,\mathrm{A}_\mu(x)\,dr,$$

and make the substitution $t \to -t$. Since in accordance with (24.28)

$$\mathrm{V}(-t) = \mathrm{U}^{-1}\mathrm{V}^{\mathrm{T}}(t)\,\mathrm{U},$$

we have

$$\left(\Phi_f,\ T\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}(t)\,dt\right]\right)\Phi_i\right)$$

$$= \left(\Phi_f,\ \bar{T}\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}(-t)\,dt\right]\right)\Phi_i\right)$$

$$= \left(\Phi_f,\ \mathrm{U}^{-1}\bar{T}\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}^{\mathrm{T}}(t)\,dt\right]\right)\mathrm{U}\Phi_i\right)$$

where $\bar{T}$ is an antichronological operator which arranges the field operators from right to left in the order not of increasing, but of decreasing time.

The last expression can evidently be rewritten in the form

$$\left(\Phi_f,\ \mathrm{U}^{-1}\bar{T}\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}^{\mathrm{T}}(t)\,dt\right]\right)\mathrm{U}\Phi_i\right)$$

$$=\left(\mathrm{U}\Phi_f,\ \bar{T}\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}^{\mathrm{T}}(t)\,dt\right]\right)\mathrm{U}\Phi_i\right)$$

$$=\left(\mathrm{U}\Phi_f,\ \left(T\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}(t)\,dt\right]\right)^{\mathrm{T}}\mathrm{U}\Phi_i\right)\cdot$$

Finally, on going over from the transposed scattering matrix to the ordinary one, and in the course of this interchanging the initial and the final states we obtain

$$\left(\Phi_f,\ T\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}(t)\,dt\right]\right)\Phi_i\right)$$

$$=\left((\mathrm{U}\Phi_i)^*,\ T\left(\exp\left[-i\int_{-\infty}^{\infty}\mathrm{V}(t)\,dt\right]\right)(\mathrm{U}\Phi_f)^*\right)\cdot \qquad (24.29)$$

This is the relation which expresses the symmetry property of the scattering matrix under time reversal. It can be formulated as follows: the element of the scattering matrix between certain states $\Phi_i$ and $\Phi_f$ is equal to the element of the scattering matrix between the time-reversed states

$$\Phi_i' = (\mathrm{U}\Phi_f)^* \quad \text{and} \quad \Phi_f' = (\mathrm{U}\Phi_i)^*,$$

i.e.,

$$(\Phi_f,\ \mathrm{S}\Phi_i) = (\Phi_f',\ \mathrm{S}\Phi_i'). \qquad (24.30)$$

We investigate the physical meaning of the time-reversed states. In order to do this we consider a Hermitian operator $\mathrm{F}\{\varphi(x)\}$ constructed from the components of the fields $\varphi(x)$ hand aving the form

$$\mathrm{F}\{\varphi(x)\} = \int\sum_P A_P\varphi_{i_1}(x)\varphi_{i_2}(x)\ldots\varphi_{i_n}(x)\,d\mathbf{r},$$

where each of the field operators $\varphi_i$ belongs to one of the classes $+1$, $-1$, $+\varepsilon$, $-\varepsilon$, while the summation is taken over all the permutations $P=\begin{pmatrix}1, 2, \ldots\\ i_1, i_2, \ldots\end{pmatrix}$ and, finally, the constants $A_P$ are chosen in such a way that the summation appearing under the integral sign is symmetric in the boson, and antisymmetric in the fermion operators, i.e., it does not change sign on interchanging any pair of the former, and reverses its sign on interchanging any pair of the latter operators.

We now investigate how $\mathrm{F}\{\varphi(x)\}$ behaves under the transformation of inversion $x \to -x$. By utilizing the notation

$$\varphi'(x) \equiv \mathrm{O}\varphi(-x),$$

where O is the matrix of the transformation of $\varphi(x)$ defined by formulas (21.28) for the field components belonging to the classes $\pm 1$, $\pm\varepsilon$, we obtain

$$\mathrm{F}\{\varphi'(x)\} = \int \sum_P A_P \varphi'_{i_1}(x)\varphi'_{i_2}(x) \dots \varphi'_{i_n}(x)\,dr.$$

But in accordance with (21.27)

$$\varphi'(x) = \mathrm{U}^{-1}\varphi^{\mathrm{T}}(x)\mathrm{U},$$

and therefore

$$\mathrm{F}\{\varphi'(x)\} = \mathrm{U}^{-1}\Big(\int \sum_P A_P \varphi^{\mathrm{T}}_{i_1}(x)\varphi^{\mathrm{T}}_{i_2}(x) \dots \varphi^{\mathrm{T}}_{i_n}(x)\,dr\Big)\mathrm{U}$$

$$= \mathrm{U}^{-1}\int\Big(\sum_P A_P \varphi_{i_n}(x) \dots \varphi_{i_2}(x)\varphi_{i_1}(x)\Big)^{\mathrm{T}} dr\,\mathrm{U},$$

but since

$$\sum_P A_P \varphi_{i_n}(x) \dots \varphi_{i_2}(x)\varphi_{i_1}(x) = \pm \sum_P A_P \varphi_{i_1}(x)\varphi_{i_2}(x) \dots \varphi_{i_n}(x),$$

where the + or the − sign is chosen in accordance with whether F contains an even or an odd number of pairs of fermion operators, we have

$$\mathrm{F}\{\varphi'(x)\} = \pm\,\mathrm{U}^{-1}\big(\mathrm{F}\{\varphi(x)\}\big)^{\mathrm{T}}\mathrm{U}. \tag{24.31}$$

We now consider a state $\Phi$ in which F has the value $\lambda$:

$$\mathrm{F}\{\varphi(x)\}\,\Phi = \lambda\Phi,$$

and determine the result of operating with $\mathrm{F}\{\varphi(x)\}$ on the time-reversed state vector $(\mathrm{U}\Phi)^*$. Since F has been assumed to be a Hermitian operator we have

$$\mathrm{F}\{\varphi(x)\}\,(\mathrm{U}\Phi)^* = \big(\mathrm{F}^*\{\varphi(x)\}\,\mathrm{U}\Phi\big)^* = \big(\mathrm{F}^{\mathrm{T}}\{\varphi(x)\}\,\mathrm{U}\Phi\big)^*$$

$$= \big(\mathrm{U}\mathrm{U}^{-1}\mathrm{F}^{\mathrm{T}}\{\varphi(x)\}\,\mathrm{U}\Phi\big)^*.$$

On substituting (24.31) into this expression we obtain

$$\mathrm{F}\{\varphi(x)\}\,(\mathrm{U}\Phi)^* = \pm\big(\mathrm{UF}\{\varphi'(x)\}\Phi\big)^*.$$

We must distinguish two cases: when the integrand in $\mathrm{F}\{\varphi(x)\}$ is a tensor quantity belonging to the class $-1$, and when it is a tensor quantity belonging to the class $+1$. We begin by examining the former case. If, for the sake of simplicity, we assume that $\mathrm{F}\{\varphi(x)\}$ contains the two operators $\varphi_1^{+\varepsilon}$ and $\varphi_2^{-\varepsilon}$, we obtain, in accordance with (21.28),

$$\varphi_1^{+\varepsilon'}(x)\varphi_2^{-\varepsilon'}(x) = \varphi_1^{+\varepsilon}(x)\varphi_2^{-\varepsilon}(x),$$

i.e.,

$$\mathrm{F}\{\varphi'(x)\} = \mathrm{F}\{\varphi(x)\},$$

and therefore

$$\mathrm{F}\{\varphi(x)\}\,(\mathrm{U}\Phi)^* = -\big(\mathrm{UF}\{\varphi(x)\}\Phi\big)^* = -\lambda(\mathrm{U}\Phi)^*. \qquad (24.32)$$

Thus, if $\mathrm{F}\{\varphi(x)\}$ belongs to the class $-1$, and the state vector $\Phi$ is an eigenfunction of $\mathrm{F}\{\varphi(x)\}$ belonging to the eigenvalue $\lambda$, then the time-reversed state vector $(\mathrm{U}\Phi)^*$ will also be an eigenfunction of the operator $\mathrm{F}\{\varphi(x)\}$ belonging to the eigenvalue $-\lambda$.

In particular, if in a state $\Phi$ the charge of the field is equal to $q$, and the component of the spin is equal to $\mu$, then in the time-reversed state $(\mathrm{U}\Phi)^*$ the values of these two quantities will be $-q$ and $-\mu$.

In a similar manner it can be easily shown that if $\mathrm{F}\{\varphi(x)\}$ is a tensor quantity belonging to the class $+1$, and the state vector $\Phi$ is an eigenfunction of the operator $\mathrm{F}\{\varphi(x)\}$ belonging to the eigenvalue $\lambda$, then the time-reversed state vector $(\mathrm{U}\Phi)^*$ will also be an eigenfunction of the operator $\mathrm{F}\{\varphi(x)\}$ belonging to the same eigenvalue $\lambda$.

In particular, if in state $\Phi$ the energy-momentum vector is equal to $p(\mathbf{p}, i\varepsilon)$, then this quantity will have the same value in the time-reversed state $(\mathrm{U}\Phi)^*$.

Thus, if

$$\mathrm{F}\{\varphi(x)\}\Phi = \lambda\Phi,$$

then

$$\mathrm{F}\{\varphi(x)\}\,(\mathrm{U}\Phi)^* = \begin{cases} \lambda(\mathrm{U}\Phi)^*, & (\mathrm{F} \text{ belongs to the class } +1), \\ -\lambda(\mathrm{U}\Phi)^*, & (\mathrm{F} \text{ belongs to the class } -1). \end{cases}$$

By utilizing these expressions we can write formula (24.30) which expresses the symmetry property of the scattering matrix under time reversal in the form

$$(p_i, e_i, \mu_i|\mathrm{S}|p_f, e_f, \mu_f) = (p_f, -e_f, -\mu_f|\mathrm{S}|p_i, -e_i, -\mu_i), \quad (24.33)$$

where $p_i$ and $p_f$ are the four-momenta of the particles, $e_i$ and $e_f$ are the charges of the particles and $\mu_i$, $\mu_f$ are the components of the particle spins in the initial and final states.

Thus, if we interchange the momenta, the charges, and the spin components of the particles in the initial and the final states, and at the same time reverse the signs of all the charges and of all the spin components, then the element of the scattering matrix will remain unchanged.

Since the whole theory is invariant under the transformation of charge conjugation, the elements of the scattering matrix are unaltered when the signs of all the charges are reversed:

$$(p_i, e_i, \mu_i | \mathrm{S} | p_f, e_f, \mu_f) = (p_i, -e_i, \mu_i | \mathrm{S} | p_f, -e_f, \mu_f). \quad (24.34)$$

By combining this relation with (24.33) we obtain

$$(p_i, e_i, \mu_i | \mathrm{S} | p_f, e_f, \mu_f) = (p_f, e_f, -\mu_f | \mathrm{S} | p_i, e_i, -\mu_i). \quad (24.35)$$

Finally, we note that from the invariance of the equations of quantum electrodynamics under the transformations P and T the following relations are obtained:

$$\begin{aligned} (\boldsymbol{p}_i, \varepsilon_i, e_i, \mu_i | \mathrm{S} | \boldsymbol{p}_f, \varepsilon_f, e_f, \mu_f) &= (-\boldsymbol{p}_i, \varepsilon_i, e_i, \mu_i | \mathrm{S} | -\boldsymbol{p}_f, \varepsilon_f, e_f, \mu_f), \\ & \qquad (24.36) \\ (\boldsymbol{p}_i, \varepsilon_i, e_i, \mu_i | \mathrm{S} | \boldsymbol{p}_f, \varepsilon_f, e_f, \mu_f) &= (-\boldsymbol{p}_f, \varepsilon_f, e_f, -\mu_f | \mathrm{S} | -\boldsymbol{p}_i, \varepsilon_i, e_i, -\mu_i) \end{aligned}$$

where $\boldsymbol{p}$ is the momentum and $\varepsilon$ is the energy of the particle.

## § 25. Graphical Representation of the Elements of the Scattering Matrix. The Scattering Matrix in Momentum Space

### 25.1. *Graphical Representation of Normal Products*

In the preceding section it was shown that the individual terms in the expansion of the scattering matrix in powers of the electron charge are given by integrals of mixed $T$-products, and that these $T$-products can be expanded into a sum of normal products of the field operators.

We shall represent each such normal product, and consequently every scattering process, graphically[1], viz., we shall represent the four-vectors, over which the integration in $S^{(n)}$ is carried out, by points in a diagram, and we shall represent the operators by lines passing through these points—the *vertices* of the diagram.

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[1] The graphical representation of the elements of the scattering matrix was introduced by R. Feynman (62).

The representation of operators is carried out in the following manner. We shall call those operators *free* which remain unpaired in products of the type (24.15) and (24.15′); and we shall agree to represent a free operator $A_\mu(x)$ by an (undirected) dotted line leading from the vertex $x$ beyond the boundaries of the diagram (and to "infinity"), to represent a free operator $\overline{\psi}(x)$ by a solid line directed from the vertex $x$ towards "infinity", and, finally, to represent a free operator $\psi(x)$ by a solid line directed from "infinity" towards the point $x$.

Since $\overline{\psi}(x)$ represents an electron annihilation or a positron creation operator, the solid line leading from "infinity" to the vertex $x$ can be used to represent graphically either an electron which existed prior to the scattering process, or a positron created during the scattering process.

Similarly, since $\overline{\psi}(x)$ is a positron annihilation or an electron creation operator, the solid line leading from the vertex $x$ towards "infinity" can be used to represent either an electron formed as a result of scattering, or a positron which existed prior to scattering.

Finally, since $A_\mu(x)$ is a photon annihilation or creation operator, a dotted line joining the vertex $x$ to "infinity" can be used to represent a photon created or annihilated as a result of scattering.

We shall also use a dotted line to represent an external electromagnetic field acting at the point $x$.

Pairings between operators are represented by internal lines of the diagram connecting its vertices, viz., a pairing between the photon operators $A_\mu(x)$ and $A_\nu(y)$ is represented by an (undirected) dotted line connecting the vertices $x$ and $y$, while a pairing between the electron operators $\psi(x)$ and $\overline{\psi}(y)$ is represented by a solid line joining the vertices $x$ and $y$ and directed from $y$ towards $x$.

Since pairings represent products of creation and annihilation operators averaged in a certain way, the internal lines of a diagram connecting its vertices can be made the counterparts of virtual "particles" created and annihilated in the scattering process.

These "particles" are characterized, as we shall see later, by the fact that for them there exists no definite relation between the time and the spatial components of the momentum: the momentum of a virtual electron, in contrast to that of a real electron, does not satisfy the condition $p^2+m^2 = 0$, while the momentum of a virtual photon, in contrast to that of a real photon, does not satisfy the relation $k^2 = 0$.

An internal electron line is directed from the point of creation of a virtual electron towards the point of its annihilation (or from the point of annihilation of a virtual positron towards the point of its creation).

Since in the expression for $S^{(n)}$ each point $x$ appears as the argument of three operators—one photon and two electron operators— it follows that one photon and two electron lines pass through each vertex of the diagram.

Diagrams representing individual normal products appearing in the matrix $S^{(n)}$ contain $n$ vertices. We shall call them diagrams of the $n$th order. They can be used to represent the effects of the $n$th order approximation of perturbation theory.

We shall give several examples.

### 25.2. *Various Interaction Processes between Fields*

We begin with first order effects. In this case, obviously, there exists only the one diagram shown in Fig. 12. It represents the scattering of an electron or a positron by an external field, the creation or annihilation of a photon by an electron (positron), and also the creation or annihilation of an electron-positron pair. (On the right hand side of the diagram we have symbolically shown the integrand of the matrix element of $S^{(1)}$ omitting the symbol for the normal product.)

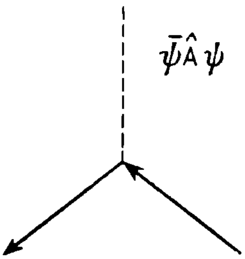


Fig. 12.

We now consider second order effects. In this case it is possible to have only the six topologically different diagrams shown in Fig. 13.

Beside the diagrams we have symbolically indicated the integrand of the matrix element of $S^{(2)}$ (omitting the normal product symbol). The lines connecting the different factors $\bar{\psi}, \psi$, and A indicate the pairings between the operators.

Fig. 14 shows the 15 diagrams giving all possible third order effects. Beside each diagram we have symbolically indicated the integrand of the matrix element of $S^{(3)}$ and the pairings between the operators.

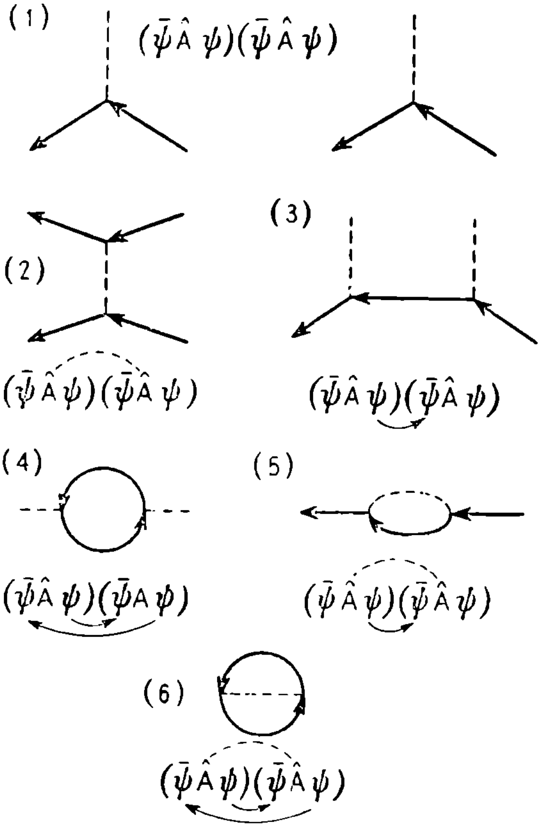
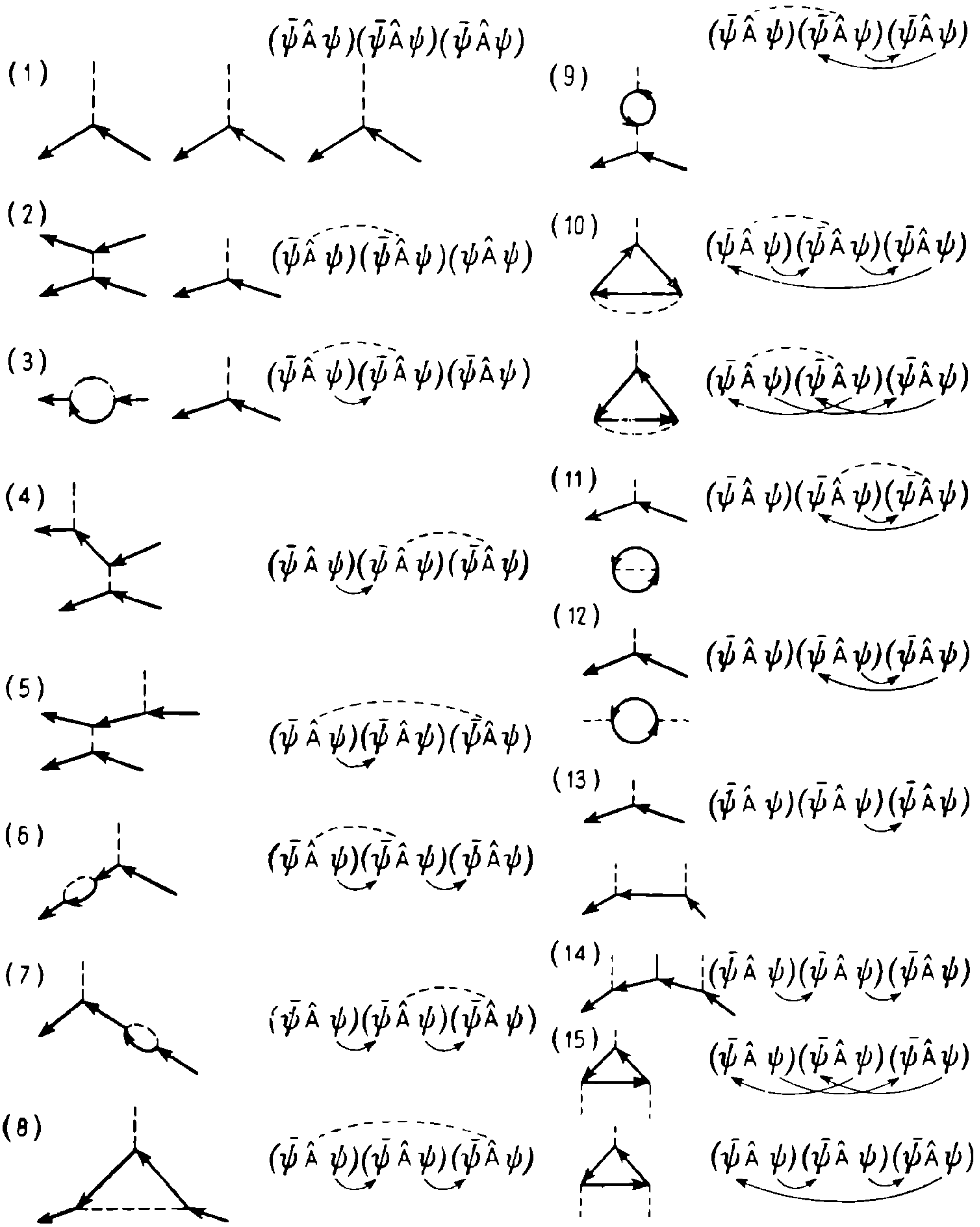

(1)
$(\bar{\psi}\hat{A}\psi)(\bar{\psi}\hat{A}\psi)$
(2)
$(\bar{\psi}\hat{A}\psi)(\bar{\psi}\hat{A}\psi)$
(3)
$(\bar{\psi}\hat{A}\psi)(\bar{\psi}\hat{A}\psi)$
(4)
$(\bar{\psi}\hat{A}\psi)(\bar{\psi}\hat{A}\psi)$
(5)
$(\bar{\psi}\hat{A}\psi)(\bar{\psi}\hat{A}\psi)$
(6)
$(\bar{\psi}\hat{A}\psi)(\bar{\psi}\hat{A}\psi)$

Fig. 13.

We examine in greater detail diagram 8 of Fig. 14. If the external lines in this diagram represent an electron and an external electromagnetic field, then this diagram can be used to represent the following process: an electron emits a virtual photon (vertex $x_1$), then the electron undergoes scattering by the external field $A^e(x_3)$ (the vertex $x_3$), and, finally, the electron absorbs the previously emitted virtual photon (vertex $x_2$).


(1)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(2)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(3)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(4)
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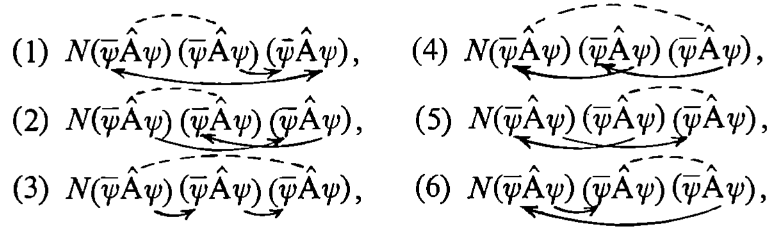
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(7)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(8)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(9)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(10)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(11)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(12)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(13)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(14)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(15)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)
(ψ̄Âψ)(ψ̄Âψ)(ψ̄Âψ)

Fig. 14.

In the expansion of $S^{(3)}$ into normal products this process corresponds to the six terms

$$-\frac{e^3}{3!}\int d^4x_1\int d^4x_2\int d^4x_3\sum_P N\big(\overline{\psi}^a(x_i)\hat{A}^b(x_i)\psi(x_i)$$
$$\times\overline{\psi}(x_j)\hat{A}^b(x_j)\psi^d(x_j)\overline{\psi}^d(x_k)\hat{A}(x_k)\psi^a(x_k)\big),$$

where the subscripts $i, j, k$, assume the values 1, 2, 3, with $i \neq j$, $i \neq k$, $k \neq j$, and the summation is taken over all the permutations of these subscripts. On omitting the integral sign, and on suppressing the arguments of the operators, we can rewrite these terms in the form



$$(1)\ N(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi),\qquad (4)\ N(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi),$$
$$(2)\ N(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi),\qquad (5)\ N(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi),$$
$$(3)\ N(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi),\qquad (6)\ N(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi)(\overline{\psi}\hat{A}\psi),$$

and, in order to be explicit we have indicated both in these expressions and in the diagram the pairings between electron operators by solid lines with arrows directed from $\psi$ toward $\overline{\psi}$, and the pairings between photon operators by dotted lines.

On noting that we can interchange the electron operators in an $N$-product, provided we multiply the $N$-product by the parity of the permutation performed, and on utilizing the formulas which define pairings between operators, we can easily show that all these six terms are equal to one another.

We can say that the different normal products corresponding to a definite $n$th order scattering process, which are analogous to those just discussed, and which differ among themselves only by rearrangements of subscripts specifying the points, are *equivalent*, since they are equal to one another. The number of such equivalent normal products is evidently equal to $z = n!/g$ where $g$ is the number of permutations of subscripts of points which leave the form of the $N$-product unaltered.

If the free operators describe different particle states, then $g = 1$ and $z = n!$. In this most important case we can in evaluating the matrix element consider only one of the equivalent $N$-products, and disregard in $S^{(n)}$ the factorial term $1/n!$ which cancels against the number of equivalent $N$-products.

The diagrams shown in Figs. 13 and 14 represent normal products of field operators in general form, and describe a number of processes simultaneously. These operators are sums of creation and annihilation operators for particles in different states; therefore the normal product corresponding to some specific physical process can also be represented in the form of a sum of several terms which contain products of creation and annihilation operators for particles taking part in the process under consideration, and which differ from one another only by the order of the arrangement of the operators. These terms can be represented by diagrams which are topologically equivalent, and which differ from one another only by the order of the arrangement of the electron and photon lines.

For example, we consider the emission of a photon by an electron in an external field. This is a second order process, and the normal product corresponding to it in the integrand of $S^{(2)}$ is

$$N\left(\overline{\psi}(x_1)\,\hat{\mathrm{A}}(x_1)\,\overrightarrow{\psi(x_1)\overline{\psi}}(x_2)\,\hat{\mathrm{A}}(x_2)\,\psi(x_2)\right).$$

In place of $\hat{\mathrm{A}}(x)$ we must substitute here

$$\hat{\mathrm{A}}(x)=\hat{\mathrm{A}}^{e}(x)+\hat{\mathrm{A}}_{k}(x),$$

where $A^e$ is the external field, and $\mathrm{A}_k$ is the operator for the emission of the photon $k$. The normal product can be decomposed into two terms:

$$N\left(\overline{\psi}(x_1)\hat{A}^{e}(x_1)\,\overrightarrow{\psi(x_1)\overline{\psi}}(x_2)\,\hat{\mathrm{A}}_{k}(x_2)\,\psi(x_2)\right)$$

$$+N\left(\overline{\psi}(x_1)\,\hat{\mathrm{A}}_{k}(x_1)\,\overrightarrow{\psi(x_1)\overline{\psi}}(x_2)\,\hat{A}^{e}(x_2)\,\psi(x_2)\right),$$

which correspond to the two diagrams shown in Fig. 15, and which differ from one another only by the order of arrangement of the photon lines representing $A^{(e)}$ and $\mathrm{A}_k$.

If $n$ photons take part in the process, then after the normal product is decomposed into terms containing annihilation and creation operators for individual photons, we obtain $n!$ terms which correspond to diagrams which differ from one another only by the order of arrangement of the photon lines.

Similarly, if several electrons and positrons take part in the process, then the normal product can be represented in the form of a sum of terms which contain the same electron and positron creation and annihilation operators, and which differ only by the order of arrangement of these operators, while the diagrams corresponding to these terms differ from one another only by the order of arrangement of the electron lines.

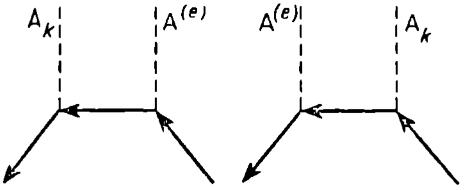
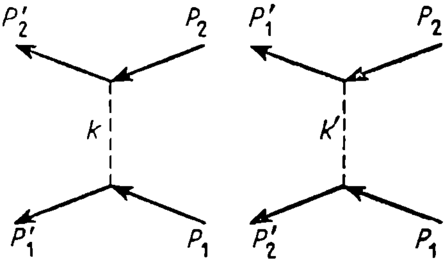


Fig. 15.



(a) (b)

Fig. 16.

For example, the process of electron-electron scattering corresponds to the two diagrams shown in Fig. 16; in these diagrams $p_1, p_2$ and $p_1', p_2'$ denote the electron momenta before and after scattering, while $k$ and $k'$ are the momenta of the virtual photons which are exchanged by the two electrons.

In contrast to processes in which several photons take part and for which the individual diagrams correspond to matrix elements having the same sign, in the case of processes in which several electrons take part, the individual diagrams correspond to matrix elements which can have opposite signs. This is associated with the fact that the creation and annihilation operators for electrons and positrons in different states do not commute, in contrast to the commuting photon creation and annihilation operators.

For example, the two diagrams of Fig. 16 correspond to matrix elements of opposite signs. Indeed, the integrand of that part of the $S^{(2)}$ matrix which describes electron-electron scattering is equal to

$$N\left(\overline{\psi}(x_1)\overbrace{\hat{A}(x_1)\psi(x_1)\overline{\psi}(x_2)\hat{A}}(x_2)\psi(x_2)\right).$$

On substituting into this expression in place of $\overline{\psi}$ and $\psi$,

$$\overline{\psi}(x) = a^{+}_{p_1'}\overline{\psi}_{p_1'}(x) + a^{+}_{p_2'}\overline{\psi}_{p_2'}(x)\,,$$
$$\psi(x) = a_{p_1}\psi_{p_1}(x) + a_{p_2}\psi_{p_2}(x),$$

where $a_p^{+}$ and $a_p$ are the creation and annihilation operators for an electron of momentum $p$, and $\psi_p(x)$ is the wave function of an electron of momentum $p$, we obtain four nonvanishing terms

$$\begin{aligned} &N(a^{+}_{p_1'}a_{p_1}a^{+}_{p_2'}a_{p_2'})\,\overline{\psi}_{p_1'}(x_1)\psi_{p_1}(x_1)\overline{\psi}_{p_2'}(x_2)\psi_{p_2}(x_2) \\ &\quad + N(a^{+}_{p_2'}a_{p_2}a^{+}_{p_1'}a_{p_1})\,\overline{\psi}_{p_1'}(x_2)\psi_{p_1}(x_2)\overline{\psi}_{p_2'}(x_1)\psi_{p_2}(x_1) \\ &\quad + N(a^{+}_{p_1'}a_{p_2}a^{+}_{p_2'}a_{p_1})\,\overline{\psi}_{p_1'}(x_1)\psi_{p_2}(x_1)\overline{\psi}_{p_2'}(x_2)\psi_{p_1}(x_2) \\ &\qquad + N(a^{+}_{p_2'}a_{p_1}a^{+}_{p_1'}a_{p_2})\,\overline{\psi}_{p_2'}(x_1)\psi_{p_1}(x_1)\,\overline{\psi}_{p_1'}(x_2)\psi_{p_2}(x_2). \end{aligned}$$

The first two terms are two equivalent normal products which are represented by diagram (a) of Fig. 16, while the second two terms are two equivalent normal products which correspond to Fig. 16(*b*).

On interchanging in the third term the operators a and $a^{+}$ it can be easily shown that

$$a^{+}_{p_1'}a_{p_2}a^{+}_{p_2'}a_{p_1} = -a^{+}_{p_1'}a_{p_1}a^{+}_{p_2'}a_{p_2}\,,$$

and, therefore, as was stated previously, the matrix elements corresponding to the two diagrams have opposite signs.

This result can be generalized in the following manner. Suppose that $n$ electrons take part in the process and that their momenta before and after scattering are respectively equal to $p_1, p_2, \ldots$ and $p_1', p_2', \ldots$; then the process will correspond to $n!$ diagrams which differ by the designation of the electron lines after scattering. If in two such diagrams the electron lines after scattering (which are the continuation of the electron lines before scattering) are denoted respectively by $p_1', p_2', \ldots$ and $p_{i_1}', p_{i_2}', \ldots$, then the relative sign of the matrix elements corresponding to the two diagrams will be determined by the parity of the permutation

$$\delta_p = \begin{pmatrix} 1, 2, \ldots, n \\ i_1, i_2, \ldots, i_n \end{pmatrix}.$$

Thus, the matrix element corresponding to some physical process can always be represented after the decomposition of the normal product in the form

$$(f|S^{(n)}|i) \equiv S^{(n)}_{i\to f} = \Sigma\, M^{(n)}_{i\to f}, \tag{25.1}$$

where the individual terms $M^{(n)}_{i\to f}$ differ from one another by the order of the arrangement of the annihilation and creation operators for the particles participating in the process; the diagrams representing these terms are topologically equivalent, and differ only in the order of the arrangement of the electron and photon lines.

### 25.3. *Transition to Momentum Space*

For the evaluation of the probabilities of different processes with the aid of the scattering matrix, and also for a general investigation of its singularities, it is convenient to go over to momentum space.

We now determine the form which the quantities $M^{(n)}_{i\to f}$ have in momentum space. In order to do this, we express the pairings between operators, and also the external potentials $A^{(e)}_{\mu}(x)$ in the form of Fourier integrals, and we substitute them, together with the expressions for the matrix elements of the particle creation and annihilation operators, into the normal product (in the expansion of the integrand of $S^{(n)}$) which corresponds to $M^{(n)}_{i\to f}$.

The pairings between the operators are given in accordance with (17.21) and (19.18) by

$$\begin{aligned}
D^c_{\mu\nu}(x) &= \frac{1}{(2\pi)^4}\int D^c_{\mu\nu}(k)\, e^{ikx}\, d^4k,\\
D^c_{\mu\nu}(k) &= \frac{1}{i}\,\frac{1}{k^2-i0}\left(\delta_{\mu\nu}+(d_l-1)\frac{k_\mu k_\nu}{k^2}\right),\\
S^c_{\alpha\beta}(x) &= \frac{1}{(2\pi)^4}\int S^c_{\alpha\beta}(p)\, e^{ipx}\, d^4p,\\
S^c_{\alpha\beta} &= i\,\frac{(i\hat{p}-m)_{\alpha\beta}}{p^2+m^2-i0},
\end{aligned} \tag{25.2}$$

where $d_l(k^2)$ is an arbitrary function of $k^2$.

We write the Fourier expansion of the external potential $A_\mu^{(e)}(x)$ treated as a $c$-number in the form

$$A_\mu^{(e)}(x) = \frac{1}{(2\pi)^4}\int A_\mu^{(e)}(q)e^{iqx}d^4q, \quad A_\mu^{(e)}(q) = \int A_\mu^{(e)}(x)e^{-iqx}d^4x. \tag{25.3}$$

Finally, the matrix elements of the field operators are defined by the following formulas. The matrix elements of the operators $\psi(x)$ and $\bar{\psi}(x)$ corresponding to the annihilation and creation of an electron or a positron of momentum $p$ and polarization $r$ are respectively given by (cf. (24.9), (24.10))

$$\begin{aligned}
\left(0_{pr}^+ | \psi(x) | 1_{pr}^+\right) &= \psi_{pr}^{(+)}(x) = u^r(p)e^{ipx},\\
\left(1_{pr}^+ | \bar{\psi}(x) | 0_{pr}^+\right) &= \bar{\psi}_{pr}^{(+)}(x) = \bar{u}^r(p)e^{-ipx},\\
\left(0_{pr}^- | \bar{\psi}(x) | 1_{pr}^-\right) &= \bar{\psi}_{pr}^{(-)}(x) = \bar{v}^r(-p)e^{ipx},\\
\left(1_{pr}^- | \psi(x) | 0_{pr}^-\right) &= \psi_{pr}^{(-)}(x) = v^r(-p)e^{-ipx},
\end{aligned} \tag{25.4}$$

where $u_\alpha^r(p)$ and $v_\alpha^r(-p)$ are spinors satisfying the following normalization conditions $u^r u^{r*} = v^r v^{r*} = 1$.

The matrix elements of the operator $A_\mu(x)$ corresponding to the annihilation and the creation of a photon of momentum $k$ and polarization $\lambda$ are respectively given by

$$\begin{aligned}
\left(0_{k\lambda} | A_\mu(x) | 1_{k\lambda}\right) &= \frac{1}{\sqrt{2\omega}} e_\mu^{(\lambda)} e^{ikx},\\
\left(1_{k\lambda} | A_\mu(x) | 0_{k\lambda}\right) &= \frac{1}{\sqrt{2\omega}} e_\mu^{(\lambda)} e^{-ikx}
\end{aligned} \tag{25.5}$$

(in all these formulas for the sake of simplicity we take the normalization volume equal to unity).

On substituting these expressions into the normal product corresponding to $M_{i\to f}^{(n)}$ (in the integrand of the expression for $S^{(n)}$) we first of all carry out the integration over $x_1, x_2, \ldots, x_n$. On collecting the factors $\exp(ipx_j)$ with a definite $x_j$ ($p$ denotes here the four-momenta of the free particles, and also the variables of integration in the Fourier expansions of the functions $A_\mu^{(e)}(x)$, $S^c(x)$ and $D^c(x)$), we obtain

$\exp(ix_j \sum p)$, where the number of vectors in the sum $\sum p$ is obviously equal to three (the number of lines passing through the vertex $x_j$ of the diagram). The integral of $\exp(ix_j \sum p)$ is equal to $(2\pi^4)\,\delta(\sum p)$, where $\delta(p)$ is a four-dimensional $\delta$-function. Therefore, on carrying out the integration over $x_1, x_2, \ldots, x_n$, we obtain the product of $n$ four-dimensional $\delta$-functions.

In momentum space, to each line of the diagram there will correspond a certain four-vector $p$. Since the functions $D^c$ and $S^c$ associated with the internal lines of the diagram depend on the difference in the coordinates of the ends of these lines, then in the two $\delta$-functions referring to some internal line the vector $p$ associated with this line will appear with opposite signs. This enables us to interpret the vectors $p$ corresponding to the internal lines of the diagram as four-momenta of the virtual "particles," "created" at one end and "annihilated" at the other end of the internal lines. In this case, obviously, there exists no relation between the time and the spatial components of the four-momentum of the virtual particle, as has been already mentioned.

Insofar as the external lines of the diagram are concerned, i.e., lines going outside the boundaries of the diagram, they obviously correspond to four-momenta of the real particles taking part in the process.

In order to obtain the final expression for $\mathrm{M}^{(n)}_{i\to f}$ it is necessary to arrange the matrices operating on the spinor indices in a definite order, carry out the integration over the momenta of the virtual particles and the variable $q$ (arising from the expansion of the external potentials into Fourier integrals), and sum the expression so obtained over the polarizations of the virtual photons.

We first of all determine the order in which the spinor matrices must be arranged. In order to do this we consider an example—the scattering of an electron by an external field, shown in Fig. 17. The normal product corresponding to this process (in the integrand of the expression for $\mathrm{S}^{(3)}$) is given by

$$F = N\Big(\overline{\psi}_\alpha(x_1)(\gamma_\mu)_{\alpha\beta}\mathrm{A}_\mu(x_1)\psi_\beta(x_1)\overline{\psi}_\lambda(x_2)(\gamma_\nu)_{\lambda\varepsilon}\mathrm{A}_\nu$$

$$(x_2)\psi_\varepsilon(x_2)\overline{\psi}_\varrho(x_3)(\gamma_\delta)_{\varrho\sigma}A^{(e)}_\delta(x_3)\psi_\sigma(x_3)\Big)\cdot$$

On interchanging $\overline{\psi}_\lambda$ and $\psi_\beta$, and then $\overline{\psi}_\lambda$ and $\overline{\psi}_a$, which will leave the sign of F unaltered, and on utilizing (17.10) and (19.16), we rewrite F in the form

$$\mathrm{F} = N\big(\overline{\psi}_\lambda(x_2)(\gamma_\nu)_{\lambda\varepsilon} S^c_{\varepsilon\varrho}(x_2-x_3)\, A^{(e)}_\delta(x_3)\,(\gamma_\delta)_{\varrho\sigma} S^c_{\sigma a}(x_3-x_1)(\gamma_\mu)_{a\beta}\,\psi_\beta(x_1)\big)$$
$$\times D^c_{\mu\nu}(x_1-x_2) = N\big(\overline{\psi}(x_2)\gamma_\nu S^c(x_2-x_3)\,\hat{\mathrm{A}}^{(e)}(x_3)$$
$$\times S^c(x_3-x_1)\gamma_\mu\psi(x_1)\big)\, D^c_{\mu\nu}(x_1-x_2)$$

(the interchange of $\overline{\psi}_a$ and $\psi_\beta$ leads to additional multiplication by $-1$). This expression shows that the internal photon line corresponds to the function $D^c_{\mu\nu}$ and to the two matrices $\gamma_\nu$ and $\gamma_\mu$ which must be taken to correspond to its ends (the vertices $x_2$ and $x_1$). In this case the matrices operating on the spinor indices, i.e., the matrices $\gamma$ and $S^c$, must be arranged in the same order read from left to right as that in which they occur if we move in the direction opposite to the electron line (cf. Fig. 17).

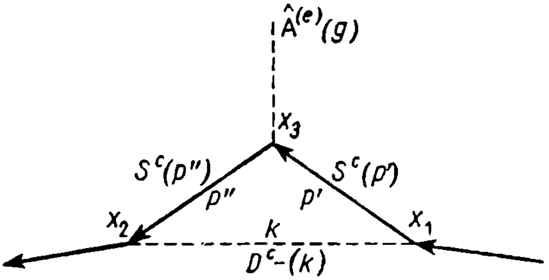


Fig. 17.

Having determined the order in which the spinor matrices must be arranged we can now write the following general expression for $M^{(n)}_{i\to f}$ in the form of an integral in momentum space:

$$M^{(n)}_{i\to f} = \delta_P(-1)^{n+l} e^n (2\pi)^{4(n-F)} \int d^4p_1 d^4p_2 \ldots d^4p_{F_e} \int d^4k_1 d^4k_2 \ldots$$
$$\ldots d^4k_{F_p} \int d^4q_1 d^4q_2 \ldots d^4q_s$$
$$\times \sum_\nu \prod_1^n \delta\Big(\sum p\Big)\, O\Big\{\prod\Big(u(p_i)\overline{v}(-p_i)\frac{\hat{e}_i}{\sqrt{2\omega_i}}\Big)\prod\Big(\overline{u}(p_f)v(-p_f)\frac{\hat{e}_f}{\sqrt{2\omega_f}}\Big)$$
$$\times \prod_1^s \frac{\hat{A}^{(e)}(q)}{(2\pi)^4} \prod_1^{F_e} i\frac{i\hat{p}-m}{p^2+m^2} \prod_1^{F_p} \gamma_\nu \frac{1}{ik^2}\Big(\delta_{\mu\nu}+(d_e-1)\frac{k_\mu k_\nu}{k^2}\Big)\gamma_\mu\Big\}. \qquad (25.6)$$

Here the integration is taken over the four-momenta of the virtual particles, i.e., over the $4F_e$ variables $p_1, p_2, \ldots, p_{F_e}$ arising from factors of the type $S^c$, and over the $4F_p$ variables $k_1, k_2, \ldots, k_{F_p}$ arising from factors of the type $D^c$, and also over the $4s$ variables $q_1, q_2, \ldots, q_s$ arising from the expansion of the external potentials into Fourier integrals ($F_e$ and $F_p$ are the numbers of the internal electron and photon lines, i.e., the numbers of virtual electrons and photons, $F = F_e + F_p$, $s$ is the number of vertices at which external potentials exist, $l$ is the number of closed electron loops with an even number of electron lines, $\delta_p$ is the parity of the permutation of the electron operators, cf. below).

In addition to integration a summation is carried out in (25.6) over the four values of the subscripts $\nu$ which denote the different polarizations of the virtual photons, with each virtual photon corresponding to its own subscript $\nu$, which assumes the values $\nu = 1, 2, 3, 4$.

The individual factors in (25.6) have the following meaning. The product $\prod\left(u(p_i)\bar{v}(-p_i)\,[\hat{e}_i/\sqrt{(2\omega_i)}]\right)$ is a product of the spinors $u(p_i)$, $\bar{v}(-p_i)$ and the matrices $\hat{e}_i/\sqrt{2\omega_i}$ describing electrons, positrons and photons in the initial state $i$ ($p_i$ is the three-dimensional momentum of an electron or a positron, $e_i$ is the unit polarization vector of a photon of momentum $k_i$, $\hat{e} = e_\mu \gamma_\mu$, $e_\mu e_\mu = 1$); $\prod\left(\bar{u}(p_f)v(-p_f)[\hat{e}_f/\sqrt{2\omega_f}]\right)$ denotes a similar product for the final state $f$.

The factors $i(i\hat{p} - m/p^2 + m^2)$ and $\gamma_\nu(1/ik^2)\ (\delta_{\mu\nu} + (d_l - 1)[k_\mu k_\nu/k^2])\,\gamma_\mu$ arise from the pairings of electron and photon operators with the number of the former being equal to $F_e$, and of the latter being equal to $F_p$.

Finally, the symbol $O$ denotes a definite order in the arrangement of the spinor matrices in (25.6), viz., the matrices $\gamma$ and $S^c$ operating on the spinor indices must be arranged reading from right to left in the same sequence in which they occur if we move in the direction of an electron line of the diagram.

A special feature is present in the case when the diagram contains closed electron loops with an even number of electron lines (in the case of an odd number of electron lines the matrix element vanishes in accordance with Furry's theorem, cf. below). In this case to each electron loop there corresponds in the expression $M^{(n)}_{i\to f}$ the negative trace of the product of the matrices $\gamma$ and $S^c$ associated with this loop.

In order to show this we consider the part of the diagram containing the closed electron loop shown in Fig. 18 (the squares A and B denote certain complex diagrams whose explicit form is not needed for this discussion). The factor associated with this loop in the general expression for the normal product corresponding to Fig. 18 has the form

$$\mathrm{G} = N\big(\overline{\psi}{}^a_\alpha(x_1)\,(\gamma_\mu)_{\alpha\beta}\psi^b_\beta(x_1)\,\overline{\psi}{}^b_\delta(x_2)\,(\gamma_\nu)_{\delta\varrho}\psi^a_\varrho(x_2)\big).$$

On utilizing expressions (19.16) for pairings of operators we obtain

$$G = -(\gamma_\mu)_{\alpha\beta} S^c_{\beta\delta}(x_1-x_2)\,(\gamma_\nu)_{\delta\varrho} S^c_{\varrho\alpha}(x_2-x_1)$$
$$= -\mathrm{Sp}\big(\gamma_\mu S^c(x_1-x_2)\,\gamma_\nu S^c(x_2-x_1)\big),$$

which is what was asserted earlier.

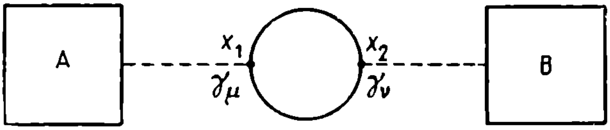


Fig. 18.

If the diagram contains $l$ closed electron loops with an even number of lines, then $M^{(n)}_{i\to f}$ acquires the factor $(-1)^l$ which appears in (25.6) together with the factor $(-1)^n\delta_p$, where $\delta_p$ determines the relative sign of $M^{(n)}_{i\to f}$ in the case when several electrons participate in the process. As has been explained earlier, $\delta_p$ represents the parity of the permutation of the electron indices

$$\delta_p = \begin{pmatrix} 1, 2, \dots \\ j_1, j_2, \dots \end{pmatrix},$$

where the numbers $1, 2, \dots$ enumerate in the diagram the momenta of the electrons before scattering, while the numbers $j_1, j_2$ do the same after scattering.

As an example of the application of the general formula (25.6) we give the expression for the matrix element which corresponds to the diagram of Fig. 17:

$$M^{(3)}_{i\to f} = i^3 e^3 \overline{u}_2 \left(\int \gamma_\nu \frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}\,\frac{\hat{A}^{(e)}(q)}{(2\pi)^4}\,\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\gamma_\nu \frac{d^4k}{k^2}\right)u_1. \tag{25.7}$$

Here $u_1$ and $\bar{u}_2$ are the spinor amplitudes of electrons of momenta $p_1$ and $p_2$ before and after scattering, and $q = p_2 - p_1$. We have eliminated the three four-dimensional $\delta$-functions appearing in the general expression for $M^{(3)}_{i\to f}$ by integrating over $q$ and over the momenta of the virtual electrons, having replaced $q$ by $p_2 - p_1$ and the momenta of the internal electron lines by $p_1 - k$ and $p_2 - k$. For this reason the integration in (25.7) is carried out only over the four-momentum of the virtual photon $k$; moreover in (25.7) the summation is carried out over the four values of the indices $\nu$, and this corresponds to summing over the four polarization states of the virtual photon. The arbitrary quantity $d_l(k^2)$ has been taken equal to $d_l = 1$.

### 25.4. *Closed Electron Loops with an Odd Number of Vertices*

We now show that diagrams containing closed internal electron loops consisting of an odd number of lines correspond to a total matrix element which vanishes (*Furry's theorem*).

If a diagram $G_1$ contains an internal closed electron loop I (Fig. 19), then in addition to this diagram we must, obviously, also consider another diagram $G_2$ which differs from the former only by the direction of circulation around the closed electron loop (loop II). The sum of the matrix elements corresponding to these diagrams, $M_{G_1}$ and $M_{G_2}$, determines the total matrix element for the process

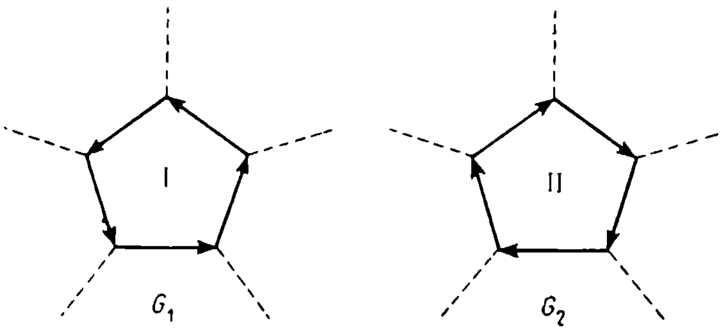


Fig. 19.

$$M = M_{G_1} + M_{G_2}.$$

In accordance with (25.6) the loops I and II correspond to the following parts of the matrix elements $M_{G_1}$ and $M_{G_2}$:

$$M_I = \int d^4p \, \mathrm{Sp}\big(\gamma_{\mu_1} S^c(p+k_1)\gamma_{\mu_2} S^c(p+k_1+k_2) \ldots S^c(p-k_N)\gamma_{\mu_N} S^c(p)\big),$$

$$M_{II} = \int d^4p' \, \mathrm{Sp}\big(S^c(p')$$
$$\times \gamma_{\mu_N} S^c(p'+k_N) \ldots S^c(p'-k_2-k_1)\gamma_{\mu_2} S^c(p'-k_1)\gamma_{\mu_1}\big),$$

which appear in $M_{G_1}$ and $M_{G_2}$ in the form of factors ($N$ is the number of vertices in the loop).

On noting that

$$\mathrm{Sp}\,\mathrm{A} = \mathrm{Sp}\,\mathrm{C}^{-1}\mathrm{AC},$$

where A and C are arbitrary matrices, we write $M_{II}$ in the form

$$M_{II} = \int d^4p' \, \mathrm{Sp}\,(\mathrm{C}^{-1} S^c(p')\,\mathrm{C}$$
$$\times \mathrm{C}^{-1}\gamma_{\mu_N}\mathrm{CC}^{-1} S^c(p'+k_N)\,\mathrm{C} \ldots \mathrm{C}^{-1} S^c(p'-k_1)\mathrm{CC}^{-1}\gamma_{\mu_1}\mathrm{C}).$$

We now choose for C the charge conjugation matrix. Then, in accordance with (10.2) we have

$$\mathrm{C}^{-1}\gamma_\mu \mathrm{C} = -\tilde{\gamma}_\mu,$$

and, consequently,

$$\mathrm{C}^{-1} S^c(p)\,\mathrm{C} = \tilde{S}^c(-p).$$

On utilizing these relations we obtain

$$M_{II} = (-1)^N \int d^4p' \, \mathrm{Sp}\big(\tilde{S}^c(-p')\,\tilde{\gamma}_{\mu_N} \tilde{S}^c(-p'-k_N) \ldots \tilde{S}^c(-p'+k_1)\,\tilde{\gamma}_{\mu_1}\big).$$

Further, on introducing the change of variable $p' = -p$, and on noting that for an arbitrary matrix A the relation $\mathrm{Sp}\,\mathrm{A} = \mathrm{Sp}\,\tilde{\mathrm{A}}$ holds, we obtain finally

$$M_{II} = (-1)^N \int d^4p \, \mathrm{Sp}\big(\gamma_{\mu_1} S^c(p+k_1)$$
$$\times \gamma_{\mu_2} S^c(p+k_1+k_2) \ldots S^c(p-k_N)\gamma_{\mu_N} S^c(p)\big).$$

Since the total matrix element $M$ contains as a factor the sum of the expressions $M_I$ and $M_{II}$, it follows from the last formula that in the case of odd $N$ the matrix element $M$ vanishes.

Thus, diagrams containing closed loops consisting of an odd number of electron lines may be left out of consideration entirely.

We note that Furry's theorem is a consequence of the law of conservation of charge parity (cf. subsection 23.7). Indeed, a closed electron loop with $N$ vertices, i.e., with $N$ photon lines approaching it, corresponds to a neutral system of real and virtual photons whose total number is equal to $N$. Since for an individual photon $\Lambda = -1$, an odd value of $N$ would mean that the initial and final states of the system would have opposite charge parity and this is impossible.

### 25.5. *Rules for Writing Down Matrix Elements*

In concluding this section we state the rules by means of which we may directly write down the matrix element corresponding to any diagram. These rules were formulated by Feynman (62) and consist of the following.

1. To each external electron line there corresponds a spinor of one of the following types: $u^r(p), \bar{u}^r(p), \bar{v}^r(-p), v^r(-p)$, where $u^r(\boldsymbol{p})$ and $\bar{u}^r(\boldsymbol{p})$ correspond to the annihilation or creation of an electron of momentum $\boldsymbol{p}$ and polarization $r$, while $\bar{v}^r(-\boldsymbol{p})$ and $v^r(-\boldsymbol{p})$ correspond to the annihilation or creation of a positron of momentum $\boldsymbol{p}$ and polarization $r$ (the four-momentum of the positron is equal to the negative of the four-momentum which defines the spinor $v$ corresponding to negative frequency).

2. To each external photon line representing a photon there corresponds the matrix $\hat{e}/\sqrt{2\omega}$, where $\omega$ is the frequency and $e_\mu$ is the unit polarization vector ($\hat{e} = e_\mu \gamma_\mu$, $e_\mu e_\mu = e^2 - e_0^2 = 1$).

To each external photon line representing an external electromagnetic field there corresponds the matrix $\hat{A}^{(e)}(q)/(2\pi)^4$.

3. To each internal electron line of momentum $p$ there corresponds the matrix $-i(1/i\hat{p}+m)$.

4. To each internal photon line of momentum $k$ there corresponds the factor $\frac{-i}{k^2}\left((\delta_{\mu\nu}+(d_l-1)\frac{k_\mu k_\nu}{k^2}\right)$, and to its ends there correspond the matrices $\gamma_\mu$ and $\gamma_\nu$.

5. To each vertex of the diagram there corresponds a $\delta$-function containing the momenta of all the lines converging at this vertex, with the momenta at the two ends of an internal line being taken of opposite sign.

6. All the matrices operating on the spinor indices are arranged reading from right towards left in the sequence in which they occur if we move in the direction of the electron line.

7. If the diagram contains a closed electron loop with an even number of electron lines, then in $M_{i\to f}^{(n)}$ there occurs as a factor the

negative trace of the product of the matrices $-i/i\hat{p}+m$ and $\gamma_\mu$, which are associated with the individual lines of the loop and with its vertices.

The matrix elements of diagrams containing closed electron loops with an odd number of electron lines vanish.

8. The numerical factor which appears in $M^{(n)}_{i\to f}$ in front of the product of the spinors $u$, $v$, $\bar{u}$, $\bar{v}$ and the matrices $\gamma$, $(-i/i\hat{p}+m)$ is equal to $(-1)^{n+l}\delta_P(2\pi)^{4(n-F)}$, where $F$ is the total number of internal lines, $l$ is the number of electron loops with an even number of electron lines, and $\delta_P$ is the character of the permutation of the indices of the electron momenta $\delta_P=\begin{pmatrix}1,2,\ldots\\ j_1,j_2,\ldots\end{pmatrix}$ (1, 2, ... enumerate the initial, and $j_1, j_2, \ldots$ enumerate the final electron momenta).

9. Integration is carried out over the four-momenta of the internal lines representing virtual particles, and over the variables $q$ associated with the external potentials, and summation is carried out over the four-dimensional polarizations of the virtual photons.

As an illustration of Feynman's rules we shall give the matrix elements corresponding to the scattering of a photon by an electron (the Compton effect), and to the conversion of an electron-positron pair into two photons (two-photon pair annihilation).

As has been explained earlier, each of these matrix elements is made up of two terms which differ in the order of operation of the photon operators. The diagrams representing these terms are shown in Fig. 20 (adjacent to the electron lines are shown the four-momenta and the spinor amplitudes for the electrons and the positrons, and adjacent to the photon lines are shown the photon four-momenta).

We see that the two processes are represented by topologically identical diagrams, and the difference consists only of the fact that the electron line of momentum $p_2$ represents in the case of the Compton effect the emission of an electron of four-momentum $p_e=p_2$, and in the case of pair annihilation the annihilation of a positron of four-momentum $p_p=-p_2$ (the corresponding spinor amplitudes are designated in the diagrams by $\bar{u}_2(p_e)$ and $v(-p_p)$).

In accordance with Feynman's rules and the general formula (25.6), the matrix elements corresponding to the scattering of a photon by an electron, and to pair annihilation, have the following form:

*For the case of photon-electron scattering*

$$S^{(2)}_{i\to f} = M^{(2)}_{\text{scat I}} + M^{(2)}_{\text{scat II}}$$

$$= ie^2(2\pi)^4\bar{u}_2\left(\frac{\hat{e}_2}{\sqrt{2\omega_2}}\frac{i(\hat{p}_1+\hat{k}_1)-m}{(p_1+k_1)^2+m^2}\frac{\hat{e}_1}{\sqrt{2\omega_1}}\right.$$

$$\left.+\frac{\hat{e}_1}{\sqrt{2\omega_1}}\frac{i(\hat{p}_1-\hat{k}_2)-m}{(p_1-k_2)^2+m^2}\frac{\hat{e}_2}{\sqrt{2\omega_2}}\right)u_1\delta(p_2-p_1+k_2-k_1). \tag{25.8}$$

*For the case of pair annihilation*

$$S^{(2)}_{i\to f} = M^{(2)}_{\text{ann I}} + M^{(2)}_{\text{ann II}}$$

$$= ie^2(2\pi)^4\bar{v}\left(\frac{\hat{e}_2}{\sqrt{2\omega_2}}\frac{i(\hat{p}_1-\hat{k}_1)-m}{(p_1-k_1)^2+m^2}\frac{\hat{e}_1}{\sqrt{2\omega_1}}\right.$$

$$\left.+\frac{\hat{e}_1}{\sqrt{2\omega_1}}\frac{i(\hat{p}_1-\hat{k}_2)-m}{(p_1-k_2)^2+m^2}\frac{\hat{e}_2}{\sqrt{2\omega_2}}\right)u\delta(p_2-p_1+k_1+k_2), \tag{25.9}$$

where $e_1$ and $e_2$ are the polarizations of the photons of momenta $k_1$ and $k_2$; the subscripts I and II denote the two diagrams corresponding to each process. We have eliminated one of the two $\delta$-functions appearing in accordance with (25.6) in $M^{(n)}_{i\to f}$ by integrating over the momentum of the internal electron line after replacing it by $p_1+k_1$ and $p_1-k_2$ in the Compton effect diagrams, and by $p_1-k_1$ and $p_1-k_2$ in the diagrams for pair annihilation.

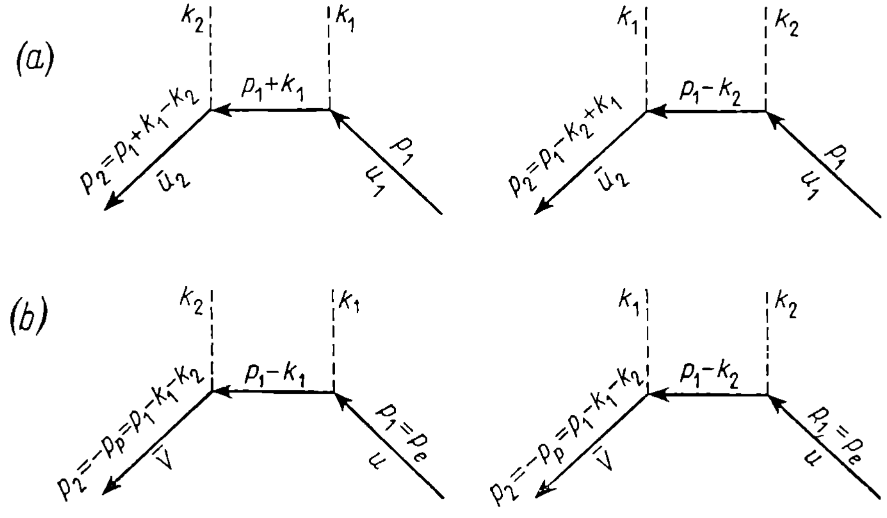


Fig. 20.

## § 26. Probabilities of Various Processes

### 26.1 *General Formula for the Probability*

Having defined in accordance with the rules of the preceding section the matrix elements corresponding to some process $i \to f$ we can obtain the probability of the process $P = |S_{i\to f}|^2$.

If only free electrons and photons participate in this process, then the matrix element obviously has the following form

$$S_{i\to f} = M_{if}\,\delta\left(\sum p_i - \sum p_f\right),$$

where $\delta\left(\sum p_i - \sum p_f\right)$ is the four-dimensional $\delta$-function containing the difference between the initial and final four-momenta of the particles. Therefore $P = |M_{if}|^2 \delta^2\left(\sum p_i - \sum p_f\right)$. We replace one of the $\delta$-functions in this expression by the integral

$$\delta(q) = \frac{1}{(2\pi)^4}\int e^{iqx} d^4x.$$

We take the region of integration here to be finite. On assuming that the spatial volume, i.e., the volume to which the particle wave functions are normalized, is equal to unity, and on denoting the interval of time variation by $T$, we obtain the following expression for $P$:

$$P = \frac{1}{(2\pi)^4}|M_{if}|^2\,\delta\left(\sum p_i - \sum p_f\right)T.$$

Thus, the probability of the transition $i \to f$ is proportional to the time. Expressed per unit time it is given by

$$W = \frac{1}{(2\pi)^4}|M_{if}|^2\,\delta\left(\sum p_i - \sum p_f\right). \tag{26.1}$$

In the processes under consideration the initial and the final states belong to the continuous spectrum, and therefore it is necessary to evaluate the probability that the three-dimensional particle momenta $\mathbf{p}_f$ in the final state lie within the ranges $d\mathbf{p}_f$. In order to obtain this differential probability, which we denote by $dw$, we must multiply $W$ by the product $\prod\left((2\pi)^{-3} d\mathbf{p}_f\right)$, where $d\mathbf{p}_f/(2\pi)^3$ is the number of states of a particle of definite polarization whose momentum $\mathbf{p}_f$ lies within the interval $d\mathbf{p}_f$ (the normalization volume is taken to be equal to unity).

Thus we have

$$dw = \frac{1}{(2\pi)^4} |M_{if}|^2 \delta\left(\sum p_i - \sum p_f\right) \prod_f \frac{d\mathbf{p}_f}{(2\pi)^3}. \tag{26.2}$$

On carrying out the integration over one of the momenta $\mathbf{p}_f$ we eliminate from $dw$ three spatial $\delta$-functions:

$$dw = \frac{1}{(2\pi)^7} |M_{if}|^2 \delta\left(\sum \varepsilon_i - \sum \varepsilon_f\right) \prod_f{}' \frac{d\mathbf{p}_f}{(2\pi)^3}. \tag{26.2'}$$

Here $\prod_f{}'$ denotes the product $\prod_f$ from which one of the factors $d\mathbf{p}_f/(2\pi)^3$ has been omitted ($\varepsilon_i$ and $\varepsilon_f$ are the energies of the particles in the initial and final states).

In order to eliminate from (26.2′) the $\delta$-function involving the energy, we represent any one of the factors appearing in $\prod_f{}'$ in the form

$$d\mathbf{p}_f/(2\pi)^3 = \varrho_\varepsilon d\varepsilon_f do,$$

where $do$ is the element of solid angle containing $\mathbf{p}_f$, and $\varrho_\varepsilon$ is the density of particle states per unit energy and per unit solid angle. On carrying out the integration over $d\varepsilon_f$ we eliminate the $\delta$-function containing the energy and obtain

$$dw = \frac{1}{(2\pi)^7} |M_{if}|^2 \varrho_f do \prod_f{}'' \frac{d\mathbf{p}_f}{(2\pi)^3}, \tag{26.3}$$

where

$$\varrho_f = \int \varrho_\varepsilon \delta\left(\sum \varepsilon_i - \sum \varepsilon_f\right) d\varepsilon_f$$

and $\prod_f{}''$ denotes the product $\prod_f{}'$ in which one of the factors $d\mathbf{p}_f/(2\pi)^3$ has been omitted.

### 26.2. *Effective Cross Section*

On dividing the differential probability $dw$ by the flux density of the colliding particles $I$ we obtain the directly observed quantity—the differential effective cross section for the process

$$d\sigma = \frac{dw}{I}. \tag{26.4}$$

The current density $I$ is most naturally defined in the laboratory coordinate system in which one of the two particles is at rest before the collision:

$$I^{(0)} = n_1^{(0)} v_1^{(0)}$$

where $n_1$ is the density, $v_1$ is the velocity of the moving particle, and the superscript zero refers to quantities in the laboratory coordinate system, $v_2^{(0)} = 0$. We note that this definition is inapplicable to the case of photon-photon scattering, when there exists no laboratory coordinate system. However, in this case we can repeat the following argument by using the center of mass system from the outset.

In the case of an arbitrarily moving scatterer the definition of current density is, generally speaking, not unique. However, we can sensibly define it in such a way that $d\sigma$ will be invariant.

In order to do this we make use of the fact that the number of scattering events occurring in the volume $\Omega$ during the time $T$ is invariant. In the laboratory system of coordinates this is given by $d\sigma^{(0)} n_1^{(0)} v_1^{(0)} n_2^{(0)} \Omega^{(0)} T^{(0)}$, while in the arbitrary system it is given by $d\sigma n_1 \tilde{v} n_2 \Omega T$, where $d\sigma$ is the cross section in this system, while $I = n_1 \tilde{v}$ is the flux density with $\tilde{v}$ being the quantity that must be suitably defined. Since $\Omega T$ is invariant, we have

$$d\sigma^{(0)} n_1^{(0)} n_2^{(0)} v_1^{(0)} = d\sigma n_1 n_2 \tilde{v} \,. \tag{26.5}$$

On the other hand, the quantities $(n_1 \boldsymbol{v}_1, in_1)$ and $(n_2 \boldsymbol{v}_2, in_2)$ are four-vectors. Therefore

$$n_1 n_2 (1 - \boldsymbol{v}_1 \boldsymbol{v}_2) = n_1^{(0)} n_2^{(0)}.$$

We see that if we define $\tilde{v}$ by

$$\tilde{v} = v_1^{(0)} (1 - \boldsymbol{v}_1 \boldsymbol{v}_2)$$

the cross section will be invariant

$$d\sigma = d\sigma^{(0)}.$$

We note that in the special case, when $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are colinear, $\boldsymbol{v}_1^{(0)} = (\boldsymbol{v}_1 - \boldsymbol{v}_2)/(1 - \boldsymbol{v}_1 \boldsymbol{v}_2)$, and it follows from (26.5) that

$$\tilde{v} = |\boldsymbol{v}_1 - \boldsymbol{v}_2| \,.$$

In the case of a collision involving a photon ($v_1 = 1$), the expression for $\tilde{v}$ assumes the form

$$\tilde{v} = 1 - v_2 \cos \vartheta,$$

where $\vartheta$ is the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

On utilizing the expression for $v_1^{(0)}$ in terms of invariant quantities

$$v_1^{(0)} = -\left((p_1 p_2)^2 - m_1^2 m_2^2\right)^{1/2} / (p_1 p_2),$$

where $p_1$, $p_2$ are the four-momenta and $m_1$, $m_2$ are the masses of the colliding particles, we can eliminate from the expression for $\tilde{v}$ any reference to the laboratory system

$$\tilde{v} = \left((p_1 p_2)^2 - m_1^2 m_2^2\right)^{1/2} / \varepsilon_1 \varepsilon_2. \tag{26.5'}$$

This formula is applicable to any case, including the case of two colliding photons.

Formulas (26.2), (26.4) and (26.5) yield the following expression for the effective differential cross section:

$$d\sigma = \frac{1}{(2\pi)^4} |M_{if}|^2 \delta(p_1 + p_2 - \sum_f p_f) \frac{\varepsilon_1 \varepsilon_2}{\left((p_1 p_2)^2 - m_1^2 m_2^2\right)^{1/2}} \prod_f \frac{d^3 p_f}{(2\pi)^3}, \tag{26.6}$$

where the subscripts 1,2 refer to the particles in the initial state, and the subscript $f$ refers to final states.

Formula (26.6) can be rewritten in such a form that $d\sigma$ is expressed only in terms of invariant quantities. In order to do this we explicitly pick out from the matrix element $M_{if}$ those factors which are not invariant. The following factors are of this kind: to each photon in the initial and final states there corresponds the normalizing factor $(2\varepsilon)^{-1/2}$, where $\varepsilon$ is the photon energy. To each electron there also corresponds a noninvariant factor associated with the noninvariance of the amplitude normalization $u^* u = \bar{u} \gamma_4 u = 1$. It corresponds to the normalization $\bar{u} u = m/\varepsilon$. Therefore, we can pick out from $M_{if}$ the factor $(m/\varepsilon)^{1/2}$ or $\varepsilon^{-1/2}$ corresponding to each electron state. Thus, we can write $M_{if}$ in the form

$$M_{if} = A \prod_k (1/\sqrt{\varepsilon_k}),$$

where the product is taken over all particles both in the initial and in the final states. The quantity $A$ is called the invariant scattering amplitude.

On substituting this expression into (26.6) and on utilizing the fact that

$$\int \frac{d^3 p_f}{\varepsilon_f} = 2 \int d^4 p_f \, \delta(p_f^2 + m_f^2) \theta(\varepsilon_f)$$

where $\theta(\varepsilon_f) = \begin{cases} 1 \text{ for } \varepsilon_f > 0 \\ 0 \text{ for } \varepsilon_f < 0 \end{cases}$, and $\theta(\varepsilon_f)$ is relativistically invariant, we obtain

$$d\sigma = \frac{|A|^2}{(2\pi)^4\left((p_1p_2)^2 - m_1^2m_2^2\right)^{1/2}} \delta(p_1+p_2-\sum_f p_f) \prod_f \frac{2d^4p_f\,\delta(p_f^2+m_f^2)\,\theta(\varepsilon_f)}{(2\pi)^3}. \tag{26.6'}$$

Formula (26.6′) gives the invariant form of the scattering cross section.

We note that the invariant form of the statistical weights of particles utilized in (26.6′) may turn out to be practically useful for integration. As an example, we consider the case of two particles in the final state with momenta $k_1$ and $k_2$. By introducing the new variables

$$q = k_1 - k_2, \qquad p = p_1 + p_2, \qquad Q = k_1 + k_2,$$

we obtain

$$d^4k_1\, d^4k_2\, \delta(k_1^2+m_1^2)\, \delta(k_2^2+m_2^2)\, \delta(Q-p)$$

$$= \tfrac{1}{4} d^4q\, \delta(q^2+k^2+2m_1^2+2m_2^2)\, \delta(qk+m_1^2-m_2^2)\, d^4Q\,\delta(Q-p) = \frac{1}{4}\sqrt{\frac{k_1}{W}}\, do$$

where $W$ is the total energy and $k_1$ is the momentum of the particle in the center of mass system, while $do$ is the element of solid angle in that system.

### 26.3. *Summation and Averaging over Polarization States of Electrons and Photons*

In those cases when we are not interested in a definite polarization state of the product particles (i.e., the orientation of electron spins and the direction of photon polarization), expression (26.2) must be summed over all possible particle polarizations in the final state. If in the initial state particles are not polarized, then (26.2) must also be averaged over the polarizations of the particles in the initial state.

We start by carrying out the summation, and by averaging over the different orientations of the electron spin.

For the sake of simplicity we discuss the case when we have only one electron in the initial and the final states. The quantity $M_{if}$ appearing in the matrix element $S^{(n)}_{i\to f}$ has in this case the form

$$M_{if} = \bar{u}_f Q u_i,$$

where $u_i$ and $\bar{u}_f$ are the spinor amplitudes for the electron in the initial and the final states, and $Q$ is a certain matrix. We are interested in the quantity

$$\overline{\sum_{\mu_f}|M_{if}|^2} = \frac{1}{2}\sum_{\mu_f}\sum_{\mu_i}(u_f^*\gamma_4 Q u_i)(u_i^* Q^+ \gamma_4 u_f),$$

where $\sum_{\mu_f}$ and $\sum_{\mu_i}$ denote summations over two orientations of the electron spin in the final and initial states. In accordance with subsection 10.7 we have

$$\frac{1}{2}\sum_{\mu_i,\mu_f}|M_{if}|^2 = \frac{1}{8\varepsilon_i\varepsilon_f}\,\mathrm{Sp}\{Q(i\hat{p}_i-m)\bar{Q}(i\hat{p}_f-m)\}, \tag{26.7}$$

where

$$\bar{Q} = \gamma_4 Q^+ \gamma_4.$$

If there is only one positron in both the initial and the final states then the following expression holds:

$$M_{if} = \bar{v}_i Q v_f,$$

where $v_f = v(-p_f)$ and $v_i = v(-p_i)$ are the spinor amplitudes corresponding to the final and the initial positron states. In accordance with subsection 10.7 we have

$$\frac{1}{2}\sum_{\mu_i,\mu_f}|\bar{v}_i Q v_f|^2 = \frac{1}{2}\sum_{\mu_i}\sum_{\mu_f}(v_i^*\gamma_4 Q v_f)(v_f^* Q^+ \gamma_4 v_i)$$
$$= \frac{1}{8\varepsilon_f\varepsilon_i}\,\mathrm{Sp}\{Q(i\hat{p}_f+m)\,\bar{Q}(i\hat{p}_i+m)\}. \tag{26.8}$$

If one of the states is an electron state while the other is a positron state, then we have

$$M_{if} = \begin{cases} \bar{u}_f Q v_f & \text{in the case of pair creation,} \\ \bar{v}_i Q u_i & \text{in the case of pair annihilation.} \end{cases}$$

Moreover, we have

$$\frac{1}{2}\sum_{\mu_f}|\bar{u}_f Q v_f|^2 = \frac{1}{8\varepsilon_e\varepsilon_p}\,\mathrm{Sp}\{Q(i\hat{p}_p+m)\bar{Q}(i\hat{p}_e-m)\}, \tag{26.9}$$

$$\frac{1}{2}\sum_{\mu_i}|\bar{v}_i Q u_i|^2 = \frac{1}{8\varepsilon_e\varepsilon_p}\,\mathrm{Sp}\{Q(i\hat{p}_e-m)\bar{Q}(i\hat{p}_p+m)\}, \tag{26.9'}$$

where $p_e$ and $p_p$ are the momenta of the electron and the positron, and $\varepsilon_e$ and $\varepsilon_p$ are the corresponding energies.

We note that formulas (26.8), (26.9) and (26.9′) can be obtained from formula (26.7): the latter goes over into (26.8) as a result of the substitution $p_i \to -p_f, p_f \to -p_i$. As a result of the substitution $p_f \to p_e, p_i \to -p_p (\varepsilon_f \to \varepsilon_e, \varepsilon_i \to -\varepsilon_p)$ it goes over into (26.9), and as a result of the substitution $p_i \to p_e, p_f \to -p_p (\varepsilon_i \to \varepsilon_e, \varepsilon_f \to -\varepsilon_p)$ it goes over into (26.9′).

In a similar manner the summation over the polarizations can also be carried out in more eomplicated cases when several electrons participate in the process. In such a case it is possible to utilize formulas (26.7), (26.8), (26.9), (26.9′), by assuming that the matrix $Q$ itself contains spinor amplitudes. Summation over the polarizations corresponding to them can be carried out by the same method by which formulas (26.7)—(26.9) were obtained.

We now proceed to the summation over the photon polarizations. For the sake of simplicity we discuss the case when only one electron and one photon participate in the process. In this case the matrices $Q$ and $\bar{Q}$ are

$$Q = \hat{e}G, \quad \bar{Q} = \overline{G\hat{e}},$$

where $G$ does not contain the photon polarization $e_\mu$. Since

$$\hat{e} = e\boldsymbol{\gamma} + ie_0\gamma_4, \quad \hat{e}^+ = e\boldsymbol{\gamma} - ie_0\gamma_4 \quad (e_\mu e_\mu = e^2 - e_0^2 = 1),$$

we have

$$\bar{\hat{e}} = \gamma_4 \hat{e}^+ \gamma_4 = -\hat{e}.$$

Therefore

$$\bar{Q} = -\bar{G}\hat{e}$$

and

$$\frac{1}{2}\sum_{\mu_i, \mu_f} |M_{if}|^2 = -\frac{1}{8\varepsilon_i\varepsilon_f}\mathrm{Sp}\{\hat{e}G(i\hat{p}_i - m)\bar{G}\hat{e}(i\hat{p}_f - m)\}. \tag{26.10}$$

If we are not interested in a definite state of photon polarization, expression (26.10) must be summed or averaged over the two independent directions of the vector $e_\mu$.

We denote by $\boldsymbol{k}$ the space part of the photon propagation vector and choose a coordinate system with the $z$-axis directed along $\boldsymbol{k}$. Then for the two independent polarization vectors we can take the vectors

$e_\mu^{(1)}(1, 0, 0, 0)$ and $e_\mu^{(2)}(0, 1, 0, 0)$ and write the sum in which we are interested in the form

$$\frac{1}{2}\sum_{e}\sum_{\mu_i,\mu_f}|M_{if}|^2 = -\frac{1}{8\varepsilon_i\varepsilon_f}\sum_{j=1}^{2}\mathrm{Sp}\,\{\hat{e}^{(j)}G(i\hat{p}_i-m)\bar{G}\hat{e}^{(j)}(i\hat{p}_f-m)\}$$

$$= -\frac{1}{8\varepsilon_i\varepsilon_f}\sum_{j=1}^{2}\mathrm{Sp}\,\{\gamma_j G(i\hat{p}_i-m)\bar{G}\gamma_j(i\hat{p}_f-m)\}, \quad (26.11)$$

where $\sum\limits_{j=1}^{2}$ denotes summation over the two polarization states of the photon.

We now show that the summation in this formula may be carried out over four values of $j$, i.e.,

$$\frac{1}{2}\sum_{e,\mu_i,\mu_f}|M_{if}|^2 = -\frac{1}{8\varepsilon_i\varepsilon_f}\mathrm{Sp}\,\{\gamma_\nu G(i\hat{p}_i-m)\bar{G}\gamma_\nu(i\hat{p}_f-m)\}, \quad (26.12)$$

where $\nu = 1, 2, 3, 4$. In order to do this, we use the fact that the equations of quantum electrodynamics are gauge-invariant. Therefore, if in the matrix element containing the four-dimensional photon polarization $e$ we replace $e$ by $k$ we obtain zero. From this it follows that if in expression (26.10), which is quadratic in the matrix element, we replace one of the vectors $e_\mu$ by $k_\mu$, then this expression will vanish, i.e.,

$$\mathrm{Sp}\,\{(\gamma_3+i\gamma_4)G(i\hat{p}_i-m)\bar{G}\hat{e}(i\hat{p}_f-m)\} = 0,$$
$$\mathrm{Sp}\,\{\hat{e}G(i\hat{p}_i-m)\bar{G}(\gamma_3+i\gamma_4)(i\hat{p}_f-m)\} = 0,$$

while for the remaining vector $e_\mu$ we can take an arbitrary vector. If we take for this vector first (0, 0, 1, 0), and then (0, 0, 0, 1), we obtain

$$\mathrm{Sp}\,\{(\gamma_3+i\gamma_4)G(i\hat{p}_i-m)\bar{G}\gamma_3(i\hat{p}_f-m)\} = 0,$$
$$\mathrm{Sp}\,\{\gamma_4 G(i\hat{p}_i-m)\bar{G}(\gamma_3+i\gamma_4)(i\hat{p}_f-m)\} = 0.$$

By multiplying the second equation by $-i$ and adding it to the first one we obtain

$$\mathrm{Sp}\,\{\gamma_3 G(i\hat{p}_i-m)\bar{G}\gamma_3(ip_f-m)+\gamma_4 G(i\hat{p}_i-m)\bar{G}\gamma_4(i\hat{p}_f-m)\} = 0.$$

Further, on adding this equation multiplied by $-1/8\varepsilon_i\varepsilon_f$ to (26.11), we obtain formula (26.12).

In accordance with the general results of the theory this result shows that no "longitudinal" or "scalar" photons can be emitted.

A similar relation holds if several photons participate in the process. If, for example, the process involves one electron and two photons with polarizations $e_{1_\mu}$ and $e_{2_\mu}$, the matrices $Q$ and $\bar{Q}$ can be written

$$Q = \hat{e}_2 A \hat{e}_1 + \hat{e}_1 B \hat{e}_2 \quad \text{and} \quad \bar{Q} = \hat{e}_1 \bar{A} \hat{e}_2 + \hat{e}_2 \bar{B} \hat{e}_1 .$$

By repeating the preceding arguments, it can be easily shown that on summing $\frac{1}{2} \sum\limits_{\mu_i, \mu_f} |M_{if}|^2$ over the different polarization states of the two photons we shall obtain the expression:

$$\frac{1}{2} \sum_{e_1, e_2} \sum_{\mu_i, \mu_f} |M_{if}|^2 = \frac{1}{8\varepsilon_i \varepsilon_f} \sum_{r, s=1}^{2} \mathrm{Sp}\, \{(\hat{e}_2^{(r)} A \hat{e}_1^{(s)} + \hat{e}_1^{(s)} B \hat{e}_2^{(r)}) (i\hat{p}_i - m)$$

$$\times (\hat{e}_1^{(s)} \bar{A} \hat{e}_2^{(r)} + \hat{e}_2^{(r)} \bar{B} \hat{e}_1^{(s)}) (i\hat{p}_f - m)\} \tag{26.13}$$

$$= \frac{1}{8\varepsilon_i \varepsilon_f} \mathrm{Sp}\, \{(\gamma_\mu A \gamma_\nu + \gamma_\nu B \gamma_\mu) (i\hat{p}_i - m) (\gamma_\nu \bar{A} \gamma_\mu + \gamma_\mu \bar{B} \gamma_\nu) (i\hat{p}_f - m)\}.$$

The summation over the four photon polarizations can be carried out with the aid of the following formulas (cf. subsection 10.7):

$$\gamma_\nu s \gamma_\nu = 4s, \quad \gamma_\nu \hat{a} \gamma_\nu = -2\hat{a}, \quad \gamma_\nu \hat{a} \hat{b} \gamma_\nu = 4ab, \quad \gamma_\nu \hat{a} \hat{b} \hat{c} \gamma_\nu = -2\hat{c} \hat{b} \hat{a}, \tag{26.14}$$

where $s$ is a scalar and $a_\nu$, $b_\nu$, $c_\nu$ are four-vectors, $ab = a_\nu b_\nu$. As a result of this summation the number of matrices following the trace symbol is evidently reduced by $2g$ where $g$ is the number of photons.

In evaluating traces of matrices it is convenient to utilize

$$\begin{aligned} &\tfrac{1}{4} \mathrm{Sp}\, \hat{A}_1 \hat{A}_2 = A_1 A_2 , \\ &\tfrac{1}{4} \mathrm{Sp}\, \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 = (A_1 A_2)(A_3 A_4) + (A_1 A_4)(A_2 A_3) - (A_1 A_3)(A_2 A_4), \end{aligned} \tag{26.15}$$

where $A_1$, $A_2$, $A_3$, $A_4$ are arbitrary four-vectors.

The trace of a product of an odd number of factors $\hat{A}_i$ is obviously equal to zero.

We also note the following useful formula:

$$\tfrac{1}{4} \mathrm{Sp}\, (\hat{A}_1 + a_1)(\hat{A}_2 + a_2)(\hat{A}_3 + a_3)(\hat{A}_4 + a_4) = (A_1 A_2 + a_1 a_2)(A_3 A_4 + a_3 a_4)$$

$$+ (A_1 A_4 + a_1 a_4)(A_2 A_3 + a_2 a_3) - (A_1 A_3 - a_1 a_3)(A_2 A_4 - a_2 a_4), \tag{26.16}$$

where $a_i$ are numbers not involving the matrices $\gamma_\mu$.

### 26.4. *Probabilities of Processes Involving Polarized Particles*

A complete solution of the collision problem involves obtaining the probability of the process as a function of the polarization state of the particles in the initial state, and determining the polarization states of the particles in the final state.

For the sake of simplicity we consider, as we have done earlier, the case when only a single electron is present both in the initial and in the final states, but the methods which we employ can be easily generalized to the case of an arbitrary number of electrons and positrons.

In the case under discussion the quantity $M_{if}$ appearing in the matrix element $S_{i\to f}^{(n)}$ has the form

$$M_{if} = \bar{u}_f Q u_i .$$

The square of the absolute value of $M_{if}$ averaged over the initial states is given by

$$\overline{|M_{if}|^2} = \overline{(\bar{u}_f Q u_i)(\bar{u}_i \bar{Q} u_f)} = (\bar{u}_f Q)_\alpha \overline{u_{i\alpha} \bar{u}_{i\beta}} (\bar{Q} u_f)_\beta$$

or by

$$\overline{|M_{if}|^2} = \bar{u}_f Q \mathcal{P}^{(i)} \gamma_4 \bar{Q} u_f , \tag{26.17}$$

where $\mathcal{P}^{(i)}$ is the polarization density matrix for the initial state, defined by formula (10.21). The matrix $\mathcal{P}^{(i)}$ can be expressed in terms of the electron polarization vector in the initial state in accordance with formulas (10.31), (10.31′).

On summing (26.17) over the polarizations of the final state we obtain

$$\sum_{\mu_f} \overline{|M_{if}|^2} = -\frac{1}{2\varepsilon_f} \mathrm{Sp}\{Q \mathcal{P}^{(i)} \gamma_4 \bar{Q} (i\hat{p}_f - m)\} . \tag{26.18}$$

If the initial state is unpolarized, then in accordance with (10.26)

$$\mathcal{P}^{(i)} = -\frac{1}{4\varepsilon_i} (i\hat{p}_i - m) \gamma_4$$

and formula (26.18) reduces to (26.7).

For the determination of the polarization state of the electron in the final state we obtain the corresponding density matrix.

The general expression for the density matrix corresponding to transitions from the initial state $i$ has the form

$$\varrho_{ff'} = S_{i\to f}^{(n)} S_{i\to f'}^{(n)*} .$$

The elements of the matrix $\varrho_{ff'}$, corresponding to a given value of the electron momentum $p_f = p_{f'}$ and to different polarizations $\mu_f$, $\mu_{f'} = \pm\frac{1}{2}$ give rise to the polarization density matrix for the electron in the final state

$$\varrho^{(f)}_{\mu_f\mu_{f'}} = \frac{S^{(n)}_{i\to f}S^{(n)*}_{i\to f'}}{\sum\limits_{\mu_f}|S_{i\to f}|^2} = \frac{M_{if}M^*_{if'}}{\sum\limits_{\mu_f}|M_{if'}|^2} \quad \text{where} \quad p_f = p_{f'} \tag{26.19}$$

normalized in accordance with

$$\operatorname{Sp}\varrho^{(f)} = \sum_{\mu_f=\pm 1/2} \varrho^{(f)}_{\mu_f\mu_f'} = 1.$$

On substituting into (26.19) $M_{if} = \bar{u}_f Q u_i$, and on averaging over the initial polarizations, we obtain, in analogy with (26.17), the following expression for the polarization matrix $\varrho^f$:

$$\varrho^{(f)}_{\mu_f\mu_{f'}} = \frac{1}{N}\,\bar{u}^{\mu_f}(p_f)\mathcal{P}^{(i)}\gamma_4\bar{Q}u^{\mu_{f'}}(p_f), \tag{26.20}$$

where

$$N = -\frac{1}{2\varepsilon_f}\operatorname{Sp}\{Q\mathcal{P}^{(i)}\gamma_4 Q(i\hat{p}_f - m)\}.$$

This matrix is a two-dimensional polarization matrix corresponding to the definition (10.18). In order to obtain the four-dimensional polarization matrix we utilize expression (10.27)

$$\mathcal{P}^{(f)}_{\alpha\beta} = \sum_{\mu_f\mu_{f'}} \varrho^{(f)}_{\mu_f\mu_{f'}} u^{\mu_f}_\alpha(p_f) u^{\mu_{f'}*}_\beta(p_f).$$

On substituting into this formula expression (26.20) in place of $\varrho^{(f)}_{\mu_f\mu_{f'}}$, and on utilizing the summation formulas (10.28) we finally obtain

$$\mathcal{P}^{(f)} = -\frac{1}{2\varepsilon_f}\,\frac{(i\hat{p}_f - m)Q\mathcal{P}^{(i)}\gamma_4\bar{Q}(i\hat{p}_f - m)\gamma_4}{\operatorname{Sp}(i\hat{p}_f - m)Q\mathcal{P}^{(i)}\gamma_4\bar{Q}}. \tag{26.21}$$

If the initial state is unpolarized, then we have

$$\mathcal{P}^{(i)} = -\frac{1}{4\varepsilon_i}(i\hat{p}_i - m)\gamma_4$$

and

$$\mathcal{P}^{(f)} = -\frac{1}{8\varepsilon_i\varepsilon_f}\,\frac{(i\hat{p}_f - m)Q(i\hat{p}_i - m)\bar{Q}(i\hat{p}_f - m)\gamma_4}{\operatorname{Sp}(i\hat{p}_f - m)Q(i\hat{p}_i - m)\bar{Q}}. \tag{26.22}$$

By utilizing this formula we can, in accordance with (10.32), obtain the electron polarization in the final state

$$a_\mu = -\frac{i\varepsilon_f}{m}\,\mathrm{Sp}\,\mathcal{P}^{(f)}\gamma_4\gamma_\mu\gamma_5.$$

We now investigate the dependence of the probabilities of the various processes on the photon polarization. If in the initial state we have a polarized photon, then the averaged value of $|M_{if}|^2$ is expressed in terms of the polarization matrix for the photon in the initial state. On writing $M_{if}$ in the form

$$M_{if} = a_\alpha^* e_\alpha^{(i)},$$

where $e^{(i)}$ is the polarization unit vector for the photon in the initial state, we obtain

$$\overline{|M_{if}|^2} = a_\alpha^* \varrho_{\alpha\beta}^{(i)} a_\beta, \tag{26.23}$$

where $\varrho^{(i)}$ is the photon density matrix defined by formula (2.13).

In order to obtain the polarization density matrix for the photon in the final state $\varrho^{(f)}$ we write $M_{if}$ in the form

$$M_{if} = e_\alpha^{(f)*} b_\alpha.$$

Then we have

$$\varrho_{\nu\nu'}^{(f)} = \frac{M_{if} M_{if'}^*}{\sum\limits_\nu |M_{if}|^2}, \qquad k_f = k_{f'},$$

where $\nu$ determines the photon polarization in the final state (the photon momenta $k_f$ and $k_{f'}$ are the same in the states $f$ and $f'$).

Since $e_\alpha^{(f)} = \delta_{\nu\alpha}$, it follows that

$$\varrho_{\nu\nu'}^{(f)} = \frac{b_\nu b_{\nu'}^*}{\sum\limits_\nu |b_\nu|^2}. \tag{26.24}$$

If we know $\varrho^{(f)}$, we can obtain the photon polarization parameters with the aid of formula (2.15).

The vectors $a$ and $b$ obviously depend on the polarization states of the other particles over which appropriate averages and summations must be taken.

The knowledge of the density matrix for the particle in the final state enables us to determine the probability of the process which results in the particle being left in a given polarization state.

Indeed, if we know the wave function of the final state $\Phi_f$, then the probability of a certain state $\Phi_d$ is given by $W^{(d)} = |(\Phi_f, \Phi_d)|^2$. Similarly, if a given state selected by the detector is characterized by the density matrix $\varrho^{(d)}$, while the state of the particle is characterized by the density matrix $\varrho^{(f)}$, then we have

$$W^{(d)} = \mathrm{Sp}\, \varrho^{(f)} \varrho^{(d)}. \tag{26.25}$$

Since the density matrix can be written (both for electrons and for photons) in the form

$$\varrho^{(f)} = \tfrac{1}{2}(1+\xi_j^f \sigma_j) \quad \text{and} \quad \varrho^{(d)} = \tfrac{1}{2}(1+\xi_j^d \sigma_j),$$

we have

$$W^{(d)} = \tfrac{1}{2}(1+\xi_j^f \xi_j^d). \tag{26.26}$$

Thus, if the cross section for a process summed over the polarizations of the particle in the final state is equal to $\sigma$, then the cross section corresponding to the production of the particle with given polarization parameters is given by

$$\sigma^{(d)} = \sigma(1+\xi_j^f \xi_j^d). \tag{26.27}$$

### 26.5. *Probabilities of Processes in the Presence of an External Field*

We now show how the probabilities of various processes are to be determined in those cases when an external electromagnetic field is present.

If a single electron participates in the process, then the matrix element can be given in the first approximation in the form

$$S_{i\to f}^{(1)} = -e\bar{u}_f \left(\int e^{-iqx} \hat{A}^e(x)\, d^4x\right) u_i,$$

where $q = p_f - p_i$. We first assume that the external potential does not depend on the time, $\hat{A}^e(x) = \hat{A}^e(\boldsymbol{r})$. Then we have

$$S_{i\to f}^{(1)} = -e\bar{u}_f \hat{A}^e(\boldsymbol{q}) u_i 2\pi\, \delta(\varepsilon_f - \varepsilon_i),$$

where

$$\hat{A}^e(\boldsymbol{q}) = \int \hat{A}^e(\boldsymbol{r}) e^{-i\boldsymbol{q}\boldsymbol{r}}\, d\boldsymbol{r}$$

and $\varepsilon_i$ and $\varepsilon_f$ are the electron energies in the initial and the final states. The probability of the process is equal to $|S_{i\to f}^{(1)}|^2$. By proceeding in the same manner as in the case of the derivation of (26.1) we obtain the following expression for the transition probability per unit time:

$$W = 2\pi |\bar{u}_f e A^e(q) u_i|^2 \delta(\varepsilon_f - \varepsilon_i). \tag{26.28}$$

Formula (26.28) determines the probability of electron scattering by the field $A^e_\mu(r)$ in the first Born approximation.

In order to obtain the differential scattering cross section $d\sigma$ we must divide $W$ by the initial electron velocity $v_i = p_i/\varepsilon_i$ and multiply it by $d\boldsymbol{p}_f/(2\pi)^3$—the number of final states for which the electron momentum $\boldsymbol{p}_f$ after scattering lies within the interval $d\boldsymbol{p}_f$:

$$d\sigma = \frac{2\pi}{v_i} |\bar{u}_f e\hat{A}^e(q) u_i|^2 \delta(\varepsilon_f - \varepsilon_i) \frac{d\boldsymbol{p}_f}{(2\pi)^3}.$$

On noting that

$$d\boldsymbol{p}_f = p_f^2 dp_f\, do = p_f \varepsilon_f\, d\varepsilon_f\, do,$$

where $do$ is the element of solid angle containing $\boldsymbol{p}_f$, and on eliminating the $\delta$-function by means of integration over $d\varepsilon_f$, we obtain

$$d\sigma = \frac{p_i^2}{v_i^2} |\bar{u}_f e\hat{A}^e(q) u_i|^2 \frac{do}{(2\pi)^2}. \tag{26.29}$$

In particular, if $\boldsymbol{A}^{(e)} = 0$, $A_4^{(e)} = iA_0$, then we have

$$d\sigma = \frac{p_i^2}{v_i^2} |u_f^* e A_0(q) u_i|^2 \frac{do}{(2\pi)^2}. \tag{26.30}$$

This expression is formally equivalent to the corresponding expression for the scattering cross section in nonrelativistic quantum mechanics in the Born approximation.

If we are not interested in definite electron polarization states before and after scattering, then formula (26.30) should be summed over the electron spin orientations after scattering, and averaged over the electron spin orientations before scattering. This may be done with the aid of relation (26.7).

In the case when $\boldsymbol{A}^{(e)} = 0$, the matrix $Q$ is of the form

$$Q = \gamma_4 e A_0(q)$$

and formula (26.7) yields

$$\frac{1}{2} \sum_{\mu_i, \mu_f} |\bar{u}_f Q u_i|^2 = \frac{1}{8\varepsilon_i^2} \operatorname{Sp}\{\gamma_4(i\hat{p}_i - m)\gamma_4(i\hat{p}_f - m)\} |eA_0(q)|^2. \tag{26.31}$$

The trace of the product of the matrices $\gamma_\mu$ can be evaluated in accordance with formula (26.15):

$$\begin{aligned}
&\tfrac{1}{4}\operatorname{Sp}\{\gamma_4(i\hat{p}_i-m)\gamma_4(i\hat{p}_f-m)\}\\
&\quad=\tfrac{1}{4}\operatorname{Sp}\{\gamma_4(i\boldsymbol{p}_i\boldsymbol{\gamma}-\varepsilon\gamma_4-m)\gamma_4(i\boldsymbol{p}_f\boldsymbol{\gamma}-\varepsilon\gamma_4-m)\}\\
&\quad=\tfrac{1}{4}\operatorname{Sp}\{(-i\boldsymbol{p}_i\boldsymbol{\gamma}-\varepsilon-m)(i\boldsymbol{p}_f\boldsymbol{\gamma}-\varepsilon-m)\}\\
&\quad=\tfrac{1}{4}\operatorname{Sp}\{(\boldsymbol{p}_i\boldsymbol{\gamma})(\boldsymbol{p}_f\boldsymbol{\gamma})+\varepsilon^2+m^2\}=p^2\cos\vartheta+m^2+\varepsilon^2\\
&\qquad\qquad\qquad\qquad=2\varepsilon^2\left(1-v^2\sin^2\frac{\vartheta}{2}\right),
\end{aligned}$$

where $\vartheta$ is the scattering angle, $\varepsilon=\varepsilon_i=\varepsilon_f$, $|\boldsymbol{p}|=|\boldsymbol{p}_i|=|\boldsymbol{p}_f|$, $v=v_i$. On substituting this expression into (26.31) and (26.30), we obtain the following formula for the averaged scattering cross section:

$$d\sigma=\frac{p^2}{v^2}\left(1-v^2\sin^2\frac{\vartheta}{2}\right)\left|\int e^{i(\boldsymbol{p}_i-\boldsymbol{p}_f)\boldsymbol{r}}eA_0(\boldsymbol{r})d\boldsymbol{r}\right|^2\frac{do}{(2\pi)^2}. \qquad (26.32)$$

This expression differs from the corresponding expression of nonrelativistic quantum mechanics by the additional factor $1-v^2\sin^2(\vartheta/2)$.

We examine in greater detail the scattering of electrons by the Coulomb field of the nucleus. In this case we have

$$eA_0(r)=-(Ze^2/r),$$

where $Ze$ is the nuclear charge (in Gaussian units) and

$$eA_0(q)=-\int\frac{Ze^2}{r}e^{i\boldsymbol{q}\boldsymbol{r}}d\boldsymbol{r}=-\frac{4\pi Ze^2}{q^2}. \qquad (26.33)$$

Indeed, we consider the integral

$$J_\eta=\int\frac{1}{r}e^{i\boldsymbol{q}\boldsymbol{r}-\eta r}d\boldsymbol{r},$$

where the factor $\exp(-\eta r)$ $(\eta>0)$ is introduced in order to guarantee the convergence of the integral for large $r$. On carrying out the integration over the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{q}$ we obtain

$$J_\eta=\frac{2\pi}{iq}\int_0^\infty(e^{-(iq+\eta)r}-e^{(iq-\eta)r})dr=\frac{4\pi}{q^2+\eta^2}.$$

By letting $\eta$ tend to zero we obtain formula (26.33).

The substitution of (26.33) into (26.32) leads to the following expression for the cross section for the scattering of electrons by the nuclear Coulomb field:

$$d\sigma = \left(\frac{Ze^2}{2pv}\right)^2 \left(1 - v^2 \sin^2\frac{\vartheta}{2}\right) \frac{do}{\sin^4\frac{\vartheta}{2}}. \tag{26.34}$$

## 26.6. *Feynman's Notation*

Throughout this book we use a Euclidean metric and four-dimensional space in which $x_4 = it$. The scalar product of two four-vectors $a$ and $b$ is defined by

$$ab = \sum_{\mu=1}^{4} a_\mu b_\mu = \boldsymbol{ab} + a_4 b_4.$$

If we use the pseudo-Euclidean metric with a real time coordinate $t = x_0 = -ix_4$, then we have

$$ab = \boldsymbol{ab} - a_0 b_0.$$

Usually in this case the scalar product is defined with the opposite sign. We denote by $(ab)$ the quantity

$$(ab) = a_0 b_0 - \boldsymbol{ab} = -ab \quad \text{and} \quad (a^2) = -a^2 = a_0^2 - \boldsymbol{a}^2.$$

In using the pseudo-Euclidean metric it is convenient to introduce (first of references (62)) in place of the matrices $\gamma_\mu (\mu = 1, 2, 3, 4)$ other matrices which we denote by $\gamma_\mu^F (\mu = 0, 1, 2, 3)$ and which are defined in the following manner:

$$\gamma_0^F = \gamma_4 = \beta, \quad \gamma_j^F = i\gamma_j = \beta\alpha_j \quad \text{where} \quad j = 1, 2, 3.$$

The matrix $\gamma_0^F$ is Hermitian, while the matrices $\gamma_j^F$ are anti-Hermitian

$$\gamma_0^{F+} = \gamma_0^F \quad \text{and} \quad \gamma_j^{F+} = -\gamma_j^F.$$

The commutation properties of the matrices $\gamma_\mu^F$ are defined by the following relation:

$$\gamma_\mu^F \gamma_\nu^F + \gamma_\nu^F \gamma_\mu^F = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3. \end{cases}$$

In analogy with the quantity $\hat{a} = \gamma a = \boldsymbol{\gamma a} + \gamma_4 a_4$ we define

$$\check{a} = (\gamma^F a) = \gamma_0^F a_0 - \boldsymbol{\gamma}^F \boldsymbol{a}.$$

It can easily be shown that

$$i\hat{a} = -\check{a}.$$

Application of the commutation relations for the matrices $\gamma^F$ leads to the following relations:

$$(\gamma^F\gamma^F) = 4, \qquad \check{a}\check{b}+\check{b}\check{a} = 2(ab),$$

$$(\gamma^F\check{a}\gamma^F) = -2\check{a}, \qquad \tfrac{1}{4}\,\mathrm{Sp}\,\check{a}\check{b} = (ab),$$

$$(\gamma^F\check{a}\check{b}\gamma^F) = 4(ab), \qquad \tfrac{1}{4}\,\mathrm{Sp}\,\check{a}\check{b}\check{c}\check{d} = (ab)\,(cd)+(ad)(cb)-(ac)\,(bd),$$

$$(\gamma^F\check{a}\check{b}\check{c}\gamma^F) = -2\check{c}\check{b}\check{a}, \qquad \overline{\gamma}_\mu^F \equiv \gamma_0^F\gamma_\mu^F\gamma_0^F = \gamma_\mu^F.$$

The properties of the matrices $\gamma_\mu^F$ under charge conjugation are the same as in the case of the matrices $\gamma_\mu$:

$$\mathrm{C}\gamma_\mu^F = -\tilde{\gamma}_\mu^F\mathrm{C}.$$

In this notation the Dirac equation assumes the form

$$(\check{\mathrm{p}}-e\check{A}-m)\,\psi = 0,$$

$$\overline{\psi}(\check{\mathrm{p}}+e\check{A}+m) = 0,$$

where

$$\check{\mathrm{p}} = -i\left(\gamma^F\frac{\partial}{\partial x}\right),$$

it being understood that in the last equation the operation of differentiation acts on quantities standing to the left of it.

For a free electron we have

$$\psi = ue^{ikx},$$

$$(\check{k}-m)\,u = 0 \quad \text{and} \quad \overline{u}(\check{k}-m) = 0.$$

The current density vector has the form

$$s_\mu = \overline{\psi}\gamma_\mu^F\psi \quad \text{and} \quad j_\mu = eN[\overline{\psi}, \gamma_\mu^F\psi].$$

The bilinear expressions $S$, $V$, $T$, $A$, $P$ may be defined by formulas (9.19) with $\gamma_\mu$ replaced by $\gamma_\mu^F$ and with

$$\gamma_5^F = \gamma_5 = i\gamma_1^F\gamma_2^F\gamma_3^F\gamma_0^F.$$

The projection operator $\eta(p)$ given by (10.22) can now be written in the form

$$\eta(p)=\frac{\check{p}+m}{2m},$$

while formulas (26.7)-(26.9) for summing over the polarizations and the density matrix (10.34) can be written in the form

$$\sum|\bar{u}_f Q u_i|^2=\frac{1}{4\varepsilon_i\varepsilon_f}\,\mathrm{Sp}\{Q(\check{p}_i+m)\,\bar{Q}(\check{p}_f+m)\},$$

$$\sum|\bar{u}_f Q v_f|^2=\frac{1}{4\varepsilon_e\varepsilon_p}\,\mathrm{Sp}\{Q(\check{p}_p-m)\,\bar{Q}(\check{p}_e+m)\},$$

$$\mathcal{P}=\frac{1}{4\varepsilon}\,(1+\check{a}\gamma_5)\,(\check{p}+m)\gamma_4,$$

where $\bar{Q}=\gamma_4 Q^{+}\gamma_4$. If $Q=\gamma_\alpha^F\gamma_\beta^F\dots\gamma_\lambda^F$, then we have $\bar{Q}=\gamma_\lambda^F\dots\gamma_\beta^F\gamma_\alpha^F$.

In this notation the Green's functions for the photon and for the electron assume the form

$$D^c(k)=\frac{i}{(k^2)},$$

$$S^c(p)=\frac{i}{\check{p}-m}=i\,\frac{\check{p}+m}{(p^2)-m^2}.$$

The rules formulated in subsection 25.5 for writing down the matrix elements can be easily put into Feynman's notation by using the identities

$$\hat{e}=i\check{e},\quad i\hat{p}=-\check{p},\quad \frac{1}{k^2}=-\frac{1}{(k^2)}.$$

CHAPTER V

# Interaction of Electrons with Photons

## § 27. Emission and Absorption of a Photon

### 27.1. *General Expression for the Matrix Element*

At this point let us initiate a systematic investigation of various specific processes which arise from the interaction between electrons and the electromagnetic field. We use perturbation theory, and in this chapter we carry out calculations only in the first nonvanishing approximation. Corrections associated with higher order approximations will be discussed in Chapter VIII.

We begin by considering first order processes. The first order scattering matrix is of the form

$$\mathrm{S}^{(1)} = -e\int N\big(\bar{\psi}(x)\hat{A}(x)\psi(x)\big)d^4x. \tag{27.1}$$

If $A_\mu(x)$ is the potential of the external electromagnetic field then the matrix elements of $\mathrm{S}^{(1)}$ determine the scattering of an electron by this field in the first Born approximation, which we have discussed in the preceding section.

But if $\mathrm{A}_\mu(x)$ is the operator of the quantized electromagnetic field then the matrix elements of $\mathrm{S}^{(1)}$ will determine photon emission and absorption, since the operator $\mathrm{A}_\mu(x)$ appears in $\mathrm{S}^{(1)}$ linearly, and the nonvanishing matrix elements of $\mathrm{A}_\mu$ correspond to photon emission and absorption.

However, it can be easily shown that the matrix elements of $\mathrm{S}^{(1)}$ vanish if the electron taking part in the process is free. Indeed, if the electron and the photon are both in states of definite momenta, then the matrix element corresponding to photon absorption contains as a factor the four-dimensional $\delta$-function $\delta(p_1+k-p_2)$, where $p_1(\boldsymbol{p}_1, i\varepsilon_1)$, $p_2(\boldsymbol{p}_2, i\varepsilon_2)$ are the four-momenta of the electron in the initial and final states, and $k(\boldsymbol{k}, i\omega)$ is the four-momentum of the photon. Therefore the matrix element will differ from zero only if the laws of conservation of energy and momentum are satisfied:

$$p_1+k = p_2. \tag{27.2}$$

But these laws cannot be satisfied simultaneously since on squaring (27.2) and noting that $p_1^2 = p_2^2 = -m^2$, $k^2 = 0$, we obtain

$$p_1 k = \mathbf{p}_1 \mathbf{k} - \omega \sqrt{\mathbf{p}_1^2+m^2} = 0,$$

which is impossible.

On replacing in (27.2) $k$ by $-k$ we obtain the conservation laws for the emission of a photon by a free electron, which also cannot be satisfied simultaneously.

On replacing $k$ by $\sum k_i$ we can easily show that no process is possible in which a free electron absorbs or emits any arbitrary number of photons.

Finally, we note that from the conservation laws it also follows that a single photon cannot give rise to a free electron-positron pair, and a pair cannot give rise to a single photon. This can be seen directly in the coordinate system in which the center of mass of the pair is at rest: in this system the total momentum of the pair is equal to zero, but their energy is different from zero (it is not less than $2m$), while in the case of a photon the energy vanishes whenever the momentum vanishes.

In order for the conservation laws to be satisfied the participation of a third body is necessary. Absorption or emission of a photon can occur only as a result of a "triple collision" in which the interaction of the electron with the "third" body plays an essential role.

In a number of important cases this interaction can be described with the aid of the concept of an external field appearing in the Hamiltonian for the electron. The operator of the electron-positron field can be expanded in terms of the eigenfunctions of this Hamiltonian (cf. (18.7)), and the concept of the electron state will in such a case automatically take into account the interaction of the electron with other bodies. We shall say that such electron states are not free.

Such an approach enables us to study the processes of emission and absorption of a photon by means of the first order scattering matrix, since its elements between electron states that are not free are, generally speaking, different from zero.

We now proceed to determine the probability of emission and absorption of a photon by an electron in an external field.

We write down the general expression for the matrix element $S^{(1)}_{i\to f}$ which determines the emission of a photon.

If $\psi_1(x)$ and $\psi_2(x)$ are the wave functions of the initial and the final electron states and $A_\mu(x)$ is the potential of the electromagnetic field corresponding to a definite state of the photon, then we have

$$S^{(1)}_{i\to f} = -e\int \bar{\psi}_2(x)\hat{A}^*(x)\psi_1(x)d^4x\sqrt{N_k+1} \tag{27.3}$$

where $N_k$ is the number of photons of the kind in which we are interested before emission has taken place. In the future we shall discuss only the case $N_k = 0$.

We assume that $\psi_1(x)$ and $\psi_2(x)$ describe stationary states of an electron of energy $\varepsilon_1$ and $\varepsilon_2$. Then the matrix element corresponding to the emission of a photon of frequency $\omega$ can be written in the form

$$\begin{aligned} S^{(1)}_{i\to f} &= -2\pi i U_{i\to f}\delta(\varepsilon_1-\varepsilon_2-\omega), \\ U_{i\to f} &= -ie\int \bar{\psi}_2(\boldsymbol{r})\hat{A}^*(\boldsymbol{r})\psi_1(\boldsymbol{r})d\boldsymbol{r}, \end{aligned} \tag{27.4}$$

where $\psi_1(\boldsymbol{r}), \psi_2(\boldsymbol{r}), A(\boldsymbol{r})$ are functions only of $\boldsymbol{r}$ (without time dependent factors).

In particular, if we are investigating the emission of a photon of definite momentum $\boldsymbol{k}$ and polarization $\boldsymbol{e}$, then the potential is given by formula (6.15) and the quantity $U_{i\to f}$ has the form

$$U_{i\to f} = -\frac{e}{\sqrt{2\omega}}\int \psi_2^*(\boldsymbol{r})\boldsymbol{\alpha}\boldsymbol{e}e^{-i\boldsymbol{k}\boldsymbol{r}}\psi_1(\boldsymbol{r})d\boldsymbol{r}. \tag{27.5}$$

### 27.2. *Electric Multipole Radiation*

We now discuss the radiation in the case when the initial and the final states of the electron are bound, and the wavelength $\lambda$ of the emitted photon is large compared to the dimensions $a$ of the region to which the electron motion is confined. As we shall see later, the probability of emission in this case is related in a simple manner to the electric or the magnetic multipole moment of the electron and therefore the radiation from such a system is called multipole radiation.

We determine the probability of emission of a photon of definite angular momentum $L$, definite component of angular moment $M$, and definite parity $P = (-1)^{L+\lambda+1}$. We recall that $\lambda = 1$ corresponds to states of electric type, while $\lambda = 0$ corresponds to states of magnetic type. Therefore, transitions involving the emission of such a photon are called *electric* $2^L$*-pole* or *EL-transitions* when $\lambda = 1$, and *magnetic* $2^L$*-pole* or *ML-transitions* when $\lambda = 0$.

We begin by discussing the emission of a photon of electric type.

The potential corresponding to a photon in a state of electric type is given by (6.17). The product of the components of this potential and the matrices $\gamma_\mu$ is equal to (omitting the time dependent factor)

$$\hat{A}_{LM} = \frac{1}{4\pi}\sqrt{\frac{\omega}{R}}\,\{\boldsymbol{\gamma}(\boldsymbol{a}^{(1)}_{LM} + C\boldsymbol{a}^{(-1)}_{LM}) + iC\gamma_4\Phi_{LM}\}, \tag{27.6}$$

where

$$\boldsymbol{a}^{(1)}_{LM} = \sqrt{\frac{L}{2L+1}}\,g_{L+1}(\omega r)\,\boldsymbol{Y}_{L,L+1,M} + \sqrt{\frac{L+1}{2L+1}}\,g_{L-1}(\omega r)\,\boldsymbol{Y}_{L,L-1,M},$$

$$\boldsymbol{a}^{(-1)}_{LM} = -\sqrt{\frac{L+1}{2L+1}}\,g_{L+1}(\omega r)\,\boldsymbol{Y}_{L,L+1,M} + \sqrt{\frac{L}{2L+1}}\,g_{L-1}(\omega r)\,\boldsymbol{Y}_{L,L-1,M},$$

$$\Phi_{LM} = g_L(\omega r)\,Y_{LM}, \tag{27.6$'$}$$

$R$ is the radius of the normalizing sphere and $C$ is an arbitrary constant. This expression must be substituted in place of $\hat{A}(\boldsymbol{r})$ in formula (27.4).

Since the size of the system is assumed to be small compared to the wavelength, the principal contribution to the integral $U_{i\to f}$ comes from small $r$ for which $\omega r \ll 1$. Therefore, we can restrict ourselves to the first term in the expansion of the functions $g_L(\omega r)$ in powers of $\omega r$:

$$g_L(\omega r) \approx 4\pi\frac{(i\omega r)^L}{(2L+1)!!}, \tag{27.6$''$}$$

where $(2L+1)!! = 1\cdot 3\cdot 5\ \dots\ (2L+1)$.

Since the function $g_L(\omega r)$ is proportional to $(\omega r)^L$, we can keep in (27.6′) only those terms which contain $g_L(\omega r)$ with the lowest value of $L$, i.e., $g_{L-1}(\omega r)$. As a result we obtain

$$\boldsymbol{a}^{(1)}_{LM} = \sqrt{\frac{L+1}{L}}\,\boldsymbol{a}^{(-1)}_{LM}, \qquad \omega r \ll 1. \tag{27.7}$$

Further, on taking into account that $\boldsymbol{a}^{(-1)}_{LM}$ and $\Phi_{LM}$ are related in consequence of (4.10) by the expression

$$\boldsymbol{a}^{(-1)}_{LM} = -\frac{i}{\omega}\nabla\Phi_{LM}, \tag{27.8}$$

we can write $U_{i\to f}$ in the form

$$U_{i\to f} = -\frac{e}{4\pi}\sqrt{\frac{\omega}{R}}\int \psi_2^* \left[\sqrt{\frac{L+1}{L}}\,\frac{i}{\omega}\nabla\Phi_{LM}^*\boldsymbol{\alpha} - C\left(\Phi_{LM}^* - \frac{i}{\omega}\boldsymbol{\alpha}\nabla\Phi_{LM}^*\right)\right]\psi_1\,d\mathbf{r}. \tag{27.9}$$

We now show that

$$U_{i\to f} = -e\sqrt{\frac{\omega}{R}}(-i)^L\sqrt{\frac{L+1}{L}}\,\frac{\omega^L}{(2L+1)!!}\int \psi_2^*(\mathbf{r})r^L Y_{LM}^*\psi_1(\mathbf{r})\,d\mathbf{r}. \tag{27.9'}$$

In order to do this we note that

$$\int \psi_2^*(\mathbf{r})(-i\boldsymbol{\alpha}\nabla\Phi_{LM}^*)\psi_1(\mathbf{r})\,d\mathbf{r} = \int \psi_2^*(\mathbf{r})(\boldsymbol{\alpha}\mathbf{p}\Phi_{LM}^* - \Phi_{LM}^*\boldsymbol{\alpha}\mathbf{p})\psi_1(\mathbf{r})\,d\mathbf{r}$$
$$= \int \psi_2^*(\mathbf{r})(\mathrm{H}\Phi_{LM}^* - \Phi_{LM}^*\mathrm{H})\psi_1(\mathbf{r})\,d\mathbf{r},$$

where $\mathbf{p}$ is the momentum operator and H is the Hamiltonian for an electron in the external field $A^{(e)}(x)$, $\mathrm{H} = \boldsymbol{\alpha}\mathbf{p}+\beta m - ie\beta\hat{A}^{(e)}$. Since H is a self-conjugate operator and $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$ are its eigenfunctions:

$$\mathrm{H}\psi_1 = \varepsilon_1\psi_1 \quad \text{and} \quad \mathrm{H}\psi_2 = \varepsilon_2\psi_2,$$

we have

$$\int \psi_2^*(-i\boldsymbol{\alpha}\nabla\Phi_{LM}^*)\psi_1\,d\mathbf{r} = -\omega\int \psi_2^*\Phi_{LM}^*\psi_1\,d\mathbf{r}, \tag{27.10'}$$

where $\omega = \varepsilon_1 - \varepsilon_2$. In virtue of this relation the terms containing the arbitrary constant $C$ drop out from expression (27.9), as might have been expected from considerations of gauge invariance, and $U_{i\to f}$ assumes the form

$$U_{i\to f} = -\frac{e}{4\pi}\sqrt{\frac{\omega}{R}}\int \psi_2^*\sqrt{\frac{L+1}{L}}\,\Phi_{LM}^*\psi_1\,d\mathbf{r}.$$

On substituting into this equation expression (27.6′) for $\Phi_{LM}^*$ and expansion (27.6″), we obtain formula (27.9′).

We note that the condition $\lambda \gg a$ is equivalent to the condition $v \ll 1$ where $v$ is the ratio of the electron velocity to the velocity of light. (This follows from the fact that in order of magnitude we have $v \sim \omega a \ll 1$.) Therefore, for the functions $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$ in formula (27.9′) we can utilize the nonrelativistic Pauli wave functions $\varphi$ (cf. § 15).

The matrix element $U_{i\to f}$ can be expressed in terms of the operator for the electric multipole moment of the electron defined in the following manner:

$$\mathrm{Q}^{(1)}_{LM} = \frac{e}{\sqrt{4\pi}}\sqrt{\frac{4\pi}{2L+1}}\, r^L Y^*_{LM}, \tag{27.10}$$

where $e/\sqrt{4\pi}$ is the electron charge in Gaussian units. Indeed, on introducing the matrix element of the electric multipole moment

$$(Q^{(1)}_{LM})_{21} = \int \varphi^*_2(r)\, \mathrm{Q}^{(1)}_{LM}\varphi_1(r)\, dr,$$

we can rewrite formula (27.9′) in the form

$$U_{i\to f} = -\sqrt{\frac{\omega}{R}}\,(-i)^L\sqrt{\frac{L+1}{L(2L+1)}}\frac{\omega^L}{(2L-1)!!}(Q^{(1)}_{LM})_{21}. \tag{27.11}$$

In accordance with the definition of the multipole moment operator (27.10) we obtain in the case $M = 0$

$$\mathrm{Q}^{(1)}_{L0} = \frac{e}{\sqrt{4\pi}}\, r^L P_L(\cos\vartheta), \tag{27.12}$$

where $P_L(\cos\vartheta)$ is a Legendre polynomial (the polar axis $z$ coincides with the axis of quantization along which the component of the angular momentum $M$ has been taken equal to zero). For $L = 1$ and $L = 2$ we obtain from the foregoing

$$\mathrm{Q}^{(1)}_{10} = \frac{e}{\sqrt{4\pi}}z, \quad \mathrm{Q}^{(1)}_{20} = \frac{e}{\sqrt{4\pi}}\frac{1}{2}(3z^2-r^2),$$

where $z = r\cos\vartheta$. These formulas coincide with the usual definitions of the dipole and quadrupole moments.

We now obtain the probability of photon emission. By utilizing expression (27.4) for the matrix element $S^{(1)}_{i\to f}$, and by proceeding in the same manner as in deriving (26.3′), we obtain the following formula for the transition probability per unit time:

$$W = 2\pi|U_{i\to f}|^2\delta(\varepsilon_1-\varepsilon_2-\omega).$$

The quantity $W$ must be multiplied by the number of photon states whose energies lie within the range $d\omega$, and it must then be integrated over $\omega$. In accordance with (1.21) the number of photon states of a given value of angular momentum $L$ and of energy lying within the range $d\omega$ is given by $\varrho_f = (Rd\omega/\pi)$. Therefore, after integrating the quantity $W\varrho_f$ over $\omega$ we obtain

$$\omega = 2\pi |U_{i\to f}|^2 \frac{R}{\pi}. \tag{27.13}$$

On substituting into this equation expression (27.11) in place of $U_{i\to f}$ we obtain the emission probability per unit time for a photon of angular momentum $L$, component of angular momentum $M$, and parity $(-1)^L$:

$$w_{LM}^{(1)} = \frac{2(L+1)}{L(2L+1)\,[(2L-1)!!]^2}\omega^{2L+1}|(Q_{LM}^{(1)})_{21}|^2. \tag{27.14}$$

In particular, for $L = 1$ we obtain the formula for the probability of dipole radiation

$$w_{1M}^{(1)} = \tfrac{4}{3}\omega^3|(Q_{1M}^{(1)})_{21}|^2. \tag{27.15}$$

### 27.3. *Magnetic Multipole Radiation*

We now determine the probability of emission of a photon in a state of magnetic type. In this case in accordance with (6.16) we have

$$\hat{A}_{LM}^{(0)} = \frac{1}{4\pi}\sqrt{\frac{\omega}{R}}\gamma a_{LM}^{(0)}, \tag{27.16}$$

where $a_{LM}^{(0)} = g_L(\omega r) Y_{LM}^{(0)}$.

On substituting this expression into (27.4) and on utilizing expression (27.6″) for $g_L(\omega r)$ we obtain

$$\begin{aligned} U_{i\to f} &= -\frac{e}{4\pi}\sqrt{\frac{\omega}{R}}\int \psi_2^* \boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(0)*}\psi_1 d\boldsymbol{r} \\ &= -e\sqrt{\frac{\omega}{R}}\frac{(-i\omega)^L}{(2L+1)!!}\int \psi_2^* \boldsymbol{\alpha} \boldsymbol{Y}_{LM}^{(0)*} r^L \psi_1 d\boldsymbol{r}. \end{aligned} \tag{27.17}$$

We have seen earlier that if the wavelength of the emitted photon is large compared to the size of the region within which the electron motion is confined, then the velocity of the electron is small compared to the velocity of light. Therefore, in formula (27.17) the functions $\psi_1$ and $\psi_2$ can be expressed in terms of the nonrelativistic two-component Pauli wave functions $\varphi$ (cf. § 15).

In order to introduce the functions $\varphi$ into the integrand of (27.17) we note that in accordance with (15.11) the term $-e\boldsymbol{\alpha}\boldsymbol{A}$ in the Hamiltonian of the Dirac equation corresponds to the term $-(e/2m)\{(\mathbf{p}A+A\mathbf{p})+\mu \operatorname{curl} A\}$ in the Hamiltonian of the Pauli equation (if we neglect in it the term quadratic in $A$). Therefore, the matrix element of the operator for the interaction energy $e\alpha A$ evaluated using the Dirac functions must be equal to the matrix element of the interaction energy $(e/2m)\{(\mathbf{p}A+A\mathbf{p})+\mu \operatorname{curl} A\}$ evaluated using the Pauli functions:

$$\int \psi_2^* e\alpha A \psi_1 \, dr = \int \varphi_2^* \left[\frac{e}{2m}(\mathbf{p}A+A\mathbf{p})+\mu \operatorname{curl} A\right]\varphi_1 \, dr, \quad \mu = \frac{e\sigma}{2m} \cdot \tag{27.18}$$

If $\operatorname{div} A = 0$, then we have $\mathbf{p}A+A\mathbf{p} = 2A\mathbf{p}$.

In the case of interest to us we have $A = a_{LM}^{(0)}$, with $\operatorname{div} a_{LM}^{(0)} = 0$. On noting that in accordance with (4.16)

$$Y_{LM}^{(0)} = \frac{-i}{\sqrt{L(L+1)}}\,[r\nabla Y_{LM}],$$

we can express $a_{LM}^{(0)}$ in terms of the function $\Phi_{LM} = g_L(\omega r)\, Y_{LM}$:

$$a_{LM}^{(0)} = \frac{-i}{\sqrt{L(L+1)}}\,[r\nabla\Phi_{LM}].$$

Therefore, we have

$$2a_{LM}^{(0)}\mathbf{p} = \frac{2i}{\sqrt{L(L+1)}}\,\nabla\Phi_{LM}\mathbf{L},$$

where $\mathbf{L} = [r\mathbf{p}]$ is the operator for the electron orbital angular momentum. Further, on taking into account the fact that in accordance with (6.17) and (4.29)

$$\operatorname{curl} a_{LM}^{(0)} = -\omega a_{LM}^{(1)},$$

and on utilizing (27.7) and (27.8), we have

$$\operatorname{curl} a_{LM}^{(0)} = i\sqrt{\frac{L+1}{L}}\,\nabla\Phi_{LM}.$$

On substituting these formulas into (27.17) and (27.18), we obtain

$$U_{i\to f} = i\sqrt{\frac{\omega}{R}}\int \varphi_2^*(\nabla\Phi_{LM}^*)\left[\frac{e}{m\sqrt{L(L+1)}}\,\mathbf{L}+\sqrt{\frac{L+1}{L}}\,\mu\right]\varphi_1 \, dr. \tag{27.19}$$

We now define the operator for the magnetic multipole moment of the electron:

$$\mathrm{Q}_{LM}^{(0)} = \sqrt{\frac{4\pi}{2L+1}}\,\nabla(r^L Y_{LM})\left(\frac{e}{m(L+1)}\right)\mathbf{L}+\mu\frac{1}{\sqrt{4\pi}} \cdot \tag{27.20}$$

For the case $M=0$ this definition leads to

$$\mathbf{Q}^{(0)}_{L0}=\nabla\big(r^L P_L(\cos\vartheta)\big)\left[\frac{e}{m(L+1)}\,\mathbf{L}+\boldsymbol{\mu}\right]\frac{1}{\sqrt{4\pi}}\cdot$$

If $\mathbf{L}=1$, then

$$\mathbf{Q}^{(0)}_{10}=\left(\frac{e}{2m}\mathbf{L}_z+\mu_z\right)\frac{1}{\sqrt{4\pi}}$$

in accordance with the usual definition of the magnetic dipole moment.

On introducing the matrix element of the magnetic multipole moment $(\mathbf{Q}^{(0)}_{LM})_{21}=\int\varphi_2^*\mathbf{Q}^{(0)}_{LM}\varphi_1\,dr$, we can write $U_{i\to f}$ in the form

$$U_{i\to f}=i\sqrt{\frac{\omega}{R}}(-i\omega)^L\sqrt{\frac{L+1}{L(2L+1)}}\,\frac{1}{(2L-1)!!}\,(\mathbf{Q}^{(0)}_{LM})_{21}.\qquad(27.21)$$

This expression differs from (27.11) only by replacing $Q^{(1)}_{LM}$ by $iQ^{(0)}_{LM}$. Therefore, by utilizing (27.4) we can directly write the expression for the emission probability for a photon in a state of magnetic type of angular momentum $L$, angular momentum component $M$, and parity $(-1)^{L+1}$:

$$w^{(0)}_{LM}=\frac{2(L+1)}{L(2L+1)\,[(2L-1)!!]^2}\,\omega^{2L+1}\,|(Q^{(0)}_{LM})_{21}|^2.\qquad(27.22)$$

We have derived formulas (27.14) and (27.22) for the probability of multipole radiation only for the case of a single particle, but they are also applicable to an arbitrary system of interacting particles, as long as the velocities of the particles are small compared to the velocity of light, and the wavelength of the emitted photon is large compared to the size of the system, $\omega a \ll 1$ ($a$ is the effective size of the region within which the particles move).

In the case of a system of particles $Q^{(1)}_{LM}$ should be interpreted as the multipole moment of the system which is equal to the sum of the multipole moments of the individual particles.

In particular, formulas (27.14) and (27.22) can also be used to estimate the probability of emission of radiation by nucleons moving within a nucleus, but in this case we must take into account the anomalous magnetic moments of the proton and the neutron. Moreover, the neutron must be treated as a particle without charge, but possessing a magnetic moment.

We should keep in mind that while for electrons moving inside an atom the following order of magnitude expression holds: $\omega a \sim v \sim (e^2/4\pi) = (1/137)$, the quantities $a$, $\omega$, $v$ are not related in any definite manner in the case of nucleons inside a nucleus. Therefore, in order that the formulas (27.14), (27.22) should be applicable to the determination of the probability of emission of radiation from nuclei, the two relations $\omega a \ll 1$ and $v \ll 1$ must be satisfied independently.

In order to obtain a rough estimate of the absolute value of the emission probability we can assume that in order of magnitude the matrix elements of the multipole moments are given by

$$Q^{(1)}_{LM} \sim ea^L, \qquad Q^{(0)}_{LM} \sim eva^L. \tag{27.23}$$

Formulas (27.14) and (27.22) can be utilized for determining the probability of emission of a photon of definite momentum and polarization. For this we must make use of expansion (4.32) of a plane polarized wave in terms of spherical waves

$$\boldsymbol{e}e^{i\boldsymbol{k}\boldsymbol{r}} = \sum_{L,M,\lambda} \left(\boldsymbol{e}\boldsymbol{Y}^{(\lambda)}_{LM}\left(\frac{\boldsymbol{k}}{\omega}\right)\right)\boldsymbol{a}^{(\lambda)}_{LM}$$

and substitute this expression into formula (27.5) for the matrix element which determines the emission of a photon of momentum $\boldsymbol{k}$ and polarization $\boldsymbol{e}$. If the wavelength is large compared to the size of the radiating system, then only one term in the expansion of $\boldsymbol{e}\exp(i\boldsymbol{k}\boldsymbol{r})$ will play an essential role. Let this term be characterized by certain values of $L$, $M$, $\lambda$ (generally speaking, if there are no special considerations forbidding this, we have $L = 1$, $\lambda = 1$ which corresponds to electric dipole radiation). Then the probability of emission of a photon of momentum $\boldsymbol{k}$ lying within the solid angle $do$, and of polarization $\boldsymbol{e}$, will be given by

$$w_{k,e}do = \left|\boldsymbol{e}\boldsymbol{Y}^{(\lambda)}_{LM}\left(\frac{\boldsymbol{k}}{\omega}\right)\right|^2 w^{(\lambda)}_{LM}do\,. \tag{27.24}$$

If we integrate this expression over $do$ and sum over the two independent photon polarizations $\boldsymbol{e}$ we obtain the total probability of emission equal to the quantity $w^{(\lambda)}_{LM}$.

In all the preceding formulas it was assumed that the number of photons in the initial state (before emission occurs) is equal to zero.

If there are some photons in the initial state, and their number is equal to $N$, then the preceding formulas for the emission probabilities must be multiplied by $N+1$.

### 27.4. *Selection Rules*

In order that the matrix element of a particular multipole moment should differ from zero, certain conditions arising from the laws of conservation of angular momentum and of parity must be satisfied. These conditions are called selection rules and consist of the following. If $j_1$ and $j_2$ are the angular momenta of the electron, and $m_1$ and $m_2$ are their components, in the initial and the final states, then the following relations must be satisfied:

$$m_1-m_2 = M \quad \text{and} \quad |j_1-j_2| \leqslant L \leqslant j_1+j_2, \tag{27.25}$$

where $L$ is the angular momentum of the photon, and $M$ is its component.

Moreover, the following condition must hold:

$$P_1 = P_2 P, \tag{27.25'}$$

where $P_1$ and $P_2$ are the parities of the initial and the final states of the electron and $P$ is the parity of the photon state given by $P = (-1)^{L+1+\lambda}$ ($\lambda = 0$ corresponds to states of magnetic type, and $\lambda = 1$ corresponds to states of electric type).

If relations (27.25) and (27.25′) are not satisfied, then the matrix element of the corresponding multipole moment vanishes.

It can be easily shown that for given $j_1$ and $j_2$ the largest emission probability occurs for a photon of angular momentum $L = |j_1-j_2|$, provided this is compatible with the parity selection rule (27.25′). Indeed, since the matrix elements $Q_{LM}^{(\lambda)}$ contain $(\omega r)^L$ in the integrand, with $\omega r \leqslant 1$, the largest value of $Q_{LM}^{(\lambda)}$ will correspond to the smallest possible $L$, i.e., $|j_1-j_2|$.

In accordance with (27.14) and (27.22) the emission probability $\omega_{LM}^{(\lambda)}$ contains the factor $(\omega a)^{2L+1}$. Therefore, if the difference in the angular momenta of the initial and the final states $j_1-j_2$ is large, then the emission probability may turn out to be very small, and the time during which the radiating system remains in the excited state may turn out to be large. Such long-lived excited states are said to be metastable.

Estimates of nuclear multipole moments made in accordance with (27.23) lead to the correct order of magnitude for the lifetime of nuclear metastable states, which for $L = 3$–$5$ can attain values of the order of minutes, hours and longer. Nuclei existing in such states are called *isomers*. Isomers behave like almost stable nuclei, and differ from nuclei in the ground state in a number of their properties, for example in their periods of $\beta$-decay.

### 27.5. *Angular Distribution and Polarization of the Radiation*

The state of a system (for example, of a nucleus) of a given energy $\varepsilon_1$ and angular momentum $j_1$ is degenerate with respect to the quantum number specifying the component of angular momentum $m_1$; $\psi_1 = \sum\limits_{m_1} c^{m_1}\psi_{j_1 m_1}$. Therefore, in the general case the state of a nucleus of given $j_1$ is described by the polarization density matrix $\varrho_1$:

$$(m_1|\varrho_1|m_1') = \overline{c^{m_1}c^{m_1'*}}$$

where the bar denotes a certain average (cf. subsection 2.4).

It is convenient to introduce the concept of nuclear *polarization moments*. The name of $l$th order polarization moment is given to the $l$-vector $\mathcal{P}_m^{(l)}$ defined in the following manner:

$$\mathcal{P}_m^{(l)} = (m_1|\varrho_1|m_1')\, C_{j_1 m_1' l m}^{j_1 m_1}, \tag{27.26}$$

where $C_{j_1 m_1' l m}^{j_1 m_1}$ are vector addition coefficients (cf. § 8); it is understood that both here and later summation is carried out over repeated $m$-indices. In particular, $\mathcal{P}^{(0)} = \mathrm{Sp}\,\varrho_1 = 1$. The polarization moment $\mathcal{P}^{(1)}$ is proportional to the expectation value of the angular momentum operator $\bar{\mathrm{J}}$,

$$\mathcal{P}_\alpha^{(1)} = \frac{1}{\sqrt{j_1(j_1+1)}}\,\bar{\mathrm{J}}_\alpha,$$

$\mathcal{P}^{(2)}$ is proportional to the expectation value of the quadrupole moment tensor

$$\mathcal{P}_{\alpha\beta}^{(2)} = \frac{\overline{3\mathrm{J}_\alpha \mathrm{J}_\beta - j_1(j+1)}}{\sqrt{j_1(j_1+1)\,(2j_1-1)\,(2j_1+3)}}.$$

If the density matrix $\varrho_1$ is diagonal $(m_1|\varrho_1|m_1') = \varrho_1(m_1)\,\delta_{m_1 m_1'}$, then the polarization moments have only the one component $\mathcal{P}_0^{(l)}$ different from zero, which we denote by $\mathcal{P}_l$;

$$\mathcal{P}_l = \sum_m \varrho_1(m)\, C_{j_1 m l 0}^{j_1 m}.$$

In particular,

$$\mathcal{P}_1 = \frac{\overline{m}_1}{\sqrt{j_1(j_1+1)}}; \quad \mathcal{P}_2 = \frac{3\overline{m_1^2} - j_1(j_1+1)}{\sqrt{j_1(j_1+1)(2j_1-1)(2j_1+3)}},$$

where $\overline{f(m_1)} = \sum_{m_1} \varrho_1(m_1) f(m_1)$.

The density matrix $(m_1|\varrho|m_1')$ can obviously be expressed in terms of the polarization moments in the following manner:

$$(m_1|\varrho_1|m_1') = \sum_{l=0}^{2j_1} C^{j_1 m_1}_{j_1 m_1' l m} \frac{2l+1}{2j_1+1} \mathcal{P}^{(l)}_m. \tag{27.27}$$

We investigate the radiation from a nucleus whose initial state is specified by the density matrix $\varrho_1$. We denote by $(j_2 m_2 LM|\mathrm{S}|j_1 m_1)$ the element of the scattering matrix corresponding to the transition of the nucleus from the state $j_1$, $m_1$ to the state $j_2$, $m_2$ accompanied by the emission of a photon whose angular momentum is defined by the quantum numbers $L$, $M$. We assume that the matrix element $(j_2 m_2 LM|\mathrm{S}|j_1 m_1)$ is normalized so that the probability of the transition $j_1 m_1 \to j_2 m_2$ is equal to the square of its absolute value. On comparing this with (27.13) we have

$$(j_2 m_2 LM|\mathrm{S}|j_1 m_1) = \sqrt{2R}\, U_{j_1 m_1 \to j_2 m_2}.$$

A complete analysis of the emission process can be given by introducing the matrix

$$w = \mathrm{S}\varrho_1 \mathrm{S}^+,$$

i.e.,

$$(m_2' M'|w|m_2 M) = (j_2 m_2 LM|\mathrm{S}|j_1 m_1)(m_1|\varrho_1|m_1')(j_2 m_2' LM'|\mathrm{S}|j_1 m_1')^*. \tag{27.28}$$

If we know the matrix $w$ we can obtain, firstly, the transition probability or the mean lifetime, $\tau$, of the excited state

$$\frac{1}{\tau} = \mathrm{Sp}\, w \tag{27.29}$$

and, secondly, the density matrix for the final state

$$\varrho_2 = \tau w. \tag{27.30}$$

We can easily obtain the dependence of the matrix element $(j_2 m_2 LM|\mathrm{S}|j_1 m_1)$ on the quantum numbers $m_1$, $m_2$, $M$. In accordance with (27.11), (27.21) it is proportional to the quantity $\int \psi^*_{j_2m_2} Q^{(\lambda)}_{LM} \psi_{j_1m_1} d\boldsymbol{r}$. Since the wave function $\psi_{jm}$ is a component of a $j$-vector, while the multipole moment $Q^{(\lambda)}_{LM}$ is a component of an $L$-vector we have (cf. § 8)

$$(j_2m_2LM|\mathrm{S}|j_1m_1) = QC^{j_1m_1}_{j_2m_2LM}, \tag{27.31}$$

where $Q$ is a quantity independent of $m_1$, $m_2$, and $M$. On substituting (27.31) into (27.28), we obtain

$$(m_2'M'|w|m_2M) = |Q|^2(m_1|\varrho_1|m_1')C^{j_1m_1}_{j_2m_2LM}C^{j_1m_1'}_{j_2m_2'LM'}. \tag{27.32}$$

The substitution of (27.32) into (27.29) yields

$$\frac{1}{\tau} = |Q|^2. \tag{27.33}$$

Thus, the lifetime of the excited state is independent of its polarization. On comparing (27.33) with (27.14) or (27.22) we see that $\sum\limits_{M} w^{(\lambda)}_{LM} = |Q|^2$ and does not depend on $m_1$.

The density matrix $\varrho_2$ describes the state of the nucleus-photon system. It is given by expressions (27.30), (27.32) in the representation in which the photon state is described by definite values of angular momentum and parity, and in accordance with (27.32) $\varrho_2$ does not depend on the photon parity. It is convenient to go over to the represnetation in which the photon states are characterized by a definite momentum $\boldsymbol{k}$ (i.e., for a given energy characterized by a definite direction of motion $\boldsymbol{n} = \boldsymbol{k}/k$) and polarization $\alpha$. On utilizing the expansion of spherical waves in terms of plane waves (subsection 4.4) we obtain

$$(m_2'\boldsymbol{n}'\alpha'|\varrho_2|m_2\boldsymbol{n}\alpha) = (m_2'M'|\varrho_2|m_2M)\left(\boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n})\right)_\alpha\left(\boldsymbol{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n}')\right)_{\alpha'},$$

or in virtue of (27.30) and (27.32)

$$(m_2'\boldsymbol{n}'\alpha'|\varrho_2|m_2\boldsymbol{n}\alpha) = (m_1|\varrho_1|m_1')C^{j_1m_1}_{j_2m_2LM}C^{j_1m_1'}_{j_2m_2'LM'}\left(\boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n})\right)_\alpha\left(\boldsymbol{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n}')\right)_{\alpha'}. \tag{27.34}$$

We now determine the angular distribution of the radiation $W(\boldsymbol{n})$. It is expressed in terms of the matrix $\varrho_2$ in the following manner:

$$W(\boldsymbol{n}) = \sum_{m_2\alpha}(m_2\boldsymbol{n}\alpha|\varrho_2|m_2\boldsymbol{n}\alpha)$$

or in virtue of (27.34) it is given by

$$W(\boldsymbol{n}) = (m_1|\varrho_1|m_1')C^{j_1m_1}_{j_2m_2LM}C^{j_1m_1'}_{j_2m_2'LM'}\boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n})\boldsymbol{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n}). \tag{27.35}$$

On writing $\varrho_1$ in the form (27.27), and on carrying out the summation over $m_1$, $m_1'$, and $m_2$ in accordance with (8.10), we obtain

$$W(\boldsymbol{n}) = \sum_l \alpha_l C^{LM}_{LM'lm} \mathcal{P}^{(l)}_m \boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n}) \boldsymbol{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n}), \tag{27.36}$$

where

$$\alpha_l = \frac{(2l+1)(2j_1+1)^{1/2}}{(2L+1)^{1/2}} W(LL, j_1 j_1; lj_2) \tag{27.36'}$$

and $W(LL, j_1 j_1; lj_2)$ are the Racah coefficients.

Further, on making use of the expansion of a product of vector spherical harmonics in terms of spherical harmonics (8.13′) we obtain

$$W(\boldsymbol{n}) = \sum_l \beta_l \mathcal{P}^{(l)}_m Y_{lm}(\boldsymbol{n}), \tag{27.37}$$

where

$$\beta_l = \alpha_l \sqrt{\frac{2l+1}{4\pi}} (2L+1) C^{L0}_{L0l0} W(LL, LL; l1).$$

If the matrix $\varrho_1$ is diagonal, then (27.37) is an expansion in terms of Legendre polynomials

$$W(\boldsymbol{n}) = \sum_l \beta_l \mathcal{P}_l \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\vartheta). \tag{27.38}$$

Since in the expansion of $\boldsymbol{Y}^{(\lambda)}_{LM} \boldsymbol{Y}^{(\lambda)*}_{LM'}$ in terms of $Y_{lm}$ $l$ are integers, then only polarization moments of even order can affect the angular distribution. The angular distribution is invariant under a replacement of $\boldsymbol{n}$ by $-\boldsymbol{n}$ which is a consequence of the law of conservation of parity in quantum electrodynamics. The even integer $l$ must, further, in consequence of (27.27) and (8.10) satisfy the two conditions $(l/2) \leqslant j_1$, and $(l/2) \leqslant L$. Therefore, in particular, the radiation from a nucleus of spin 0 or $\frac{1}{2}$ is always isotropic. If $j_1 = 1, 3/2$ or $L = 1$, then

$$W(\boldsymbol{n}) = \frac{1}{4\pi} + \beta_2 \mathcal{P}^{(2)}_m Y_{2m}(\boldsymbol{n}).$$

We now consider the photon polarization. The polarization density matrix $\varrho_{\alpha\alpha'}$ for a photon emitted in the direction $\boldsymbol{n}$ is given by

$$\varrho_{\alpha\alpha'} = \frac{1}{W(\boldsymbol{n})} \sum_{m_2} (m_2 \boldsymbol{n} \alpha | \varrho_2 | m_2 \boldsymbol{n} \alpha').$$

On utilizing (27.34) and (27.36′) we obtain

$$\varrho_{\alpha\alpha'} = \frac{1}{W(\boldsymbol{n})} \sum_{l'} \alpha_{l'} C^{LM}_{LM'l'm} \mathcal{P}^{(l')}_m \left(\boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n})\right)_\alpha \left(\boldsymbol{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n})\right)_{\alpha'}.$$

We now obtain the circular polarization of the photon (cf. subsection 2.4). In order to do this we take $\alpha$ and $\alpha'$ to be the Cartesian axes in the plane perpendicular to $\boldsymbol{n}$. On applying formula (2.15) we obtain

$$\xi_2 = \frac{i}{W(\boldsymbol{n})} \sum_{l'} a_{l'} C^{LM}_{LM'l'm} \mathcal{P}^{(l')}_m \; [\boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n}) \mathbf{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n})]\boldsymbol{n}. \qquad (27.39)$$

Since $-i[\boldsymbol{Y}^{(\lambda)}_{LM}(\boldsymbol{n})\boldsymbol{Y}^{(\lambda)*}_{LM'}(\boldsymbol{n})]\boldsymbol{n} = \boldsymbol{Y}^{(1-\lambda)}_{LM}(\boldsymbol{n})\boldsymbol{Y}^{(\lambda*)}_{LM'}-(\boldsymbol{n})$, then with the aid of the expansion (8.13′) we can write (27.39) in the form

$$\xi_2 = \frac{1}{W(\boldsymbol{n})} \sum_{l'} \gamma_{l'} \mathcal{P}^{(l')}_m Y_{l'm}(\boldsymbol{n}), \qquad (27.39')$$

where

$$\gamma_{l'} = \sqrt{\frac{2l'+1}{4\pi}} \frac{(2L+1)^{3/2}}{2l'+1} \left\{ \sqrt{L+1}\, C^{L0}_{L-10l'0} W(LL, L-1L; l'1) \right.$$
$$\left. + \sqrt{L}\, C^{L0}_{L+10l'0} W(LL, L+1L; l'1) \right\}.$$

If the density matrix $\varrho_2$ is diagonal, then

$$\xi_2 = \frac{1}{W(\boldsymbol{n})} \sum_{l'} \gamma_{l'} \mathcal{P}_{l'} \sqrt{\frac{2l'+1}{4\pi}} P_{l'}(\cos\vartheta). \qquad (27.39'')$$

In particular, if only the first order polarization moment differs from zero, then

$$\xi_2 = \frac{(2j_1+1)^{3/2}(2L+1)^{1/2}}{\sqrt{L(L+1)}} W(LL, j_1 j_1; 1 j_2)\, \mathcal{P}_1 \cos\vartheta.$$

In the expansions (27.39) $l'$ is an odd integer satisfying the inequality $l' \leqslant 2j_1$ and $l' \leqslant 2L+1$. In contrast to the angular distribution the circular polarization is determined by the polarization moments of odd order. Therefore, only a measurement both of the angular distribution and the circular polarization of the radiation completely determines the polarization state of the nucleus.

We now proceed to the investigation of the polarization of the final state, i.e., of the state of the nucleus after emission has taken place. It is determined by the density matrix

$$(m_2'|\varrho_2|m_2) = \int do\,(m_2'\boldsymbol{n}\alpha|\varrho_2|m_2\boldsymbol{n}\alpha) = (m_2'M|\varrho_2|m_2M),$$

i.e., by taking the average of the density matrix (27.30) over the photon states. On utilizing (27.32) or (27.34) we obtain

$$(m_2|\varrho_2|m_2') = (m_1|\varrho_1|m_1')\, C^{j_1m_1}_{j_2m_2LM}\, C^{j_1m_1'}_{j_2m_2'LM}. \qquad (27.40)$$

In accordance with (27.26) formula (27.40) gives the following relation between the polarization moments $\mathcal{P}^{(l)}_{\mathrm{I}m}$ of the initial and $\mathcal{P}^{(l)}_{\mathrm{II}m}$ of the final states:

$$\mathcal{P}^{(l)}_{\mathrm{II}m} = (2j_1+1)^{1/2}\,(2j_2+1)^{1/2}\, W(j_1 j_1, j_2 j_2;\, lL)\,\mathcal{P}^{(l)}_{\mathrm{I}m}. \tag{27.41}$$

If the initial state is unpolarized, then the final state will evidently also be unpolarized. However, in this case a correlation of polarization occurs, i.e., for a given direction of propagation of the emitted photon the nucleus becomes polarized after emitting the photon. Such a correlation of polarization is defined by the density matrix

$$(m_2|\varrho_2(\boldsymbol{n})|m_2') = \frac{1}{W(\boldsymbol{n})}\,(m_2\boldsymbol{n}\alpha|\varrho_2|m_2'\boldsymbol{n}\alpha).$$

On substituting into this equation expression (27.34) and on setting

$$(m_1|\varrho_1|m_1') = \frac{1}{2j_1+1}\,\delta_{m_1m_1'},$$

we obtain, after utilizing the expansion (8.13),

$$(m_2|\varrho_2(\boldsymbol{n})|m_2') = \sum_l a_l C^{j_2m_2}_{j_2m_2'lm}\, Y_{lm}(\boldsymbol{n}), \tag{27.42}$$

where

$$a_l = \frac{(2L+1)^{3/2}}{(2j_2+1)^{1/2}}\sqrt{\frac{2l+1}{4\pi}}\; W(LL, LL;\, l1)\, W(j_2 j_2, j_1 j_1;\, lL)$$

or

$$\mathcal{P}^{(l)}_{\mathrm{II}m} = a_l\,\frac{2j_2+1}{2l+1}\, Y_{lm}(\boldsymbol{n}). \tag{27.43}$$

If we choose the direction of motion $\boldsymbol{n}$ of the photon as the $z$-axis, then

$$\mathcal{P}_{\mathrm{II}l} = a_l\,\frac{2j_2+1}{\sqrt{4\pi(2l+1)}}. \tag{27.43'}$$

Since in (27.42) $l$ is an even integer, polarization moments of only even order occur with $l \leqslant 2L$ and $l \leqslant 2j_2$.

If the final state of the nucleus is also an excited one, then a subsequent emission of a photon of angular momentum $L'$ may take place with the nucleus making a transition from the state of angular momentum $j_2$ to a state of angular momentum $j_3$. In this case the correlation of polarization leads to an angular correlation between the directions of motion of the two photons emitted in cascade (83).

From (27.43′) and (27.38) we obtain the following formula for the angular correlation $W(\theta)$:

$$W(\theta) = \frac{2j_2+1}{4\pi}\sum_l a_l \beta_l P_l(\cos\theta), \tag{27.44}$$

where $\theta$ is the angle between the directions of motion of the two photons, $a_l$ is the coefficient in (27.42) and $\beta_l$ is the coefficient in (27.37), in which $j_1$ has been replaced by $j_2$, $j_2$ has been replaced by $j_3$ and $L$ has been replaced by $L'$.

We give the values for the coefficients in the expansion (27.44) in some frequently occurring cases. We rewrite (27.44) in the form (71), (127):

$$W(\theta) = \sum_{n=0}^{N} A_{2n}P_{2n}(\cos\theta), \tag{27.44′}$$

$$A_{2n} = (2n+1)b_{2n}(L)b_{2n}(L')t_{2n},$$

where

$$b_k(L) = \left(1-\frac{k(k+1)}{2L(L+1)}\right)\frac{(2L+1)!\,k!\,(L+k/2)!}{(2L+k+1)!\,(k/2)!\,(L-k/2)!},$$

$$t_k = 1 \quad \text{for} \quad j_1 = j_2 \pm L, j_3 = j_2 \mp L',$$

$$t_k = \frac{(2j_2-k)!\,(2j_2+k+1)!}{(2j_2)!\,(2j_2+1)!} \quad \text{for} \quad j_1 = j_2 - L, j_3 = j_2 - L',$$

$$t_k = \frac{(2j_2)!\,(2j_2+1)!}{(2j_2-k)!\,(2j_2+k+1)!} \quad \text{for} \quad j_1 = j_2 + L, j_3 = j_2 + L',$$

$$t_2 = -\frac{2j_2+3}{j_2} \quad \text{for} \quad j_1 = 1, j_3 = j_2 - L',$$

$$t_2 = -\frac{2j_2-1}{j_2+1} \quad \text{for} \quad j_1 = 1, j_3 = j_2 + L',$$

$$t_2 = \frac{(2j_2-1)(2j_2+3)}{j_2(j_2+1)} \quad \text{for} \quad j_1 = j_3 = 1.$$

The phenomenon of angular correlation of $\gamma$-rays is one of the experimental methods for the determination of nuclear moments. In order to be able to observe this phenomenon, it is necessary that the transition probability from the intermediate state $j_2$ into the final state $j_3$ should be sufficiently great, otherwise the polarization of the intermediate state will be destroyed by the interaction with the atomic electron shells. It is obvious that this condition reduces to the inequality $1/\tau \gg \nu$, where $1/\tau$ is the probability of the transition $j_2 \to j_3$, while $\nu$ is the magnitude of the hyperfine splitting of the atomic levels.

## § 28. Scattering of a Photon by a Free Electron

### 28.1. *Scattering Matrix Element*

In § 27 we have shown that a free electron cannot emit or absorb a photon, i.e., that first order processes are impossible for a free electron. Therefore, the simplest processes arising from the interaction between photons and free electrons are described by the second order scattering matrix, which in accordance with (24.14) has the form

$$\mathrm{S}^{(2)} = \frac{e^2}{2}\int T[N(\bar{\psi}(x_1)\hat{\mathrm{A}}(x_1)\psi(x_1))N(\bar{\psi}(x_2)\hat{\mathrm{A}}(x_2)\psi(x_2))]d^4x_1 d^4x_2.$$

Since the matrix $\mathrm{S}^{(2)}$ contains the operator for the potential $\mathrm{A}_\mu(x)$ bilinearly, its elements describe those processes in which the total number of photon states is equal either to two or to zero. The latter case refers to the interaction between electrons with no photons taking part, and will be discussed in Chapter VI.

The matrix elements of $\mathrm{S}^{(2)}$ involving two photon states may be divided into three kinds, corresponding to the total number of electront in the initial and the final states. This number may be equal to four, two, or zero. The matrix elements with four electron states are simply products of first order matrix elements of $\mathrm{S}^{(1)}$ (cf. Fig. 13, 1). The matrix element which does not involve any electron states determines the "proper mass" of the photon (cf. Fig. 13, 4) and will be discussed in § 44.

Matrix elements involving two electron states are of the greatest physical interest. Processes involving two photon and two electron states are described by that part of the second order scattering matrix which is of the form

$$\mathrm{S}^{(2)} = e^2 N\int \bar{\psi}(x_2)\hat{\mathrm{A}}(x_2)S^c(x_2-x_1)\hat{\mathrm{A}}(x_1)\psi(x_1)d^4x_1 d^4x_2, \qquad (28.1)$$

where the function $S^c(x)$ is defined by formulas (19.16), (25.2). The simplest process described by the matrix $\mathrm{S}^{(2)}$ is the scattering of a photon by a free electron, the detailed investigation of which we now undertake.

We denote by $\psi_1(x)$ and $\psi_2(x)$ the wave functions of the initial and final electron states and by $A_1(x)$ and $A_2(x)$ the potentials of the initial and final photon states. We obtain the element of the matrix $S^{(2)}$ corresponding to photon-electron scattering if we replace in (28.1) the operators $\overline{\psi}(x_2)$ and $\psi(x_1)$ by the functions $\overline{\psi}_2(x_2)$ and $\psi_1(x_1)$, and the operator $\hat{\mathrm{A}}(x_2)\hat{\mathrm{A}}(x_1)$ by the function $\hat{A}_2^*(x_2)\ \hat{A}_1(x_1)+\hat{A}_1(x_2)\hat{A}_2^*(x_1)$:

$$S_{i\to f}^{(2)}=e^2\int\overline{\psi}_2(x_2)[\hat{A}_2^*(x_2)S^c(x_2-x_1)\hat{A}_1(x_1)$$

$$+\hat{A}_1(x_2)S^c(x_2-x_1)\hat{A}_2^*(x_1)]\psi_1(x_1)d^4x_1\,d^4x_2. \qquad (28.2)$$

We choose the functions $\psi_r(x)$ and $A_r(x)$ $(r=1,2)$ in the form of plane waves, normalized per unit volume:

$$\psi_r(x)=u_r e^{ip_r x} \quad \text{and} \quad A_r(x)=\frac{e_r}{\sqrt{2\omega_r}}e^{ik_r x},$$

where $p_r$ and $k_r$ are the four-momenta of the electron and the photon, $\omega_r=|k_r|$ is the photon frequency, $u_r$ and $e_r$ are unit (spinor and vector) amplitudes.

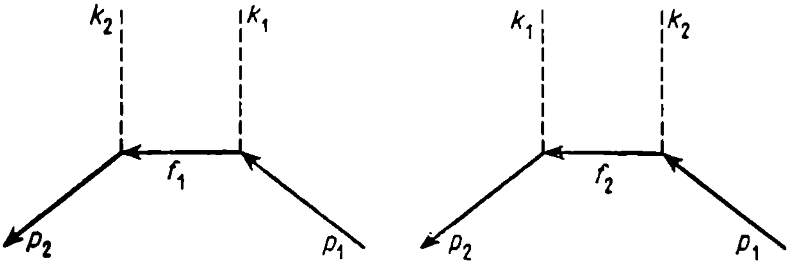


Fig. 21.

On substituting these functions and the expression (25.2) for $S^c(x)$ into (28.2) we obtain the following result which is in accordance with the general rules of subsection 25.5:

$$S_{i\to f}^{(2)}=\frac{ie^2}{2\sqrt{\omega_1\omega_2}}\overline{u}_2\left\{\hat{e}_2\frac{i\hat{f}_1-m}{f_1^2+m^2}\hat{e}_1+\hat{e}_1\frac{i\hat{f}_2-m}{f_2^2+m_2}\hat{e}_2\right\}u_1(2\pi)^4\,\delta(p_1+k_1-p_2-k_2), \qquad (28.3)$$

where $f_1=p_1+k_1=p_2+k_2$ and $f_2=p_1-k_2=p_2-k_1$. Figure 21 shows two diagrams corresponding to the two terms of the matrix element $S_{i\to f}^{(2)}$.

### 28.2. *Application of Conservation Laws*

The matrix element $S^{(2)}_{i\to f}$ differs from zero only when the laws of conservation of energy and momentum are satisfied:

$$p_1+k_1=p_2+k_2. \tag{28.4}$$

For given momenta $\boldsymbol{p}_1$ and $\boldsymbol{k}_1$ these laws enable us to obtain four of the six components of the momenta of the final state; if, in addition, we specify the direction of $\boldsymbol{k}_2$ or of $\boldsymbol{p}_2$, then we can completely determine $\boldsymbol{k}_2$ and $\boldsymbol{p}_2$.

We now obtain the dependence of the frequency of the scattered photon $\omega_2$ on the direction of its momentum. On squaring (28.4) $p_1^2+k_1^2+2p_1k_1=p_2^2+k_2^2+2p_2k_2$ and on noting that

$$p_1^2=p_2^2=-m^2 \text{ and } k_1^2=k_2^2=0,$$

we obtain $p_1k_1=p_2k_2$ or $p_1k_1=p_1k_2+k_1k_2$, whence it follows that

$$\omega_1(1-v_1\cos\vartheta_1)=\omega_2(1-v_1\cos\vartheta_2)+\frac{\omega_1\omega_2}{\varepsilon_1}(1-\cos\vartheta), \tag{28.5}$$

where $v_1=|\boldsymbol{p}_1|/\varepsilon_1$ is the initial electron velocity, $\varepsilon_1$ is the initial energy, $\vartheta_1$ and $\vartheta_2$ are the angles between the momenta of the primary and the scattered photons and the initial electron momentum, $\vartheta$ is the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$.

In the case of the scattering of a photon by an electron at rest ($v_1=0$, $\varepsilon_1=m$) we obtain from (28.5) the well-known Compton formula

$$\omega_2=\frac{\omega_1}{1+(\omega_1/m)(1-\cos\vartheta)}. \tag{28.6}$$

The quantities $f_1$ and $f_2$ appearing in $S^{(2)}_{i\to f}$ are the four-momenta of the virtual electrons for which the usual relation between the energy and the momentum does not hold. The difference between a virtual electron state and a real one can be characterized by the quantity

$$\varkappa_r=\frac{f_r^2+m^2}{m^2} \quad \text{where} \quad r=1,2, \tag{28.7}$$

which vanishes in the case of a real state. It can be easily shown that $m^2\varkappa_1=2p_1k_1=2p_2k_2$ and $m^2\varkappa_2=-2p_1k_2=-2p_2k_1$. If the electron was at rest before the collision, then

$$\varkappa_1=-\frac{2\omega_1}{m} \quad \text{and} \quad \varkappa_2=\frac{2\omega_2}{m}. \tag{28.7'}$$

### 28.3. *Differential Cross Section for Unpolarized Particles*

In accordance with the general rules of § 26 the differential cross section for scattering in which the momenta of the final states of the electron and of the photon lie respectively within the intervals $dp_2$ and $dk_2$, is given by

$$d\sigma = \frac{e^4}{4\omega_1\omega_2}|\bar{u}_2 Q u_1|^2 \frac{dp_2 dk_2}{J(2\pi)^2}\delta(p_1+k_1-p_2-k_2)\delta(\varepsilon_1+\omega_1-\varepsilon_2-\omega_2), \tag{28.8}$$

where $Q$ is the matrix appearing in the figure brackets of (28.3):

$$Q = \frac{1}{m^2\varkappa_1}\hat{e}_2(i\hat{f}_1-m)\hat{e}_1+\frac{1}{m^2\varkappa_2}\hat{e}_1(i\hat{f}_2-m)\hat{e}_2, \tag{28.9}$$

and $J$ is the flux density of the colliding particles. In accordance with (26.5′):

$$J = \tilde{v} = k_1 p_1/\varepsilon_1\omega_1 = m^2\varkappa_1/2\varepsilon_1\omega_1. \tag{28.10}$$

The presence in (28.8) of $\delta$-functions enables us to carry out the integration over $dp_2$ and $d\omega_2$. In order to do this it is sufficient merely to perform the substitution

$$\delta(p_1+k_1-p_2-k_2)\,dp_2 \to 1,$$

$$\delta(\varepsilon_1+\omega_1-\varepsilon_2-\omega_2)\,dk_2 \to \frac{\omega_2^2\,do_2}{|(\partial/\partial\omega_2)(\omega_2+\varepsilon_2)|},$$

where $do_2$ is the element of solid angle containing the vector $k_2$. In the last expression $\varepsilon_2$ is a function of $\omega_2$, since the momentum $p_2$ is related to $k_2$ by the conservation law

$$\begin{aligned}\varepsilon_2 &= \sqrt{m^2+(p_1+k_1-k_2)^2}\\ &= \sqrt{\varepsilon_1^2+\omega_1^2+2p_1\omega_1\cos\vartheta_1+\omega_2^2-2p_1\omega_2\cos\vartheta_2-2\omega_1\omega_2\cos\vartheta}.\end{aligned}$$

From this it follows that

$$\begin{aligned}\frac{\partial}{\partial\omega_2}(\omega_2+\varepsilon_2) &= 1+\frac{\omega_2-p_1\cos\vartheta_1-\omega_1\cos\vartheta}{\varepsilon_2} = 1+\frac{k_2(k_2-p_1-k_1)}{\omega_2\varepsilon_2}\\ &= 1-\frac{p_2k_2}{\omega_2\varepsilon_2} = -\frac{p_2k_2}{\varepsilon_2\omega_2} = -\frac{m^2\varkappa_1}{2\varepsilon_2\omega_2}\end{aligned} \tag{28.11}$$

and therefore

$$d\sigma = 4r_0^2\frac{\omega_2^2}{m^2}\frac{\varepsilon_1\varepsilon_2}{\varkappa_1^2}|\bar{u}_2 Q u_1|^2\,do_2, \tag{28.12}$$

where $r_0 = e^2/4\pi m$ is the classical electron "radius" $(e/\sqrt{4\pi})$, is the electron charge in Gaussian units.

If the initial electron state is unpolarized and we are interested in the result of scattering without reference to the polarization of the electron in the final state we must sum expression (28.12) over the final values, and average it over the initial values of the electron spin component. For the sake of brevity we denote this quantity, i.e., $\frac{1}{2}\Sigma_{\mu_1,\mu_2} d\sigma$ ($\mu_1$ and $\mu_2$ are the values of the electron spin components in the initial and the final states), as before, by $d\sigma$. In accordance with (26.7) such an averaged value of the differential cross section is equal to

$$d\sigma = \frac{r_0^2}{2}\frac{\omega_2^2}{m^2}\frac{1}{\varkappa_1^2}\,\mathrm{Sp}\,\{Q(i\hat{p}_1-m)\bar{Q}(i\hat{p}_2-m)\}\,do_2, \qquad (28.13)$$

where

$$\bar{Q} = \gamma_4 Q^{+}\gamma_4.$$

On substituting into this expression the value of $Q$, and on noting that for any four-vector $q$ whose spatial components are real and whose fourth component is imaginary, $\bar{\hat{q}} = -\hat{q}$, we obtain

$$\begin{aligned} &Q(i\hat{p}_1-m)\bar{Q}(i\hat{p}_2-m)\\ &\quad = \left[\frac{1}{m^2\varkappa_1}\hat{e}_2(i\hat{f}_1-m)\hat{e}_1+\frac{1}{m^2\varkappa_2}\hat{e}_1(i\hat{f}_2-m)\hat{e}_2\right](i\hat{p}_1-m)\\ &\quad \times\left[\frac{1}{m^2\varkappa_1}\hat{e}_1(i\hat{f}_1-m)\hat{e}_2+\frac{1}{m^2\varkappa_2}\hat{e}_2(i\hat{f}_2-m)\hat{e}_1\right](i\hat{p}_2-m). \qquad (28.14)\end{aligned}$$

If the primary photon is also unpolarized, we obtain the scattering cross section, irrespective of the polarization of the scattered photon, by summing (28.13) over the final polarization states, and by averaging it over the initial polarization states of the photons. We denote this quantity, i.e., $\frac{1}{2}\Sigma_{\nu_1,\nu_2} d\sigma$, where $\nu_1$ and $\nu_2$ are the photon polarizations, also by $d\sigma$.

The evaluation of this averaged cross section is considerably simplified, since the summation may be carried out not over two, but over four photon polarizations, including both the "longitudinal" and the "scalar" polarizations (cf. subsection 26.3). In doing this we must simply replace in expression (28.14) $\hat{e}_1$ by $\gamma_\nu$, $\hat{e}_2$ by $\gamma_\lambda$ and sum over $\nu$ and $\lambda$ $(\nu, \lambda = 1, 2, 3, 4)$.

Thus, we obtain

$$d\sigma = r_0^2 \frac{\omega_2^2}{m^2\varkappa_1^2}\, do_2 \frac{1}{4}\,\mathrm{Sp}\,F, \tag{28.15}$$

where

$$\begin{aligned} \mathrm{Sp}\,F = \mathrm{Sp}\Bigg\{&\left[\gamma_\lambda \frac{i\hat{f}_1 - m}{m^2\varkappa_1}\gamma_\nu + \gamma_\nu \frac{i\hat{f}_2 - m}{m^2\varkappa_2}\gamma_\lambda\right] \\ &\times (i\hat{p}_1 - m)\gamma_\nu \frac{i\hat{f}_1 - m}{m^2\varkappa_1}\gamma_\lambda (i\hat{p}_2 - m)\Bigg\} \\ +\mathrm{Sp}\Bigg\{&\left[\gamma_\lambda \frac{i\hat{f}_1 - m}{m^2\varkappa_1}\gamma_\nu + \gamma_\nu \frac{i\hat{f}_2 - m}{m^2\varkappa_2}\gamma_\lambda\right] \\ &\times (i\hat{p}_1 - m)\gamma_\lambda \frac{i\hat{f}_2 - m}{m^2\varkappa_2}\gamma_\nu (i\hat{p}_2 - m)\Bigg\}. \end{aligned} \tag{28.15'}$$

It can be easily shown that the second term in this expression is obtained from the first one by means of the substitution $k_1 \to -k_2$, $k_2 \to -k_1$ which also corresponds to the following substitution $f_1 \to f_2, f_2 \to f_1, \varkappa_1 \to \varkappa_2, \varkappa_2 \to \varkappa_1$. Obviously, each of the terms is a function of only the two invariants $\varkappa_1$ and $\varkappa_2$. Therefore, we can write Sp$F$ in the following form:

$$\begin{aligned} \mathrm{Sp}F &= P(\varkappa_1, \varkappa_2) + P(\varkappa_2, \varkappa_1), \\ P(\varkappa_1, \varkappa_2) &= h_1(\varkappa_1, \varkappa_2) + h_2(\varkappa_1, \varkappa_2), \end{aligned} \tag{28.16}$$

where

$$h_1(\varkappa_1, \varkappa_2) = \frac{1}{m^4\varkappa_1^2}\,\mathrm{Sp}\{\gamma_\lambda(i\hat{f}_1 - m)\gamma_\nu(i\hat{p}_1 - m)\gamma_\nu(i\hat{f}_1 - m)\gamma_\lambda(i\hat{p}_2 - m)\},$$

$$h_2(\varkappa_1, \varkappa_2) = \frac{1}{m^4\varkappa_1\varkappa_2}\,\mathrm{Sp}\{\gamma_\nu(i\hat{f}_2 - m)\gamma_\lambda(i\hat{p}_1 - m)\gamma_\nu(i\hat{f}_1 - m)\gamma_\lambda(i\hat{p}_2 - m)\}.$$

On carrying out in $h_1(\varkappa_1, \varkappa_2)$ the summation over $\nu$ and $\lambda$ with the aid of formula (26.14), and on dropping the terms with an odd number of matrices $\lambda$, we obtain

$$h_1(\varkappa_1, \varkappa_2) = \frac{4}{m^4\varkappa_1^2}\,\mathrm{Sp}\{\hat{f}_1\hat{p}_1\hat{f}_1\hat{p}_2 + 4m^2(\hat{f}_1\hat{p}_1 + \hat{f}_1\hat{p}_2 - \hat{f}_1\hat{f}_1) - m^2\hat{p}_1\hat{p}_2 + 4m^4\}.$$

From here we can easily obtain with the aid of formulas (26.15) (26.16)

$$h_1(\varkappa_1, \varkappa_2) = \frac{8}{\varkappa_1^2}(4 - 2\varkappa_1 - \varkappa_1\varkappa_2).$$

In an analogous manner we can show that

$$h_2(\varkappa_1, \varkappa_2) = \frac{4}{\varkappa_1 \varkappa_2}(4-\varkappa_1-\varkappa_2).$$

Therefore

$$\frac{1}{8}\,\mathrm{Sp}F \equiv U_0 = 4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2 - 4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right) - \left(\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right), \qquad (28.16')$$

and we finally obtain the following expression for the averaged cross section $d\sigma$:

$$d\sigma = r_0^2 \frac{2}{m^2 \varkappa_1^2} U_0 \omega_2^2 do_2. \qquad (28.17)$$

This expression confirms that $d\sigma$ is relativistically invariant. Indeed, $\varkappa_1$ and $U_0$ are invariants, and the quantity $\omega^2 do$ which transforms like $\omega d\omega do = d^3k/\omega = 2\delta(k^2)d^4k$, is also an invariant.

If the electron was initially at rest, i.e. $p_1 = 0$, then on using (28.6) and (28.7′) we obtain $U_0 = (\omega_1/\omega_2)+(\omega_2/\omega_1)-\sin^2\vartheta$, and the scattering cross section assumes the form (107), (193)

$$d\sigma = \frac{r_0^2}{2}\left(\frac{\omega_2}{\omega_1}\right)^2\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-\sin^2\vartheta\right)do_2. \qquad (28.18)$$

We note that the scattering cross section for the photon contains the mass of the scatterer $m$ in the denominator. Therefore as $m \to \infty$ no scattering occurs. However, this conclusion is not exact, since it was obtained only by investigating the second order scattering matrix. It turns out that the fourth order scattering matrix already contains elements corresponding to the scattering of a photon by an infinitely heavy charge. The phenomenon of coherent scattering of a photon by a nucleus corresponding to these matrix elements will be investigated in § 55.

### 28.4. *Angular Distribution and Total Cross Section*

By utilizing expression (28.6) which determines $\omega_2$ as a function of $\omega_1$ and $\vartheta$ we can express the differential scattering cross section for unpolarized photons in terms of the frequency of the incident photon and the scattering angle for the photon

$$d\sigma = \frac{r_0^2}{2}\frac{1+\cos^2\vartheta}{[1+(\omega_1/m)(1-\cos\vartheta)]^2}\left[1+\frac{(\omega_1/m)^2(1-\cos\vartheta)^2}{(1+\cos^2\vartheta)[1+(\omega_1/m)(1-\cos\vartheta)]}\right]do_2. \qquad (28.19)$$

The angular distribution of the unpolarized photons given by this formula is shown in Fig. 22 for different values of the parameter $\gamma = \omega_1/m$.

At low photon energies $\omega_1 \ll m$ formula (28.19) reduces to the classic Thomson formula

$$d\sigma = \frac{r_0^2}{2}(1+\cos^2\vartheta)\,do_2, \quad \text{where} \quad \omega_1 \ll m. \tag{28.20}$$

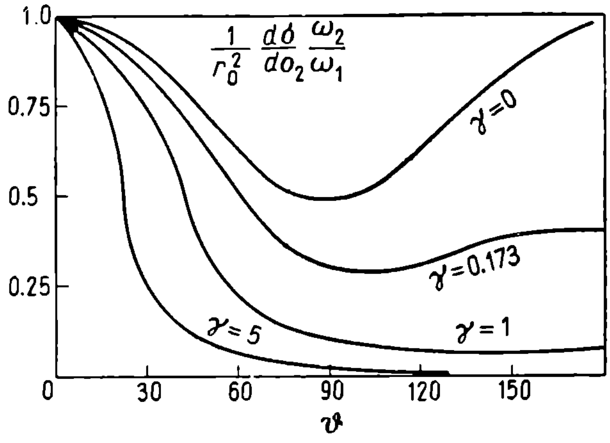


Fig. 22.

If the energy of the primary photon is large compared to the electron rest energy $\omega_1 \gg m$ then from (28.19) we can obtain simple expressions for two limiting cases, when $\vartheta \ll \sqrt{m/\omega_1}$ and when $\vartheta \gg \sqrt{m/\omega_1}$:

$$d\sigma = r_0^2\,do_2, \quad \text{where} \quad \omega_1 \gg m, \ \vartheta \ll \sqrt{\frac{m}{\omega_1}}; \tag{28.21}$$

$$d\sigma = \frac{r_0^2}{2}\frac{m}{\omega_1}\frac{do_2}{1-\cos\vartheta}, \quad \text{where} \quad \omega_1 \gg m, \ \vartheta \gg \sqrt{\frac{m}{\omega_1}}. \tag{28.21'}$$

For $\vartheta = \sqrt{m/\omega_1}(\omega_1 \gg m)$ both these formulas lead to the same results. This shows that for $\omega_1 \gg m$ formula (22.21) gives a good approximation to the angular distribution over the angular interval $0 \leqslant \vartheta \leqslant \sqrt{m/\omega_1}$, while formula (28.21′) gives a good approximation over the angular interval $\vartheta \geqslant \sqrt{m/\omega_1}$.

The differential scattering cross section can also be expressed with the aid of (28.6) in terms of the energy $\omega_2$ of the scattered photon. After integrating over the azimuthal angle we obtain

$$d\sigma = \pi r_0^2 \frac{m\,d\omega_2}{\omega_1^2}\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+\left(\frac{m}{\omega_2}-\frac{m}{\omega_1}\right)^2-2m\left(\frac{1}{\omega_2}-\frac{1}{\omega_1}\right)\right], \tag{28.22}$$

where in accordance with (28.6) $\omega_2$ varies over the range

$$\frac{\omega_1}{1+2(\omega_1/m)} \leqslant \omega_2 \leqslant \omega_1 .$$

Integration of formula (28.19) over the angle $\vartheta$ leads to the following expression for the total scattering cross section:

$$\sigma = \frac{3}{4}\sigma_0\left\{\frac{1+\gamma}{\gamma^3}\left[\frac{2\gamma(1+\gamma)}{1+2\gamma}-\ln(1+2\gamma)\right]+\frac{\ln(1+2\gamma)}{2\gamma}-\frac{1+3\gamma}{(1+2\gamma)^2}\right\}, \tag{28.23}$$

where $\sigma_0 = (8\pi/3)r_0^2$ and $\gamma = (\omega_1/m)$.

For $\omega_1 \ll m$, expansion of (28.23) in powers of $\gamma$ yields

$$\sigma = \sigma_0(1-2\gamma+\ldots), \quad \text{where} \quad \gamma \ll 1, \tag{28.24}$$

i.e., in the nonrelativistic domain the scattering cross section is almost independent of the energy.

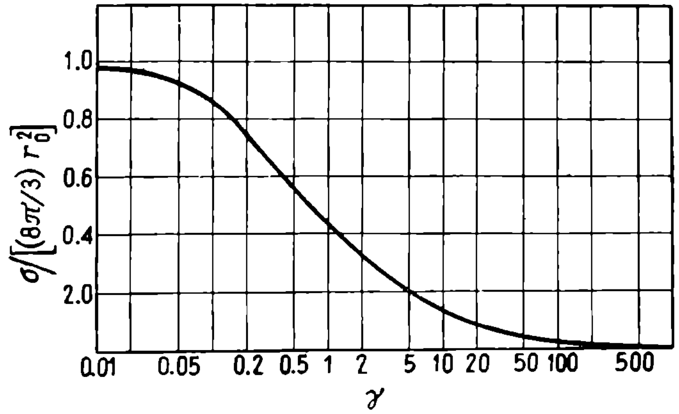


Fig. 23.

In the extreme relativistic domain $\omega_1 \gg m$ the expression for the cross section is greatly simplified:

$$\sigma = \frac{3}{8}\sigma_0\frac{m}{\omega_1}\left(\ln\frac{2\omega_1}{m}+\frac{1}{2}\right), \quad \text{where} \quad \omega_1 \gg m. \tag{28.25}$$

In this region the cross section falls off almost inversely with the photon energy.

Thus, at high energies the number of scattered photons is greatly reduced, as a result of which the penetrating power of $\gamma$-radiation increases with increasing energy.

Fig. 23 shows the dependence of $\sigma/\sigma_0$ on the energy of the primary photon. The values of $\sigma/\sigma_0$ are given in table 9.

TABLE 9

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\gamma$ | 0.05 | 0.1 | 0.2 | 0.33 | 0.5 | 1 | 2 | 3 |
| $\sigma/\sigma_0$ | 0.913 | 0.84 | 0.737 | 0.637 | 0.563 | 0.431 | 0.314 | 0.254 |
| $\gamma$ | 5 | 10 | 20 | 50 | 100 | 200 | 500 | 1000 |
| $\sigma/\sigma_0$ | 19.1 | 12.3 | 7.54 | 3.76 | 2.15 | 1.22 | 0.556 | 0.304 |

## 28.5. *Distribution of Recoil Electrons*

As a result of its collision with the photon the electron which was initially at rest acquires some energy. From the conservation laws it can be easily shown that the energy acquired by it is given by

$$E-m = \frac{\omega_1^2(1-\cos\vartheta)}{m+\omega_1(1-\cos\vartheta)}. \tag{28.26}$$

It varies between zero at $\vartheta = 0$ up to the maximum value $E_{\max}-m = 2\omega_1^2/(m+\omega_1)$ at $\vartheta = \pi$.

The angle $\beta$ between the momentum of the scattered electron and the momentum of the primary photon is related to the angle $\vartheta$ by

$$\cos\beta = (1+\gamma)\sqrt{\frac{1-\cos\vartheta}{2+\gamma(\gamma+2)(1-\cos\vartheta)}} \tag{28.27}$$

and varies from $\pi/2$ at $\vartheta = 0$ to zero at $\vartheta = \pi$.

By utilizing this relation we can express $d\sigma$ in terms of $\beta$. Thus, we obtain angular distribution of the recoil electrons

$$d\sigma = 4r_0^2\frac{(1+\gamma)^2\cos\beta\, do_e}{[1+2\gamma+\gamma^2\sin^2\beta]^2}$$
$$\times\left\{1+\frac{2\gamma^2\cos^4\beta}{[1+2\gamma+\gamma^2\sin^2\beta][1+\gamma(\gamma+2)\sin^2\beta]}-\frac{2(1+\gamma)^2\sin^2\beta\cos^2\beta}{[1+\gamma(\gamma+2)\sin^2\beta]^2}\right\}, \tag{28.28}$$

where $do_e$ is the element of solid angle containing the momentum of the scattered electron.

In the extreme relativistic case $\gamma \gg 1$ this formula yields

$$\begin{aligned} d\sigma &= \frac{\gamma r_0^2\, do_e}{\gamma^2\beta^2+1}, \quad \text{where} \quad \gamma \gg 1 \quad \text{and} \quad \gamma\beta^2 \ll 1; \\ d\sigma &= \frac{4r_0^2}{\gamma^2}\frac{\cos\beta\, do_e}{\sin^4\beta}, \quad \text{where} \quad \gamma \gg 1 \quad \text{and} \quad \gamma \sin^2\beta \gg 1. \end{aligned} \tag{28.29}$$

### 28.6. *Scattering of Polarized Photons*

We now investigate the scattering of polarized photons by unpolarized electrons. We describe the polarization state of the incident photon by the density matrix (cf. subsection 2.4)

$$\varrho_{\alpha\beta}^{(1)} = \tfrac{1}{2}(\delta_{\alpha\beta}+\xi_j^{(1)}\tau_{j,\alpha\beta}), \tag{28.30}$$

where $\tau_j$ are the Pauli matrices and $\xi_j^{(1)} (j = 1, 2, 3)$ are the Stokes parameters. By utilizing the general formulas of subsection 26.3 we obtain in place of (28.12) and (28.13) the following expression for the differential cross section:

$$d\sigma(\boldsymbol{\xi}^{(1)}) = \frac{1}{2} r_0^2 \frac{\omega_2^2}{m^2\varkappa_1^2} \operatorname{Sp}\{Q_{\alpha\mu}(i\hat{p}_1-m)\varrho_{\alpha\beta}^{(1)}\overline{Q}_{\beta\mu}(i\hat{p}_2-m)\}\, do_2 \tag{28.31}$$

where (cf. (28.9)):

$$Q_{\alpha\mu} = \frac{1}{m^2\varkappa_2}\hat{e}_\alpha^{(1)}(i\hat{f}_2-m)\hat{e}_\mu^{(2)}+\frac{1}{m^2\varkappa_1}\hat{e}_\mu^{(2)}(i\hat{f}_1-m)\hat{e}_\alpha^{(1)}$$

and $e_\alpha^{(1)} (\alpha = 1, 2)$ are unit vectors in the plane perpendicular to the vector $k_1$. We choose them in the following manner:

$$e_1^{(1)} = \frac{[k_1, k_2]}{|[k_1, k_2]|} \quad \text{and} \quad e_2^{(1)} = \frac{1}{|k_1|}[k_1, e_1^{(1)}];$$

$e_\mu^{(2)}$ are unit vectors in the plane perpendicular to $k_2 (\mu = 1, 2)$. As has been explained in subsection 26.3, in expression (28.31) the summation over $\mu$ may be extended over the values $\mu = 1, 2, 3, 4$, i.e., we can replace $\hat{e}_\mu^{(2)}$ by $\gamma_\mu$.

On evaluating the trace (over the spinor indices) in (28.31) we obtain in the case of an electron at rest

$$d\sigma(\boldsymbol{\xi}^{(1)}) = d\sigma_0+\xi_3^{(1)}d\sigma_1, \tag{28.32}$$

where $d\sigma_0$ is the scattering cross section for an unpolarized photon determined by formula (28.18), and

$$d\sigma_1 = \frac{1}{2} r_0^2 \frac{\omega_2^2}{\omega_1^2} \sin^2\vartheta \, do. \tag{28.33}$$

The expression for $d\sigma$ can also be written in the form

$$d\sigma = \frac{1}{2} r_0^2 \left(\frac{\omega_2}{\omega_1}\right)^2 F \, do, \tag{28.34}$$

where

$$F = \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + (\xi_3^{(1)} - 1)\sin^2\vartheta. \tag{28.34'}$$

We now obtain the polarization of the scattered photon. By applying the results of subsection 26.4 we obtain the expression for the Stokes parameters for the scattered photon,

$$\xi_j^{(2)} = \frac{\mathrm{Sp}\{Q_{\alpha\mu}(i\hat{p}_1 - m)\varrho_{\alpha\beta}^{(1)}\tau_{j,\mu\nu}\bar{Q}_{\beta\nu}(i\hat{p}_2 - m)\}}{\mathrm{Sp}\{Q_{\alpha\mu}(i\hat{p}_1 - m)\varrho_{\alpha\beta}^{(1)}\bar{Q}_{\beta\nu}(i\hat{p}_2 - m)\}}. \tag{28.35}$$

Here the summation over all the variables ($\alpha$, $\beta$, $\mu$, $\nu$) is carried out over the values 1 and 2. On choosing the unit vectors $e_\mu^{(2)}$

$$e_1^{(2)} = \frac{[\boldsymbol{k}_1, \boldsymbol{k}_2]}{|[\boldsymbol{k}_1, \boldsymbol{k}_2]|}, \qquad e_2^{(2)} = \frac{[\boldsymbol{k}_2, e_1^{(2)}]}{|\boldsymbol{k}_2|},$$

we obtain after evaluating the trace in (28.35)

$$\begin{aligned} \xi_3^{(2)} &= \left(\sin^2\vartheta + (1+\cos^2\vartheta)\xi_3^{(1)}\right)\frac{1}{F}, \\ \xi_1^{(2)} &= 2\xi_1^{(1)} \cos\vartheta \frac{1}{F}, \\ \xi_2^{(2)} &= \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)\xi_2^{(1)} \frac{\cos\vartheta}{F}. \end{aligned} \tag{28.36}$$

From these formulas we can see, in particular, that the unpolarized photon becomes partially polarized as a result of scattering. Indeed, on setting $\xi_j^{(1)} = 0$ in (28.36) we obtain

$$\begin{aligned} \xi_1^{(2)} &= \xi_2^{(2)} = 0, \\ \xi_3^{(2)} &= \frac{\sin^2\vartheta}{(\omega_1/\omega_2) + (\omega_2/\omega_1) - \sin^2\vartheta}. \end{aligned} \tag{28.36'}$$

Since $\xi_3^{(2)} > 0$ the photon is polarized in the direction $e_1^{(2)}$, i.e., perpendicular to the scattering plane. In the nonrelativistic approximation we have $\xi_3^{(2)} = \sin^2\vartheta/(1+\cos^2\vartheta)$ and in the case of scattering through 90° the radiation is completely polarized.

We consider the case when the incident photon is linearly polarized ($\xi_2^{(1)} = 0$, $(\xi_1^{(1)})^2+(\xi_3^{(1)})^2 = 1$), and seek the differential scattering cross section in the case when the scattered photon is also linearly polarized. In order to do this we utilize formula (26.27) and expressions (28.18) and (28.35) for $d\sigma$ and $\xi_j^{(2)}$. On expressing the parameters $\xi_j^{(2)}$ and $\xi_j^d$ in accordance with (2.15) in terms of the components of the photon polarization vectors which we denote by $e^{(1)}$ and $e^{(2)}$, we obtain the following expression for the differential cross section for the scattering of polarized photons by an electron at rest:

$$d\sigma = \frac{r_0^2}{4}\left(\frac{\omega_2}{\omega_1}\right)^2\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-2+4\cos^2\theta\right)do_2\,, \tag{28.37}$$

where $\theta$ is the angle between the polarizations of the primary and the scattered photons, $\cos\theta = e^{(1)}e^{(2)}$.

The scattering cross section as a function of $\theta$ reaches a maximum when the polarization directions of the primary and the scattered photons coincide.

We can obtain the total scattered radiation for given frequencies $\omega_1$ and $\omega_2$ by adding expressions (28.37) for the two linearly independent polarization states of the scattered photon. We choose two mutually orthogonal directions $e_\mu^{(2)}$ ($\mu = 1, 2$; $e_1^{(2)}e_2^{(2)} = 0$), corresponding to these states in the following manner: one of them is perpendicular to $e^{(1)}$: $\cos\theta^{(1)} = e^{(1)}e_1^{(2)} = 0$ and the other lies in the plane containing $k_2$ and $e^{(1)}$, $\cos^2\theta^{(2)} = (e_2^{(2)}e^{(1)})^2 = 1-\sin^2\vartheta\cos^2\varphi$, where $\varphi$ is the angle between the planes ($k_1$, $k_2$) and ($k_2$, $e^{(1)}$).

If $d\sigma^{(1)}$ and $d\sigma^{(2)}$ are the scattering cross sections corresponding to these polarizations, then the total scattering cross section will be equal to $d\sigma = d\sigma^{(1)}+d\sigma^{(2)}$. If the incident photons are unpolarized, then the sum $d\sigma^{(1)}+d\sigma^{(2)}$ averaged over the angle $\varphi$ gives the scattering cross section for unpolarized photons (28.18).

For $\omega_1 \ll m$

$$d\sigma^{(1)} = 0, \qquad d\sigma^{(2)} = r_0^2(1-\sin^2\vartheta\cos^2\varphi)do_2. \tag{28.38}$$

For $\omega_1 \gg m$ we have to distinguish between the regions of small and large scattering angles. If $\vartheta \ll \sqrt{m/\omega_1}$, then

$$d\sigma^{(1)} = 0, \quad d\sigma^{(2)} = r_0^2(1-\sin^2\vartheta\cos^2\varphi)\,do_2, \quad \omega_1 \gg m, \quad \vartheta \ll \sqrt{\frac{m}{\omega_1}}. \tag{28.39}$$

If $\vartheta \gg \sqrt{m/\omega_1}$, then

$$d\sigma^{(1)} = d\sigma^{(2)} = \frac{1}{2}d\sigma = \frac{r_0^2}{4}\frac{m}{\omega_1}\frac{do_2}{1-\cos\vartheta}, \quad \omega_1 \gg m, \quad \vartheta \gg \sqrt{\frac{m}{\omega_1}}. \tag{28.39'}$$

We see that in the extreme relativistic domain the scattering for small $\vartheta$ takes place in the same way as in the nonrelativistic domain. At high energies and large scattering angles the scattered photon is unpolarized independently of the nature of polarization of the primary photon.

### 28.7. *Scattering of Photons by Polarized Electrons*

We now discuss the scattering of photons by a polarized electron at rest (57), (126), (217). If the expectation value of the electron spin is equal to $\frac{1}{2}\boldsymbol{\zeta}^{(1)}$, then its polarization density matrix has, in accordance with subsection 10.6, the following form:

$$\mathcal{P}^{(1)} = \tfrac{1}{4}(1+\gamma_4)(1-i\hat{\zeta}^{(1)}\gamma_5). \tag{28.40}$$

By utilizing (26.18) and (28.31) we obtain for the differential scattering cross section

$$d\sigma(\boldsymbol{\xi}^{(1)}, \boldsymbol{\zeta}^{(1)}) = \frac{r_0^2}{2}\left(\frac{\omega_2}{\omega_1}\right)^2 \mathrm{Sp}\,\{Q_{\alpha\mu}\mathcal{P}^{(1)}\varrho^{(1)}_{\alpha\beta}\bar{Q}_{\beta\mu}(i\hat{p}_2-m)\}. \tag{28.41}$$

On evaluating the trace we obtain

$$d\sigma(\boldsymbol{\xi}^{(1)}, \boldsymbol{\zeta}^{(1)}) = d\sigma(\boldsymbol{\xi}^{(1)}) + r_0^2\left(\frac{\omega_2}{\omega_1}\right)^2 \xi_2^{(1)}\boldsymbol{\zeta}^{(1)}\boldsymbol{G}, \tag{28.42}$$

where

$$\boldsymbol{G} = -\frac{1}{m}(1-\cos\vartheta)(\boldsymbol{k}_1\cos\vartheta+\boldsymbol{k}_2) \tag{28.42'}$$

and $d\sigma(\boldsymbol{\xi}^{(1)})$ is the cross section for the scattering of a photon by an unpolarized electron (28.32).

We note that the second term in (28.42) contains only the component $\xi_2^{(1)}$, which describes the circular polarization of the incident photon, and the component of the polarization vector of the electron in the scattering plane.

We give the expression for the total scattering cross section in the case when the electron is polarized in the direction $k_1$, while the photon has only a circular polarization:

$$\sigma(\xi_2^{(1)}, \zeta_3^{(1)}) = \sigma_0 + \xi_2^{(1)}\zeta_3^{(1)}\sigma',$$

$$\sigma' = \pi r_0^2 \left\{ \frac{1+4\omega_1+5\omega_1^2}{\omega_1(1+2\omega_1)^2} - \frac{(1+\omega_1)\ln(1+2\omega_1)}{2\omega_1^2} \right\}, \tag{28.43}$$

where $\sigma_0$ is the scattering cross section for unpolarized photons.

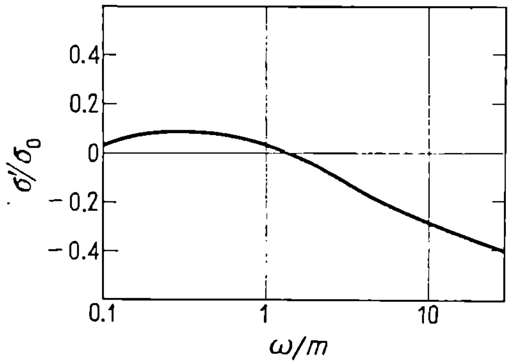


Fig. 24.

Figure 24 shows the ratio $\sigma'/\sigma_0$ as a function of the photon energy $\omega_1$ (79).

If the incident photon is unpolarized, then the polarization of the scattered photon can be obtained from

$$\xi_j^{(2)} = \frac{\mathrm{Sp}\{Q_{\alpha\mu}\mathcal{P}^{(1)}\tau_{j,\mu\nu}\bar{Q}_{\alpha\nu}(i\hat{p}_2-m)\}}{\mathrm{Sp}\{Q_{\alpha\mu}\mathcal{P}^{(1)}\bar{Q}_{\alpha\mu}(i\hat{p}_2-m)\}}. \tag{28.44}$$

In this case the scattered photon has a linear polarization determined by formula (28.36′) and, in addition to that, the circular polarization

$$\xi_2^{(2)} = \frac{1}{F} G\boldsymbol{\xi}^{(1)}, \tag{28.45}$$

where $G$ is defined by formula (28.42′), while $F$ is defined by formula (28.34′) with $\xi_3^{(1)} = 0$.

Finally, we investigate the polarization of the recoil electrons. In accordance with (26.21) and (10.32) we can find the parameter $a_j$, and then from formula (10.30′) we can find the polarization vector $\boldsymbol{\xi}^{(2)}$.

If the electron was unpolarized before the collision then

$$\boldsymbol{\zeta}^{(2)} = \xi_2^{(1)} \frac{\boldsymbol{G}'}{F}, \tag{28.46}$$

where $F$ is defined by formula (28.34′), and

$$\boldsymbol{G}' = -\frac{1}{m}(1-\cos\vartheta)\left[(\boldsymbol{k}_1\cos\vartheta+\boldsymbol{k}_2)-(1+\cos\vartheta)\frac{\omega_1+\omega_2}{m_2+\varepsilon_2}(\boldsymbol{k}_1-\boldsymbol{k}_2)\right]. \tag{28.46′}$$

We see that the electron is polarized only in the case when the photon has circular polarization.

## § 29. Bremsstrahlung

### 29.1. *Perturbation Theory for an Electron Wave Function in the Continuum. Incoming and Outgoing Waves*

When an electron collides with a charge or a system of charges, in addition to the electron being scattered photon emission can also take place. Such a process is referred to as bremsstrahlung. If the electron collides with a heavy particle (nucleus, atom) the effect of the latter can be taken into account as the effect of an external field. In this case bremsstrahlung is described by the matrix element (27.3) in which the initial state refers to the continuous spectrum, and the final state refers also to either the continuous or to the discrete spectrum.

Because of the complicated nature of the wave functions of an electron in the field of a nucleus (cf. § 13) the matrix element (27.3) can be evaluated only in the case of low energies, when the nonrelativistic approximation may be used, and also in the limiting case of high energies and small scattering angles for the electron when the wave functions of subsection 14.5 may be used.

If the external field is such that it can be treated by perturbation theory methods, then in order to determine the probability of bremsstrahlung it is sufficient to evaluate the corresponding element of the second order scattering matrix $S^{(2)}$ (28.1). This matrix element will differ from the matrix element for photon-electron scattering (28.2) only by the fact that the potential of the primary photon must be replaced by the potential of the external field $A^{(e)}_\mu(x)$:

$$S^{(2)}_{i\to f} = e^2 \int \bar{\psi}_2(x_2)\,[\hat{A}^*(x_2)\,S^c(x_2-x_1)\,\hat{A}^{(e)}(x_1) + \hat{A}^{(e)}(x_2)\,S^c(x_2-x_1)\,\hat{A}^*(x_1)]\,\psi_1(x_1)\,d^4x_1\,d^4x_2. \qquad (29.1)$$

Here $\psi_1(x)$ and $\psi_2(x)$ are the wave functions for the free electron in the initial and the final states, $A_\mu(x)$ is the potential of the emitted photon and $A^{(e)}_\mu(x)$ is the potential of the external field.

The criterion for the applicability of such treatment obviously coincides with the criterion for the applicability of the Born approximation to the case of the Coulomb field $(Ze^2/v) \ll 1$, where $Ze$ is the nuclear charge and $v$ is the electron velocity. We first consider this relatively simple case which has quite an extensive range of applicability.

Before evaluating the second order matrix element (29.1) we note that a comparison of (29.1) with the first order matrix element enables us to obtain the wave function of the electron in the external field in the first approximation of perturbation theory. Indeed, formula (29.1) must coincide with (27.3) if the wave functions in the latter take into account scattering by the external field. To distinguish them from the free electron wave functions $\psi_1$ and $\psi_2$ we denote them by $\psi_1^{(e)}$ and $\psi_2^{(e)}$ and set

$$\psi_1^{(e)} = \psi_1 + \psi_1' \quad \text{and} \quad \bar{\psi}_2^{(e)} = \bar{\psi}_2 + \bar{\psi}_2'. \qquad (29.2)$$

Substituting these expressions into (27.3), and taking into account the fact that (27.3) vanishes if the free electron wave functions are substituted into it, we obtain on comparing with (29.1)

$$\begin{aligned} \psi_1^{(e)}(x) &= \psi_1(x) - e\int S^c(x-x')\,\hat{A}^{(e)}(x')\,\psi_1(x')\,d^4x', \\ \bar{\psi}_2^{(e)}(x) &= \bar{\psi}_2(x) - e\int \bar{\psi}_2(x')\,\hat{A}^{(e)}(x')\,S^c(x'-x)\,d^4x'. \end{aligned} \qquad (29.3)$$

These formulas have a straightforward physical meaning: in the second terms the function $S^c(x)$ plays the role of the Green's function, and the product $A^{(e)}(x)\psi(x)$ plays the role of source density, while the second terms themselves represent scattered waves obtained as a result of the superposition of waves scattered by each four-dimensional volume element $d^4x'$.

By proceeding in an analogous manner we can also obtain the higher order approximations for the wave function of the electron in an external field. For example, in second order we obtain

$$\begin{aligned} \psi_1^{(e)}(x) &= \psi_1(x)-e\int S^c(x-x')\hat{A}^{(e)}(x')\psi_1(x')d^4x' \\ &\quad +e^2\int S^c(x-x')\hat{A}^{(e)}(x')S^c(x'-x'')\hat{A}^{(e)}(x'')\psi_1(x'')d^4x'd^4x'', \\ \bar{\psi}_2^{(e)}(x) &= \bar{\psi}_2(x)-e\int \bar{\psi}_2(x')\hat{A}^{(e)}(x')S^c(x'-x)d^4x' \\ &\quad +e^2\int \bar{\psi}_2(x'')\hat{A}^{(e)}(x'')S^c(x''-x')\hat{A}^{(e)}(x')S^c(x'-x)d^4x'd^4x''. \end{aligned} \tag{29.4}$$

The summation of the series obtained by perturbation theory should, in principle, lead to the exact solutions of the corresponding Dirac equations.

If we substitute into (29.3) in place of $\psi_{1,2}(x)$ the plane waves $\psi_{1,2} = u_{1,2}\exp(ip_{1,2}x)$, we obtain

$$\begin{aligned} \psi_1^{(e)}(x) &= u_1e^{ip_1x}-\frac{ie}{(2\pi)^4}\int\frac{i\hat{f}-m}{f^2+m^2}\hat{A}^{(e)}(f-p_1)u_1e^{ifx}d^4f, \\ \bar{\psi}_2^{(e)}(x) &= \bar{u}_2e^{-ip_2x}-\frac{ie}{(2\pi)^4}\int\bar{u}_2\hat{A}^{(e)}(p_2-f)\frac{i\hat{f}-m}{f^2+m^2}e^{-ifx}d^4f, \end{aligned} \tag{29.5}$$

where $\hat{A}^{(e)}(f)$ is the Fourier component of the external potential defined by formula (25.3).

In an electrostatic field

$$A^{(e)}(f)=0, \qquad A_0^{(e)}(f)=A_0^{(e)}(f)2\pi\delta(f_0), \qquad f(f,if_0)$$

and

$$\begin{aligned} \psi_1^{(e)} &= u_1e^{ip_1x}+\frac{e}{(2\pi)^3}\int\frac{i\hat{f}-m}{f^2-p^2}A_0^{(e)}(f-p_1)\gamma_4u_1e^{ifx}df, \\ \bar{\psi}_2^{(e)} &= \bar{u}_2e^{-ip_2x}+\frac{e}{(2\pi)^3}\int\bar{u}_2\gamma_4A_0^{(e)}(p_2-f)\frac{i\hat{f}-m}{f^2-p^2}e^{-ifx}df. \end{aligned} \tag{29.6}$$

In the nonrelativistic approximation small values of $|f|$ play an important role in these integrals, and therefore $i\hat{f}\approx -m$ and the function $\psi^{(e)}$ assumes the following form which is well known from the perturbation theory for the continuous spectrum:

$$\psi^{(e)}(r)=\psi_p+\frac{e}{(2\pi)^3}\int\frac{A_0^{(e)}(f-p)}{\varepsilon_p-\varepsilon_f}\psi_fdf, \tag{29.6'}$$

where $\psi_p = \exp(ipr)$ and $\varepsilon_p = p^2/2m$, and the integration is carried out in accordance with the definition of the function $S^c(x)$ by taking into account the rule for going around the pole of the integrand (cf. § 18).

It can be easily shown that the second terms in (29.6) which, as has been pointed out already, represent scattered waves, have the asymptotic form of outgoing spherical waves for large values of $r$.

Indeed, we examine the integral appearing in (29.6):

$$J = \int g(p, f) \frac{e^{ifr}}{p^2 - f^2} df,$$

where $g(p, f)$ includes all the factors other than those explicitly shown in $J$. On integrating over the angle $\vartheta$ between the vectors $f$ and $r$ we obtain

$$J = \frac{2\pi i}{r} \left\{ \int_0^\infty \frac{g_\pi e^{-ifr}}{f^2 - p^2} f\, df - \int_0^\infty \frac{g_0 e^{ifr}}{f^2 - p^2} f\, df \right\} + \frac{2\pi i}{r} \int_0^\infty f\, df \int_{-1}^{1} \frac{e^{ifr\xi}}{f^2 - p^2} \frac{\partial g}{\partial \xi} d\xi,$$

where $\xi = \cos\vartheta$ and $g_\pi$ and $g_0$ are the values of $g(p, f)$ for $\vartheta = \pi$ and 0.

The second term here is evidently of order $1/r^2$ and may, therefore, be neglected. In the first term we can make the substitution $f \to -f$, as a result of which the equation assumes the form

$$J = -\frac{i\pi}{r} \left\{ \int_{-\infty}^{\infty} g_0 \frac{e^{ifr}}{f^2 - p^2} f\, df - \int_{-\infty}^{\infty} g_\pi \frac{e^{-ifr}}{f^2 - p^2} f\, df \right\}, \quad \text{where} \quad r \to \infty.$$

In accordance with (17.18) the poles of the integrands in $J$ are given by the points $p + i0$ and $-p - i0$. Therefore, on completing the contour of integration in the first integral by a semicircle in the lower half-plane, and in the second integral by a semicircle in the upper half-plane, we obtain $J \sim \frac{1}{r} e^{ipr}$, where $r \to \infty$.

Thus, in virtue of the rule for going around the poles of the function $S^c(p)$, both the function $\bar{\psi}_1^{(e)}$ and the function $\bar{\psi}_2^{(e)}$ contain only outgoing waves at $r \to \infty$ and contain no incoming waves. This assertion is obviously valid not only for the first-order approximation of perturbation theory, but also for all higher order approximations. Therefore, the exact wave functions for an electron in an external field, which belong to the continuous spectrum, and which may be regarded as sums of series obtained by perturbation theory, must also have the same property, viz., the initial state function $\psi_1(x)$ and the complex-conjugate function for the final state $\overline{\psi}_2(x)$ must have at $r \to \infty$ the asymptotic form of the sum of a plane wave and an outgoing spherical wave (187)

$$\begin{aligned} \psi_1 &\sim e^{ip_1 r} + \frac{a_1}{r} e^{ip_1 r}, \\ \overline{\psi}_2 &\sim e^{-ip_2 r} + \frac{a_2}{r} e^{ip_2 r}. \end{aligned} \tag{29.7}$$

From this it follows that the final state wave function itself must at $r \to \infty$ have the form of a sum of a plane wave and an incoming spherical wave

$$\psi_2 \sim e^{ip_2 r} + \frac{a_2}{r} e^{-ip_2 r}. \tag{29.8}$$

This fact is of the greatest importance for the evaluation of the matrix elements using the exact wave functions for the electron in an external field. The point is that the wave functions of the continuous spectrum have a peculiar degeneracy. In order to define them it is not sufficient to specify the electron momentum, but we must also specify the nature of the asymptotic behavior at infinity, since for a given momentum the wave function at infinity may have the form either of the sum of a plane wave and an outgoing wave, or of the sum of a plane wave and an incoming wave. We now see that from the rule for going around the poles of the function $S^c$ it follows that the wave functions of the initial and the final states must be chosen in such a way that at infinity the initial state wave function must have the form of the sum of a plane wave and an outgoing wave, while the final state wave function must have the form of the sum of a plane wave and an incoming wave.

This rule can also be formulated in the following manner: the wave functions of all the particles formed as the result of any process must at $r \to \infty$ have the form of a sum of a plane wave and an incoming wave, while the wave functions of particles which disappear (i.e., appear only in the initial state) must at $r \to \infty$ have the form of a sum of a plane wave and an outgoing wave.

The nature of this asymptotic behavior admits a simple physical interpretation. If we expand a plane wave in terms of incoming and outgoing waves, then in the initial state wave function $\psi_i$ at $r \to \infty$ only the incoming waves will have amplitudes which do not depend on the nature of the scatterer, while the outgoing waves will be completely determined by the scattering force field. This corresponds to the fact that in the initial state the particles are incident on the scatterer. In the final state wave functions $\psi_f$ at $r \to \infty$ only the outgoing waves may have amplitudes which do not depend on the nature of the scatterer, while the incoming waves must be determined by the scattering force field.

### 29.2. *Effective Cross Section for Bremsstrahlung*

On substituting into (29.1) the electron wave functions and the photon potentials in the form of plane waves (cf. (25.4), (25.5)), and also the expression (25.2) for $S^c(x)$, we obtain the following form for the matrix element $S^{(2)}_{i\to f}$ which determines the bremsstrahlung cross section:

$$S^{(2)}_{i\to f} = \frac{ie^2}{\sqrt{2\omega}}\bar{u}_2\left\{\hat{e}\frac{i\hat{f}_1-m}{m^2\varkappa_1}\hat{A}^{(e)}(q)+\hat{A}^{(e)}(q)\frac{i\hat{f}_2-m}{m^2\varkappa_2}\hat{e}\right\}u_1, \quad (29.9)$$

where

$$f_1 = p_2+k, \quad f_2 = p_1-k, \quad q = p_2+k-p_1,$$
$$m^2\varkappa_1 = f_1^2+m^2 = 2p_2k,$$
$$m^2\varkappa_2 = f_2^2+m^2 = -2p_1k,$$
$$A^{(e)}(q) = \int A^{(e)}(x)e^{-iqx}d^4x.$$

This expression can also be obtained with the aid of Feynman's rules (cf. subsection 25.5) if we first draw the diagrams corresponding to the process of bremsstrahlung. These diagrams are shown in Fig. 25; they differ from the diagrams corresponding to the Comptom effect, only by the fact that corresponding to the primary photon $k_1$ we have here a "pseudophoton" of four-momentum $q$, i.e., the Fourier $q$-component of the external potential, while corresponding to the scattered photon $k_2$ we have here the emitted photon of momentum $k$ and polarization $e$. Correspondingly, the expression for the bremsstrahlung matrix element (29.1) differs from the Compton effect matrix element by the substitutions $k_1 \to q$, $k_2 \to k$, $\hat{e}_1 \to \hat{A}^{(e)}(q)$ and $\hat{e}_2 \to \hat{e}$.

From the laws of conservation of energy and momentum it follows that the four-momentum of the "pseudophoton" $q(\boldsymbol{q}, iq_0)$ taken with reversed sign is equal to the four-momentum $p_n$ given to the nucleus as a result of the bremsstrahlung process: $p_n = -q = p_1 - p_2 - k$. Thus, $-\boldsymbol{q}$ is the three-dimensional nuclear recoil momentum, while $-q_0$ is the energy imparted to the nucleus.

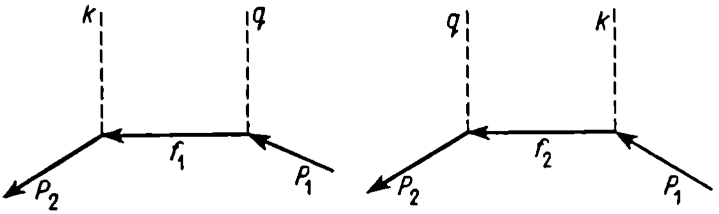


Fig. 25.

If the external field does not depend on the time and is described by the static potential $A_0^{(e)}(\boldsymbol{r})$, as is the case for a stationary nucleus, then

$$\begin{aligned} \hat{A}^{(e)}(q) &= \gamma_4 i A_0^{(e)}(q)\, 2\pi\delta(q_0), \\ A_0^{(e)}(\boldsymbol{q}) &= \int A_0^{(e)}(\boldsymbol{r})\, e^{-i\boldsymbol{q}\boldsymbol{r}}\, d\boldsymbol{r}. \end{aligned} \tag{29.9'}$$

In this case the polarization of the pseudophoton, in contrast to the polarization of a real photon, is directed along the $x_4$-axis.

In accordance with (26.33) $A_0^{(e)}(\boldsymbol{q})$ is given for the nuclear Coulomb field by $A_0^{(e)}(\boldsymbol{q}) = -(Ze/\boldsymbol{q}^2)$, where $e$ is the electron charge in Heaviside units.

We now proceed to the determination of the differential cross section for bremsstrahlung in the Coulomb field of a nucleus. In accordance with (26.3′) it can be written in the form

$$d\sigma = \frac{e^4}{2\omega} |A_0^{(e)}(\boldsymbol{q})|^2 |\bar{u}_2 Q u_1|^2 \frac{d\boldsymbol{p}_2 d\boldsymbol{k}}{J(2\pi)^5} \delta(\varepsilon_1 - \varepsilon_2 - \omega), \tag{29.10}$$

where $J$ is the electron flux density in the initial state, whose magnitude is equal to their velocity, and $Q$ is the matrix appearing in the figure brackets of formula (29.9),

$$Q = \frac{1}{m^2\varkappa_1} \hat{e}(i\hat{f}_1 - m)\gamma_4 + \frac{1}{m^2\varkappa_2} \gamma_4(i\hat{f}_2 - m)\hat{e}.$$

We first of all obtain the bremsstrahlung cross section averaged over the spin directions of the incident electron and summed over the spin directions of the scattered electron and over the polarizations of the emitted photon. Such averaging and summation over the polarizations of the particles reduces, according to (26.7), to the following replacement in (29.10):

$$|\bar{u}_2 Q u_1|^2 \to \frac{1}{8\varepsilon_1\varepsilon_2}\,\mathrm{Sp}\,F, \tag{29.11}$$

where

$$\begin{gathered}\mathrm{Sp}\,F = \mathrm{Sp}\{Q_\mu(i\hat{p}_1-m)\bar{Q}_\mu(i\hat{p}_2-m)\},\\ Q_\mu = \gamma_\mu\frac{i\hat{f}_1-m}{m^2\varkappa_1}\gamma_4+\gamma_4\frac{i\hat{f}_2-m}{m^2\varkappa_2}\gamma_\mu,\\ \bar{Q}_\mu = \gamma_4 Q_\mu^+\gamma_4 = \gamma_4\frac{i\hat{f}_1-m}{m^2\varkappa_1}\gamma_\mu+\gamma_\mu\frac{i\hat{f}_2-m}{m^2\varkappa_2}\gamma_4.\end{gathered} \tag{29.11'}$$

The quantity Sp $F$ is a function of $\varkappa_1$, $\varkappa_2$, $q^2$, $\varepsilon_1$, $\varepsilon_2$:

$$\mathrm{Sp}\,F = \mathcal{F}_1(\varkappa_1,\ \varkappa_2,\ q^2,\ \varepsilon_1,\ \varepsilon_2)+\mathcal{F}_2(\varkappa_1,\ \varkappa_2,\ q^2,\ \varepsilon_1,\ \varepsilon_2), \tag{29.12}$$

where

$$\mathcal{F}_1 = \mathrm{Sp}\left\{Q_\mu(i\hat{p}_1-m)\gamma_4\frac{i\hat{f}_1-m}{m^2\varkappa_1}\gamma_\mu(i\hat{p}_2-m)\right\},$$

$$\mathcal{F}_2 = \mathrm{Sp}\left\{Q_\mu(i\hat{p}_1-m)\gamma_\mu\frac{i\hat{f}_2-m}{m^2\varkappa_2}\gamma_4(i\hat{p}_2-m)\right\}.$$

If in the expression defining $\mathcal{F}_2$ we make the substitution $p_1 \to p_2$, $p_2 \to p_1$, $q \to -q$ and $k \to -k$, then $\varkappa_1$ and $\varkappa_2$, as well as $f_1$ and $f_2$, will be interchanged so that $\varkappa_1 \to \varkappa_2$, $\varkappa_2 \to \varkappa_1$, $f_1 \to f_2$, $f_2 \to f_1$, and we shall obtain in the brackets following the symbol Sp in the expression for $\mathcal{F}_2$ the same matrices as in the expression for $\mathcal{F}_1$, only arranged in the inverse order. From this it follows that $\mathcal{F}_2(\varkappa_1,\ \varkappa_2,\ q^2,\ \varepsilon_1,\ \varepsilon_2) = \mathcal{F}_1(\varkappa_2,\ \varkappa_1,\ q^2,\ \varepsilon_2,\ \varepsilon_1)$ and, therefore, it is sufficient to evaluate $\mathcal{F}_1$.

The evaluation of $\mathcal{F}_1$ can be carried out simply: after summing over the subscript $\mu$ in accordance with (26.14) we must, by utilizing the commutation rules for the matrices $\gamma_\nu$, arrange them in such a way that the two matrices $\gamma_4$ become adjacent. This may be accomplished with the aid of the obvious formula $\gamma_4\hat{b} = -\hat{b}^*\gamma_4$, where $b^* = b$ and $b_4^* = -b_4$, which is valid for any real four-vector $b$ (having a real spatial and an imaginary fourth component). Since $\gamma_4^2 = 1$, after such a rearrangement of the matrices $\gamma_4$ there will remain in each term a product of not more than four matrices $\gamma_\nu$ for which the trace can be easily evaluated. By proceeding in this manner we obtain

$$\begin{aligned}\frac{1}{4}\mathcal{F}_1 = \frac{1}{\varkappa_1^2}\left\{\left(\varkappa_1\varkappa_2+2\frac{q^2}{m^2}\right)-\frac{4\varepsilon_1^2}{m^2}(\varkappa_1+2)+\frac{4\varepsilon_1\varepsilon_2}{m^2}\varkappa_1\right\}\\ +\frac{1}{\varkappa_1\varkappa_2}\left\{-\frac{q^2}{m^2}\left(\varkappa_1+\varkappa_2-\frac{q^2}{m^2}-2\right)+\frac{2\varepsilon_1^2}{m^2}\left(\varkappa_2-\frac{q^2}{m^2}\right)\right.\\ \left.+\frac{2\varepsilon_2^2}{m^2}\left(\varkappa_1-\frac{q^2}{m^2}\right)-\frac{2\varepsilon_1\varepsilon_2}{m^2}(\varkappa_1+\varkappa_2+4)\right\}.\end{aligned} \tag{29.12'}$$

By interchanging in this expression $\varkappa_1$ and $\varkappa_2$, and also $\varepsilon_1$ and $\varepsilon_2$, we obtain $\mathcal{F}_2$.

Consequently,

$$\begin{aligned}d\sigma = \frac{Z^2e^6}{4\omega\varepsilon_1\varepsilon_2}\frac{dp_2dk}{J(2\pi)^5}\frac{\delta(\varepsilon_1-\varepsilon_2-\omega)}{\varkappa_1^2\varkappa_2^2q^4}\\ \times\left\{-2\varkappa_1\varkappa_2\frac{q^2}{m^2}\left[\frac{2}{m^2}(\varepsilon_1^2+\varepsilon_2^2)+\varkappa_1+\varkappa_2-\frac{q^2}{m^2}-2\right]\right.\\ \left.+(\varkappa_1^2+\varkappa_2^2)\left(\varkappa_1\varkappa_2+2\frac{q^2}{m^2}\right)-\frac{8}{m^2}(\varkappa_1\varepsilon_2+\varkappa_2\varepsilon_1)^2\right\}.\end{aligned} \tag{29.13}$$

In contrast to the Compton effeet the momentum of the scattered electron $p_2$ and the photon momentum $k$ are here independent. Therefore the $\delta$-function can be eliminated simply; on noting that $dp_2 = p_2\varepsilon_2 d\varepsilon_2 do_2$ and $dk = \omega^2 d\omega do$, where $do_2$ and $do$ are elements of solid angles containing the vectors $p_2$ and $k$, we have

$$\delta(\varepsilon_1-\varepsilon_2-\omega)dp_2dk \rightarrow \omega^2|p_2|\varepsilon_2 do do_2 d\omega.$$

On substituting instead of the primary electron flux density $J$ their velocity $J = v_1 = |p_1|/\varepsilon_1$, we obtain the expression for the bremsstrahlung cross section averaged over the polarizations of the particles (21):

$$\begin{aligned}d\sigma = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{|p_2|}{|p_1|}\frac{d\omega}{\omega}\frac{do do_2}{q^4}\frac{2\omega^2}{\varkappa_1^2\varkappa_2^2}\left\{-2\frac{q^2}{m^2}\varkappa_1\varkappa_2\left[-2+\varkappa_1+\varkappa_2-\frac{q^2}{m^2}\right.\right.\\ \left.\left.+\frac{2}{m^2}(\varepsilon_1^2+\varepsilon_2^2)\right]+(\varkappa_1^2+\varkappa_2^2)\left(\varkappa_1\varkappa_2+2\frac{q^2}{m^2}\right)-\frac{8}{m^2}(\varkappa_1\varepsilon_2+\varkappa_2\varepsilon_1)^2\right\},\end{aligned} \tag{29.14}$$

where $\alpha = e^2/4\pi = 1/137$.

On taking into account the relations

$$\begin{aligned} m^2\varkappa_1 &= 2(\mathbf{p}_2\mathbf{k}-\varepsilon_2\omega) = -2\omega(\varepsilon_2-p_2\cos\vartheta_2), \\ m^2\varkappa_2 &= -2(\mathbf{p}_1\mathbf{k}-\varepsilon_1\omega) = 2\omega(\varepsilon_1-p_1\cos\vartheta_1), \\ q^2 &= m^2(\varkappa_1+\varkappa_2-2)+2(\varepsilon_1\varepsilon_2-\mathbf{p}_1\mathbf{p}_2), \\ p_1^2 &= \varepsilon_1^2-m^2, \\ p_2^2 &= \varepsilon_2^2-m^2, \end{aligned}$$

where $\vartheta_1$ and $\vartheta_2$ are the angles between the vectors $\mathbf{k}$ and $\mathbf{p}_1$, and $\mathbf{k}$ and $\mathbf{p}_2$ we can rewrite $d\sigma$ in the form

$$\begin{aligned} d\sigma = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{p_2}{p_1}\frac{d\omega do do_2}{\omega q^4}\frac{4}{m^4}\Bigg\{4\left[\mathbf{k},\frac{\varepsilon_1\mathbf{p}_2}{\varkappa_1}+\frac{\varepsilon_2\mathbf{p}_1}{\varkappa_2}\right]^2 \\ -q^2\left[\mathbf{k},\frac{\mathbf{p}_2}{\varkappa_1}+\frac{\mathbf{p}_1}{\varkappa_2}\right]^2-\frac{2\omega^2}{\varkappa_1\varkappa_2}[\mathbf{k},\mathbf{p}_2-\mathbf{p}_1]^2\Bigg\} \end{aligned} \tag{29.15}$$

or

$$\begin{aligned} d\sigma = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{p_2}{p_1}\frac{d\omega}{\omega}\frac{do do_2}{q^4}\Bigg\{\frac{p_2^2\sin^2\vartheta_2}{(\varepsilon_2-p_2\cos\vartheta_2)^2}(4\varepsilon_1^2-q^2) \\ +\frac{p_1^2\sin^2\vartheta_1}{(\varepsilon_1-p_1\cos\vartheta_1)^2}(4\varepsilon_2^2-q^2)+2\omega^2\frac{p_1^2\sin^2\vartheta_1+p_2^2\sin^2\vartheta_2}{(\varepsilon_1-p_1\cos\vartheta_1)(\varepsilon_2-p_2\cos\vartheta_2)} \\ -2\frac{p_1p_2\sin\vartheta_1\sin\vartheta_2\cos\varphi}{(\varepsilon_1-p_1\cos\vartheta_1)(\varepsilon_2-p_2\cos\vartheta_2)}(4\varepsilon_1\varepsilon_2-q^2+2\omega^2)\Bigg\}, \end{aligned} \tag{29.16}$$

where $\varphi$ is the angle between the planes $(\mathbf{k}_1,\mathbf{p}_1)$ and $(\mathbf{k}_2,\mathbf{p}_2)$, $p = |\mathbf{p}|$, $q = |\mathbf{q}|$.

The square of the momentum $q^2$ imparted to the nucleus is related to the angles $\vartheta_1$, $\vartheta_2$, and $\varphi$ by the expression

$$\begin{aligned} q^2 = p_1^2+p_2^2+\omega^2-2p_1\omega\cos\vartheta_1+2p_2\omega\cos\vartheta_2 \\ -2p_1p_2(\cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos\varphi). \end{aligned}$$

From (29.16) it follows that $d\sigma$ vanishes when $\vartheta_1 = \vartheta_2 = 0$.

## 29.3. *Angular Distribution of the Radiation in a Coulomb Field*

Formula (29.16) determines the angular distribution of bremsstrahlung in the case of unpolarized primary electrons, averaged over the polarizations of the photon and the scattered electron..

In the general case this angular distribution of the radiation is of a very complicated nature. Simplifications are possible in the two limiting cases of low and high energies, which we shall now investigate.

In the limiting case of low primary electron energies $p_1 \ll m$, which correspond to the problem of the continuous X-ray spectrum, the photon momentum is small compared to the electron momentum, since $\omega = (p_1^2 - p_2^2)/2m \ll p_1$. Therefore $q^2 = (p_1 - p_2)^2$.

Further, on noting that in the nonrelativistic case $m^2 \varkappa_2 = -m^2 \varkappa_1 = 2m\omega$, $\varepsilon_1 = \varepsilon_2 = m$, and on retaining in the figure brackets of (29.15) only the first term, we obtain

$$d\sigma = \frac{Z^2 a^3}{\pi^2} \frac{d\omega}{\omega} \frac{p_2}{p_1} \frac{do\, do_2}{(p_1 - p_2)^4} \frac{[k, p_1 - p_2]^2}{\omega^2} \quad \text{where} \quad p_1 \ll m. \tag{29.17}$$

Since $p_2$ differs little from $p_1$, the factor

$$d\sigma_s = \frac{(2mZa)^2}{|p_1 - p_2|^4} do_2 \tag{29.18}$$

appearing in the preceding expression is the Rutherford cross section for elastic scattering of electrons. We can, therefore, say that in the nonrelativistic case the bremsstrahlung cross section is equal to the product of the cross section for elastic scattering of electrons $d\sigma_s$ and the probability of photon emission $dw_\gamma$;

$$d\sigma = d\sigma_s \, dw_\gamma,$$

$$dw_\gamma = \frac{a}{(2\pi)^2} \left[ \frac{k}{\omega}, \; \boldsymbol{v}_1 - \boldsymbol{v}_2 \right]^2 \frac{d\omega}{\omega} do. \tag{29.19}$$

This expression for the emission probability is in agreement with classical radiation theory, viz., $dw_\gamma$ is the ratio of the intensity of dipole radiation at low frequencies to $\hbar\omega$, i.e. the average number of photons emitted per unit time determined by classical methods (in connection with this cf. § 30).

The intensity of the radiation has a maximum in the direction perpendicular to the plane of the electron motion $(p_1, p_2)$.

We now discuss the angular distribution in the case of high energies, when $\varepsilon_1 \gg m$, $\varepsilon_2 \gg m$. Due to the presence in the denominator of (29.16) of the factors $1 - v_2 \cos \vartheta_2$ and $1 - v_1 \cos \vartheta_1$ the bremsstrahlung cross section has a sharp maximum in the neighborhood of the direction of the momentum of the incident electron, and the radiation is concentrated primarily within a narrow cone about this direction of angular aperture of order of magnitude $\vartheta_1 \sim m/\varepsilon_1$. The same cone also contains the momentum of the scattered electron. In the extreme relativistic case the angular distribution of the radiation is approximately given by the following formula (90), (190):

$$d\sigma(\vartheta_1) = A\frac{\vartheta_1 d\vartheta_1}{[\vartheta_1^2+(m/\varepsilon_1)^2]^2}\left[\ln\left(1+\vartheta_1^2\frac{m^2}{\varepsilon_1^2}\right)+B\right], \qquad (29.20)$$

where $\varepsilon_1 \gg m$ and $\varepsilon_2 \gg m$ and where $A$ and $B$ are constants.

### 29.4. *Polarization of the Radiation*

We now consider the nature of the polarization of the bremsstrahlung. We take the electron in the initial state also to be polarized.

By utilizing the general formula (26.18) and the expression for the bremsstrahlung matrix element (29.9), we obtain for the differential cross section formulas (29.10), (29.11), in which

$$F = Q_\mu(ip_1-m)(1-i\hat{a}\gamma_5)\bar{Q}_\mu(i\hat{p}_2-m),$$

$Q_\mu$ is defined by formula (29.11′), and the vector $a_\mu$ is related to the electron polarization vector in the initial state $\boldsymbol{\xi}$ by expressions (10.30′).

On evaluating Sp$F$ we can easily show that the terms containing $a_\mu$ vanish. Thus, the bremsstrahlung cross section is independent of the electron polarization.

In accordance with (26.24) and (29.9) the photon density matrix is of the form

$$\varrho_{\alpha\beta} = \frac{1}{\mathrm{Sp}\,F}\,\mathrm{Sp}\,\{Q_\alpha(i\hat{p}_1-m)(1-i\hat{a}\gamma_5)\bar{Q}_\beta(i\hat{p}_2-m)\}, \qquad (29.21)$$

where

$$Q_\alpha = \hat{e}_\alpha\frac{i\hat{f}_1-m}{m^2\varkappa_1}\gamma_4+\gamma_4\frac{i\hat{f}_2-m}{m^2\varkappa_2}\hat{e}_\alpha$$

and $e_\alpha(\alpha = 1, 2)$ are two mutually perpendicular unit vectors in the plane perpendicular to the vector $\boldsymbol{k}$. We chose them in the manner

$$\boldsymbol{e}_1 = \frac{[\boldsymbol{p}_1, \boldsymbol{k}]}{|[\boldsymbol{p}_1, \boldsymbol{k}]|} \quad \text{and} \quad \boldsymbol{e}_2 = \frac{1}{|\boldsymbol{k}|}[\boldsymbol{k}, \boldsymbol{e}_1].$$

On performing the calculations in accordance with formula (29.21), we obtain the following expressions for the Stokes parameters for the photon $\xi_j$ (201):

$$\xi_3 = \frac{1}{\varkappa_1^2 q^4 Q}\left\{2(4\varepsilon_1^2-q^2)(\mathbf{p}_2\mathbf{e}_1)^2-2\frac{\varkappa_1}{\varkappa_2}[(\mathbf{p}_2-\mathbf{p}_1),\mathbf{k}]^2\right\}-1,$$

$$\xi_1 = \frac{2(\mathbf{p}_2\mathbf{e}_1)}{\varkappa_1 q^4 Q}\left\{\frac{4\varepsilon_1^2-q^2}{\varkappa_1}(\mathbf{p}_2\mathbf{e}_2)+\frac{4\varepsilon_1\varepsilon_2-q^2}{\varkappa_2}(\mathbf{p}_1\mathbf{e}_2)\right\},$$

$$\begin{aligned}\xi_2 = \frac{2m}{\varepsilon_1\omega^2 q^4 Q}\Bigg\{&\left(\left(\frac{\mathbf{p}_1}{\varkappa_2}+\frac{\mathbf{p}_2}{\varkappa_1}\right)[\mathbf{k},[\mathbf{k},(\mathbf{p}_1-\mathbf{p}_2)]]\right)\\ &\times\left(\frac{\omega\mathbf{p}_1\mathbf{a}+\varepsilon_1\mathbf{k}\mathbf{a}}{\varkappa_1}+\frac{\omega\mathbf{p}_1\mathbf{a}-\varepsilon_1\mathbf{k}\mathbf{a}}{\varkappa_2}\right)\\ &+\frac{1}{2}m^2\omega\left(\frac{\varkappa_1}{\varkappa_2}-\frac{\varkappa_2}{\varkappa_1}+\frac{4\omega}{m^2}\left(\frac{\varepsilon_1}{\varkappa_1}+\frac{\varepsilon_2}{\varkappa_2}\right)\right)\\ &\times\left(\frac{\varepsilon_2}{\varkappa_2}(\omega\mathbf{p}_1\mathbf{a}-\varepsilon_1\mathbf{k}\mathbf{a})+\frac{\varepsilon_1}{\varkappa_1}(\omega\mathbf{p}_2\mathbf{a}-\varepsilon_2\mathbf{k}\mathbf{a})\right)\Bigg\},\end{aligned}\qquad (29.22)$$

where

$$Q=\frac{4}{\omega^2 q^4}\left[\mathbf{k},\left(\frac{\varepsilon_1}{\varkappa_1}\mathbf{p}_2+\frac{\varepsilon_2}{\varkappa_2}\mathbf{p}_1\right)\right]^2-\frac{1}{q^2\omega^2}\left[\mathbf{k},\left(\frac{\mathbf{p}_2}{\varkappa_1}+\frac{\mathbf{p}_1}{\varkappa_2}\right)\right]^2-\frac{2}{\varkappa_1\varkappa_2 q^4}[\mathbf{k},(\mathbf{p}_2-\mathbf{p}_1)]^2,$$

$$\mathbf{a}=(\boldsymbol{\zeta}\mathbf{n}_1)\frac{\varepsilon_1}{m}\mathbf{n}_1+\boldsymbol{\zeta}-(\boldsymbol{\zeta}\mathbf{n}_1)\mathbf{n}_1,\qquad \mathbf{n}_1=\frac{\mathbf{p}_1}{|\mathbf{p}_1|}.$$

It follows from (29.22) that independently of the nature of the polarization of the incident electron the photon turns out to be linearly polarized ($\xi_1$ and $\xi_3$ do not depend on $\boldsymbol{\zeta}$). In the case of a polarized electron the photon also turns out to have circular polarization. Thus, the determination of the degree of circular polarization of bremsstrahlung can serve as a method for the measurement of the degree of electron polarization. Circular polarization of photons manifests itself in scattering by polarized electrons (cf. subsection 28.8).

Of greater practical interest is the quantity $\bar{\xi}_2$—the degree of circular polarization of the photon averaged over the final momenta of the electron $\frac{1}{n}$ but it is not possible to obtain simple expressions for this quantity. Fig. 26 shows the dependence of $\bar{\xi}_2$ on the energy of the photon emitted in the direction $p_1$ by an electron of kinetic energy

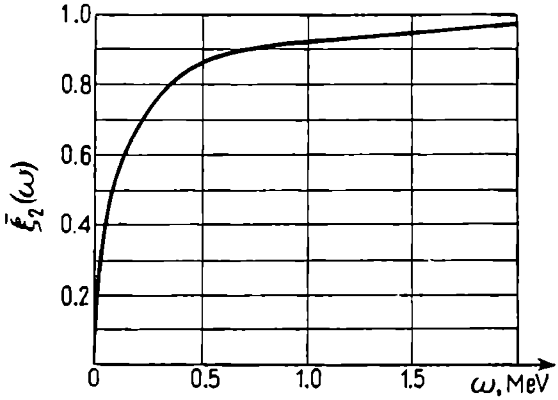


Fig. 26.

$\varepsilon_1 - m = 2$ MeV (134). We see that near the upper limit of the spectrum circular polarization becomes almost complete.

### 29.5 *Spectrum of the Radiation*

On integrating (29.16) over the angles we obtain the differential cross section $d\sigma_\omega$ for the radiation in the frequency interval $\omega$ to $\omega + d\omega$ independently of the directions of the momenta $\boldsymbol{k}$ and $\boldsymbol{p}$:

$$d\sigma_\omega = \sigma_\omega d\frac{\omega}{\varepsilon_1 - m}$$

$$\begin{aligned}\sigma_\omega = \bar{\Phi}\frac{d\omega}{\omega}\frac{p_2}{p_1}\Bigg\{&\frac{4}{3} - 2\varepsilon_1\varepsilon_2\frac{p_1^2+p_2^2}{p_1^2p_2^2} + m^2\left(\frac{\eta_1\varepsilon_2}{p_1^3} + \frac{\eta_2\varepsilon_1}{p_2^3} - \frac{\eta_1\eta_2}{p_1p_2}\right)\\ &+L\Bigg[\frac{8}{3}\frac{\varepsilon_1\varepsilon_2}{p_1p_2} + \frac{\omega^2}{p_1^3p_2^3}(\varepsilon_1^2\varepsilon_2^2+p_1^2p_2^2)\\ &\qquad + \frac{m^2\omega^2}{2p_1p_2}\left(\eta_1\frac{\varepsilon_1\varepsilon_2+p_1^2}{p_1^3} - \eta_2\frac{\varepsilon_1\varepsilon_2+p_2^2}{p_2^3} + \frac{2\omega\varepsilon_1\varepsilon_2}{p_1^2p_2^2}\right)\Bigg]\Bigg\},\end{aligned} \tag{29.23}$$

where

$$\bar{\Phi} = r_0^2 Z^2 \alpha,$$

$$L = \ln\frac{p_1^2+p_1p_2-\varepsilon_1\omega}{p_1^2-p_1p_2-\varepsilon_1\omega} = 2\ln\frac{\varepsilon_1\varepsilon_2+p_1p_2-m^2}{m\omega},$$

$$\eta_1 = \ln\frac{\varepsilon_1+p_1}{\varepsilon_1-p_1} = 2\ln\frac{\varepsilon_1+p_1}{m}, \quad \eta_2 = 2\ln\frac{\varepsilon_2+p_2}{m} \quad \text{and} \quad p_{1,2} = |\boldsymbol{p}|_{1,2}.$$

At low energies ($p_1 \ll m$) this expression is simplified to

$$\sigma_\omega d\frac{\omega}{T_1} = \frac{16}{3}\bar{\Phi}\frac{d\omega}{\omega}\frac{m^2}{p_1^2}\ln\frac{p_1+p_2}{p_1-p_2}$$

$$= \frac{8}{3}\bar{\Phi}\frac{d\omega}{\omega}\frac{m}{T_1}\ln\frac{(\sqrt{T_1}+\sqrt{T_1-\omega})^2}{\omega}, \qquad (29.24)$$

where $T_1 = p_1^2/2m = \varepsilon_1 - m$ is the initial kinetic energy of the electron.

We see that in the nonrelativistic domain the radiation cross section is approximately inversely proportional to the frequency. At the highest possible frequency $\omega = \mathrm{T}_1$ the cross section becomes equal to zero.

As $\omega \to 0$ the intensity of radiation $\omega\, d\sigma_\omega$ diverses logarithmically. This divergence is related to the divergence of the Rutherford cross section for small angles, and is characteristic of the pure Coulomb field. As we shall see later, just as in the case of elastic scattering this divergence can be removed by taking into account the screening of the nuclear field by the atomic electrons.

In the extreme relativistic domain for $\varepsilon_1 \gg m$, $\varepsilon_2 \gg m$ the radiation cross section assumes the form

$$d\sigma_\omega = \sigma_\omega d\frac{\omega}{\varepsilon_1} = 4\bar{\Phi}\frac{d\omega}{\omega}\frac{\varepsilon_2}{\varepsilon_1}\left(\frac{\varepsilon_1}{\varepsilon_2}+\frac{\varepsilon_2}{\varepsilon_1}-\frac{2}{3}\right)\left(\ln\frac{2\varepsilon_1\varepsilon_2}{m\omega}-\frac{1}{2}\right)$$

$$= 4\bar{\Phi}\frac{d\omega}{\omega}\left[1+\left(1-\frac{\omega}{\varepsilon_1}\right)^2-\frac{2}{3}\left(1-\frac{\omega}{\varepsilon_1}\right)\right]\left[\ln\left(\frac{2\varepsilon_1}{m}\frac{1-(\omega/\varepsilon_1)}{(\omega/\varepsilon_1)}\right)-\frac{1}{2}\right]. \qquad (29.25)$$

We see that the probability of an electron radiating a definite fraction of its energy, i.e., the probability of radiation for a given value of the ratio $\omega/\varepsilon_1$, increases approximately linearly with the logarithm of the ratio $\varepsilon_1/m$.

The radiation intensity, i.e., the product of the frequency and the cross section, $\omega d\sigma_\omega$, diverges logarithmically as $\omega \to 0$ both in the relativistic and in the nonrelativistic case.

## 29.6. *Screening*

In evaluating the bremsstrahlung cross section we assumed that the nucleus gives to a pure Coulomb field. Such an assumption is correct for distances from the nucleus smaller than the radius of the *K*-shell of the atom. At greater distances the field of the nucleus is partially or completely screened by the field of the atomic electrons, and this can lead to a decrease in the bremsstrahlung cross section. Therefore in order to determine the extent to which our results are correct we have to estimate the range of distances from the nucleus which is significant for the bremsstrahlung process.

From the form of the integral (26.33) defining the Fourier component of the Coulomb potential it follows that the main contributions to the integral come from distances of the order

$$R_0 \sim \frac{1}{q}, \qquad q = |\boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{k}|. \tag{29.26}$$

Indeed, large values of $r$ are unimportant because of the rapidly oscillating function $\exp(-i\boldsymbol{q}\boldsymbol{r})$, while small values are unimportant because of the smallness of the corresponding volume $dV = 4\pi r^2 dr$. Therefore, we can assume that $R_0$ determines the order of magnitude of the most important impact parameters.

On the other hand, as can be seen from formula (29.16), the differential cross section for bremsstrahlung is particularly large for small $q$ which, in accordance with (29.26), correspond to large impact parameters. Obviously, the screening effect will play no role if the maximum value of $R_0$ corresponding to the minimum value of $q$ is considerably smaller than the effective size $a$ of the atom $R_{0\max} \ll a$. On the other hand, when the inequality $R_{0\max} \gg a$ holds, complete screening of the nuclear Coulomb field must occur.

The minimum value of $q$ is, obviously, equal to $q_{\min} = p_1 - p_2 - \omega$, where $p_1$, $p_2$, and $\omega$ are related by $\sqrt{m^2 + p_1^2} = \sqrt{m^2 + p_2^2} + \omega$.

For low energies and low frequencies $p_1 \ll m$, $\omega \ll p_1$, it follows from the law of conservation of energy that

$$p_2 = \sqrt{p_1^2 - 2m\omega} \approx p_1 - \frac{m\omega}{p_1},$$

and therefore

$$q_{\min} = p_1 - p_2 - \omega \approx \frac{m\omega}{p_1} - \omega \approx \frac{m\omega}{p_1}, \qquad \text{where } p_1 \ll m, \text{ and } \omega \ll p_1.$$

In the relativistic domain we have

$$q_{\min} = \sqrt{\varepsilon_1^2 - m^2} - \sqrt{\varepsilon_2^2 - m^2} + \omega \approx \frac{m^2\omega}{2\varepsilon_1\varepsilon_2}, \quad \text{where } \varepsilon_1 \gg m \text{ and } \varepsilon_2 \gg m.$$

Thus, the maximum value of $R_0$ is determined by

$$R_{0\,\max} \sim \begin{cases} \dfrac{p_1}{m\omega}, & p_1 \ll m, \quad \omega \ll p_1, \\ \dfrac{2\varepsilon_1\varepsilon_2}{m^2\omega}, & \varepsilon_1 \gg m, \quad \varepsilon_2 \gg m, \end{cases} \tag{29.27}$$

from which it follows that at sufficiently low frequencies $R_{0_{\max}}$ can exceed the size of the atom. In other words, as $\omega \to 0$ complete screening always occurs, and this leads to a considerable decrease in the bremsstrahlung cross section. In particular, when $\omega \to 0$ the quantity $\omega d\sigma_\omega$ vanishes, in contrast to the case of the pure Coulomb field for which this quantity diverges logarithmically.

In the relativistic domain $R_{0_{\max}}$ becomes large even at frequencies of the order of the energy of the primary electron. Indeed, if $\omega \sim \varepsilon_1$, $\varepsilon_2 \sim \varepsilon_1$, $\varepsilon_1 \gg m$, then $R_{0_{\max}} \sim (\varepsilon_1/m^2)$.

If we use the Thomas-Fermi atomic model and take the effective radius of the atom equal to $a = a_0 Z^{-1/3}$, where

$$a_0 = \frac{\hbar^2}{me^2} = 0.529 \cdot 10^{-8}\,\text{cm}$$

is the radius of the hydrogen atom, then for an electron energy exceeding $\varepsilon_0 = 137\,(m/Z^{1/3})$, the value of $R_{0_{\max}}$ will be greater than $a$. Therefore, in the domain of high energies screening must always be taken into account.

In order to take into account the screening of the nuclear field by the external electrons we must in accordance with (29.10) find the Fourier component of the total potential due both to the nuclear charge and to the charge of the electrons. This potential satisfies $\Delta A_0^{(e)} = -\varrho$, where $\varrho$ is the charge density $\varrho = -Ze\delta(\boldsymbol{r}) + en(\boldsymbol{r})$ and $n(\boldsymbol{r})$ is the electron density in the atom. On going over to Fourier components, and on utilizing the notation (29.9′), we obtain $-q^2 A_0^{(e)}(\boldsymbol{q}) = e[Z - F(\boldsymbol{q})]$, whence it follows that

$$A_0^{(e)}(\boldsymbol{q}) = -\frac{e[Z - F(\boldsymbol{q})]}{q^2}, \tag{29.28}$$

where $F(\boldsymbol{q})$ is the atomic form-factor $F(\boldsymbol{q}) = \int n(\boldsymbol{r}) e^{-i\boldsymbol{q}\boldsymbol{r}} d\boldsymbol{r}$. Thus, we see that in order to take screening into account we must replace $Z$ by the quantity $Z-F(q)$. In the Thomas-Fermi model we have

$$F(q) = Zf(qZ^{-1/3}), \tag{29.28'}$$

where $f(x)$ is a certain universal function which is the same for all atoms, and whose values may be obtained by means of numerical integration.

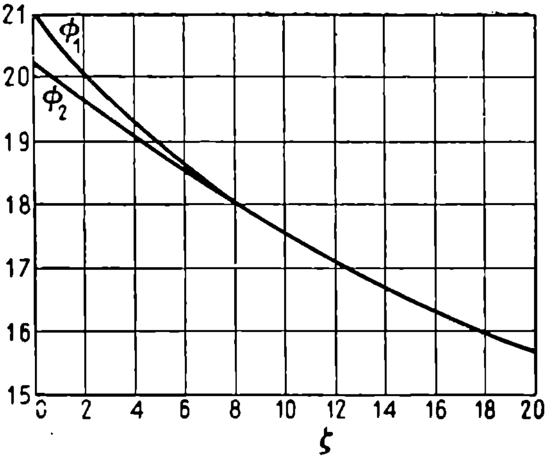


Fig. 27.

On introducing the atomic form-factor for the Thomas-Fermi model we can represent the spectrum of the radiation in the relativistic domain in the following form (21):

$$d\sigma_\omega \equiv \sigma_\omega \frac{d\omega}{\varepsilon_1 - m} = Z^2 a r_0^2 \frac{d\omega}{\omega \varepsilon_1^2}\left\{(\varepsilon_1^2+\varepsilon_2^2)\left[\Phi_1(\zeta) - \frac{4}{3}\ln Z\right] \right.$$

$$\left. -\frac{2}{3}\varepsilon_1\varepsilon_2\left[\Phi_2(\zeta) - \frac{4}{3}\ln Z\right]\right\}, \tag{29.29}$$

where

$$\zeta = 200a\frac{a}{R_{0\max}} = 100\frac{m\omega}{\varepsilon_1\varepsilon_2 Z^{1/3}}$$

and $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ are the two functions of $\zeta$ shown in Fig. 27. The quantity $\zeta$ which is proportional to the ratio $a/R_{0\max}$ determines the effect of screening. Small values of $\zeta$ correspond to large screening and large values of $\zeta$ correspond to small screening. If $\zeta = 0$ we can speak of complete screening. In this case $\Phi_1(0) = 4 \ln 183, \Phi_2(0) = 4 \ln 183 - \frac{2}{3}$ and the cross section (29.29) assumes the form

$$\sigma_\omega d\frac{\omega}{\varepsilon_1 - m} = Z^2 a r_0^2 \frac{d\omega}{\omega} \frac{4}{\varepsilon_1^2}\left\{\left(\varepsilon_1^2+\varepsilon_2^2-\frac{2}{3}\varepsilon_1\varepsilon_2\right)\ln\left(183Z^{-1/3}\right)+\frac{\varepsilon_1\varepsilon_2}{9}\right\},$$

$$\left(\frac{\varepsilon_1\varepsilon_2}{m\omega} \gg 137\,Z^{-1/3}, \qquad \varepsilon_1 \gg 137\,mZ^{-1/3}\right). \qquad (29.30)$$

Under these conditions the bremsstrahlung cross section for a given value of the ratio $\omega/\varepsilon_1$ does not depend on the primary electron energy $\varepsilon_1$. For large values of $\zeta$ this conclusion becomes incorrect.

For $\zeta \gg 1$ the screening effect plays no role. The small correction due to it can be taken into account if we replace in the last bracketed expression in formula (29.25) the term $\frac{1}{2}$ by $\frac{1}{2}+c(\zeta)$, where the values of $c(\zeta)$ are given in Table 10.

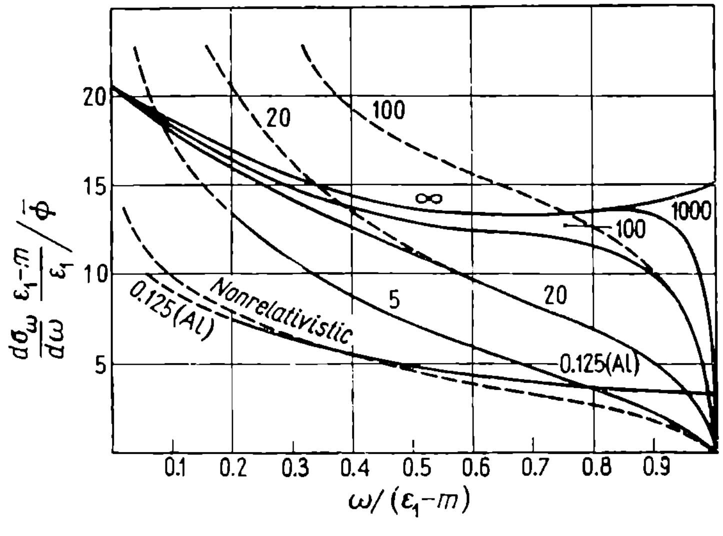


Fig. 28.

Figure 28 shows the dependence of the intensity of bremsstrahlung on the ratio of the photon energy to the electron kinetic energy $\omega/(\varepsilon_1-m)$ (21). The dotted curves do not take screening into account and are valid for all the elements (the nuclear charge $Ze$ appears only in the quantity $\Phi$). The numbers beside the curves indicate the initial energies $\varepsilon_1-m$ in units of $m$. The solid curves take screening into account, and refer to lead $Z = 82$ (with the exception of the solid curve corresponding to $(\varepsilon_1 - m)/m = 0.125$ which refers to aluminum). In the region of low frequencies they approach the curve which refers to the case of complete screening. The nonrelativistic curve for aluminum has been plotted taking into account the exact wave functions for the electron in the continuous spectrum (cf. subsection 29.8). We see that in the case of light elements screening is not important in the nonrelativistic domain.

TABLE 10

| $\zeta$ | 2 | 2.5 | 3 | 4 | 5 | 6 | 8 | 10 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| $c(\zeta)$ | 0.21 | 0.16 | 0.13 | 0.09 | 0.065 | 0.05 | 0.03 | 0.02 | 0.01 |

We note that the use of the Born approximation which we have employed can lead to an error in the case of large values of $Z$ not only in the nonrelativistic domain. This is associated with the fact that the parameter $\zeta = Ze^2/v$ is not small in the domain of large values of the energy and of $Z$, but is equal to $Z/137$. As we shall see later, the deviation from the Born approximation in the relativistic domain is noticeable only at large values of $q$ for which screening plays no role. Therefore, corrections to the bremsstrahlung cross section due to the deviation from the Born approximation and to the screening effect are simply additive.

## 29.7. *Radiative Energy Losses*

If we know the bremsstrahlung cross section we can obtain the average energy loss for an electron in passing through a medium. This energy loss, evaluated per unit path length of the electron, is obviously equal to

$$-\left(\frac{d\varepsilon_1}{dx}\right)_r = N\int_0^{\varepsilon_1 - m} \sigma_\omega \omega d\omega,$$

where $N$ is the number of atoms per unit volume of the medium. If we define the radiation damping cross section $\Phi_r$ as

$$\Phi_r = \frac{1}{\varepsilon_1}\int^{\varepsilon_1 - m} \omega\sigma_\omega d\omega, \qquad (29.31)$$

then the energy loss will be given as

$$-\left(\frac{d\varepsilon_1}{dx}\right)_r = N\varepsilon_1\Phi_r. \tag{29.32}$$

By utilizing expression (29.23) for $\sigma_\omega$ we can evaluate $\Phi_r$:

$$\Phi_r = \bar{\Phi}\left\{\frac{12\varepsilon_1^2+4m^2}{3\varepsilon_1 p_1}\ln\frac{\varepsilon_1+p_1}{m} - \frac{(8\varepsilon_1+6p_1)m^2}{3\varepsilon_1 p_1^2}\left(\ln\frac{\varepsilon_1+p_1}{m}\right)^2 - \frac{4}{3} + \frac{2m^2}{\varepsilon_1 p_1}F\left(\frac{2p_1(\varepsilon_1+p_1)}{m^2}\right)\right\}, \tag{29.33}$$

where

$$F(x) = \int_0^x \frac{\ln(1+y)}{y}\,dy.$$

On noting that

$$F(x) = [\pi^2/6]+[(\ln x^2)/2]-F(1/x)$$

and that for small $x$, $F(x) = x-(x^2/4)+(x^3/9)-(x^4/16)+\cdots$, we can obtain $\Phi_r$ in the two limiting cases of low and high energies:

$$\begin{aligned} \Phi_r &= \frac{16}{3}\bar{\Phi}, \quad \text{where} \quad \varepsilon_1 \ll m, \\ \Phi_r &= 4\left(\ln\frac{2\varepsilon_1}{m}-\frac{1}{3}\right)\bar{\Phi}, \quad \text{where} \quad \varepsilon_1 \gg m. \end{aligned} \tag{29.34}$$

We see that in the nonrelativistic domain the ratio of the radiated energy to the initial electron energy does not depend on the initial energy, while in the relativistic case $\Phi_r$ increases logarithmically with $\varepsilon_1$. However, this result holds only if the screening effect is neglected. If we take screening into account, then in place of the second formula (29.34) we obtain the constant cross section

$$\Phi_r = \bar{\Phi}\left(4\ln(183Z^{-1/3})+\tfrac{2}{9}\right), \quad \text{where} \quad \varepsilon_1 \gg 137mZ^{-1/3}. \tag{29.35}$$

The values of $\Phi_r/\bar{\Phi}$ for different energies are given in Table 11.

TABLE 11

| $(\varepsilon_1-m)/m$ | | 0 | 1 | 2 | 5 | 10 | 20 | 50 | 100 | 200 | 1000 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\Phi_r}{\bar{\Phi}}$ | $H_2O$ | 5.33 | 5.5 | 6.5 | 9.1 | 11.2 | 12.9 | 14.6 | 15.6 | 16.4 | 17.5 | 18.3 |
| | Pb | 5.33 | 5.5 | 6.5 | 8.75 | 10.3 | 11.4 | 12.6 | 13.6 | 13.8 | 14.5 | 15.2 |

The energy dependence of the cross section $\Phi_r$ taking screening into account is shown in Fig. 29 on a logarithmic scale. The straight line corresponds to the logarithmic increase in the cross section in accordance with the second formula (29.34) with screening not taken into account.

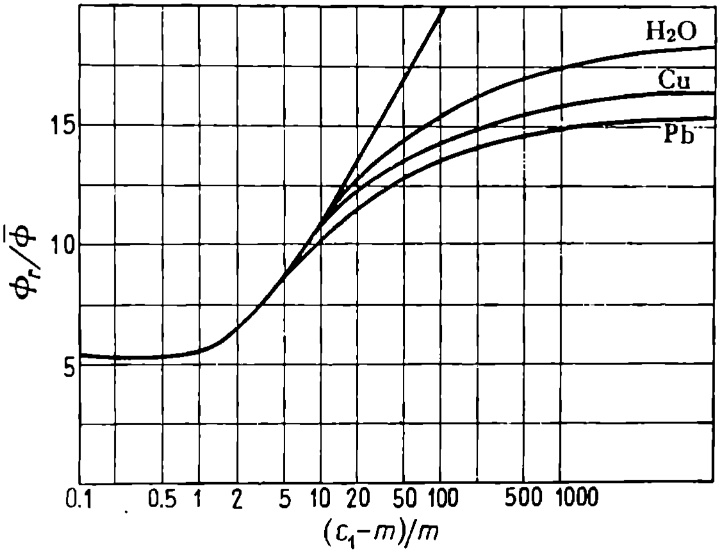


Fig. 29.

The energy lost by the electron per unit path length due to its inelastic collisions with atoms is determined by the following formula (21):

$$-\left(\frac{d\varepsilon_1}{dx}\right)_i = 2\pi r_0^2 mNZ \ln\frac{\varepsilon_1^3}{2mI^2}, \tag{29.36}$$

where $I$ is the average ionization potential which is approximately equal to $I = 13.5Z$ eV. The ratio of radiation losses to collision losses is approximately given by (21):

$$\frac{-(d\varepsilon_1/dx)_r}{-(d\varepsilon_1/dx)_i} = \frac{1}{1600}\frac{\varepsilon_1 Z}{m}. \tag{29.36'}$$

Here it is assumed that the logarithmic term in both formulas (29.34) and (29.36) has approximately the same value. It follows from formula (29.36′) that losses due to collisions and to radiation become equal at an energy given by $\varepsilon_0 = 1600(m/Z)$. At higher energies the radiation losses exceed the losses due to collisions ($\varepsilon_0$ is equal to 10 MeV for lead, 55 MeV for copper, 200 MeV for air).

### 29.8. *Exact Theory of Bremsstrahlung in the Nonrelativistic Domain*

As we have already noted earlier, the matrix element (27.3) which determines the bremsstrahlung in the Coulomb field of a nucleus can be calculated exactly in the nonrelativistic and in the extreme relativistic domains.

We shall, first of all, show how this calculation is carried out in the nonrelativistic case (187).

The energy of interaction between the electron and the electromagnetic radiation field $-\boldsymbol{\alpha A}$ corresponds in the nonrelativistic approximation (cf. § 15) to the operator $-\boldsymbol{pA}/m$, where $\boldsymbol{p}$ is the electron momentum operator. Therefore, the matrix element (27.3) assumes the following form in the nonrelativistic domain:

$$S^{(1)}_{i\to f} = \frac{ie}{\sqrt{2\omega}\, m}\int \psi_2^* e\boldsymbol{p} e^{-i\boldsymbol{kr}+i\omega t}\psi_1\, d\boldsymbol{r}\, dt.$$

The exponential factor $\exp(i\boldsymbol{kr})$ appearing in this expression can evidently be replaced by unity. Indeed, in the integral $S^{(1)}_{i\to f}$ the important values of $r$ are of the order $r\sim v\tau$, where $v$ is the electron velocity and $\tau$ is the collision time, which is given in order of magnitude by $\tau\sim 1/\omega$; therefore $kr\sim \omega v\tau \sim v \ll 1$.

Thus,

$$S^{(1)}_{i\to f} = -2\pi i U_{i\to f}\,\delta(\varepsilon_1-\varepsilon_2-\omega), \tag{29.37}$$

where

$$U_{i\to f} = -\frac{e}{\sqrt{2\omega}}\int \psi_2^*(\boldsymbol{r})\frac{e\boldsymbol{p}}{m}\psi_1(\boldsymbol{r})\, d\boldsymbol{r} \tag{29.37'}$$

and $\psi_1(\boldsymbol{r})$ and $\psi_2(\boldsymbol{r})$ are the wave functions for the electron in the initial and the final states with the time dependent factors omitted ($\varepsilon_1$ and $\varepsilon_2$ are the corresponding energies).

Since the matrix element of the operator for the velocity $\boldsymbol{v} = \boldsymbol{p}/m$ is related to the matrix element of the radius-vector $\boldsymbol{r}$ by $(\boldsymbol{v})_{12} = i\omega(\boldsymbol{r})_{12}$, $U_{i\to f}$ can be written in the form

$$U_{i\to f} = -\frac{ie\omega}{\sqrt{2\omega}}\int \psi_2^*(\boldsymbol{r})\, e\boldsymbol{r}\,\psi_1(\boldsymbol{r})\, d\boldsymbol{r}.$$

Thus, the problem is reduced to the evaluation of the matrix element of the coordinate using the continuous spectrum wave functions for the Coulomb field.

In order to determine completely the continuous spectrum wave functions it is necessary to specify their asymptotic behavior at infinity, which may be of two forms, since at infinity the function $\psi$ may have the form either of the sum of a plane wave corresponding to a definite momentum, and of an outgoing spherical wave, or the form of the sum of a plane wave and an incoming spherical wave. As has been shown in subsection 29.1, the wave function describing the initial state of the electron must have an asymptotic behavior of the first type, while the wave function of the final state must have an asymptotic behavior of the second type.

In the case of a Coulomb field the wave functions with such asymptotic behavior have the following form in the nonrelativistic approximation:

$$\begin{aligned} \psi_1(\mathbf{r}) &= N_1 e^{i\mathbf{p}_1\mathbf{r}} F[i\xi_1,\ 1,\ i(p_1 r - \mathbf{p}_1\mathbf{r})], \\ \psi_2(\mathbf{r}) &= N_2 e^{i\mathbf{p}_2\mathbf{r}} F[-i\xi_2,\ 1,\ -i(p_2 r + \mathbf{p}_2\mathbf{r})], \end{aligned} \tag{29.38}$$

where $F$ is the confluent hypergeometric function, $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of the electron in the initial and final states, $\xi_{1,2} = \alpha Z/v_{1,2}$ ($v_{1,2}$ are the electron velocities, $\alpha = e^2/\hbar c = 1/137$), $N_1$ and $N_2$ are normalizing constants. At large values of $r$ these functions are asymptotically given by

$$\psi_1(\mathbf{r}) \approx N_1 \frac{e^{-(\pi/2)\xi_1}}{\Gamma(1-i\xi_1)} \left( e^{i\mathbf{k}_1\mathbf{r}} + \frac{f_1(\theta)}{r} \cdot e^{ik_1 r} \right),$$

$$\psi_2(\mathbf{r}) \approx N_2 \frac{e^{-(\pi/2)\xi_2}}{\Gamma(1+i\xi_2)} \left( e^{i\mathbf{k}_2\mathbf{r}} + \frac{f_2(\theta)}{r} \cdot e^{-ik_2 r} \right),$$

where

$$f_1(\theta) = \frac{\xi_1}{2k_1 \sin^2 \frac{\theta}{2}} e^{2i\xi_1 \ln \sin \frac{\theta}{2}} \cdot \frac{\Gamma(1-i\xi_1)}{\Gamma(1+i\xi_1)},$$

$$f_2(\theta) = \frac{\xi_2}{2k_2 \cos^2 \frac{\theta}{2}} e^{-2i\xi_2 \ln \cos \frac{\theta}{2}} \cdot \frac{\Gamma(1+i\xi_2)}{\Gamma(1-i\xi_2)}.$$

If the constants $N_1$ and $N_2$ are taken equal to

$$\begin{aligned} N_1 &= e^{\frac{\pi}{2}\xi_1}\Gamma(1-i\xi_1), \\ N_2 &= e^{\frac{\pi}{2}\xi_2}\Gamma(1+i\xi_2), \end{aligned} \tag{29.38'}$$

then the wave functions will have unit amplitudes at infinity. In the future we shall use this normalization.

In accordance with the general rules of § 24 the differential cross section for the emission of a photon of frequency $\omega$ and polarization $e$ within the solid angle $do_\gamma$ is given by

$$d\sigma = \frac{e^2\omega^3 d\omega p_2}{2(2\pi)^4 p_1}\left|\int \psi_2^* \boldsymbol{re}\psi_1 d\boldsymbol{r}\right|^2 do_\gamma do_2, \tag{29.39}$$

where $do_2$ is the element of solid angle containing the electron momentum after scattering, $p_{1,2} \equiv |\boldsymbol{p}_{1,2}|$.

The differential bremsstrahlung cross section summed over the two photon polarizations is given by

$$d\sigma = \frac{e^2\omega^3 d\omega p_2}{2(2\pi)^4 p_1}\left(|\boldsymbol{D}|^2 - \frac{(\boldsymbol{kD})^2}{k^2}\right), \tag{29.40}$$

where

$$\boldsymbol{D} = \int \psi_2^*(\boldsymbol{r})\boldsymbol{r}\psi_1(\boldsymbol{r})d\boldsymbol{r}. \tag{29.40'}$$

In order to evaluate $\boldsymbol{D}$ we use the following general formula (144):

$$\begin{aligned} J &\equiv \int d\boldsymbol{r} e^{-\lambda r}\frac{e^{i\boldsymbol{qr}}}{r}F(i\xi_1,\ 1,\ ip_1 r - i\boldsymbol{p}_1\boldsymbol{r})F(i\xi_2,\ 1,\ ip_2 r + i\boldsymbol{p}_2\boldsymbol{r}) \\ &= \frac{2\pi}{\alpha}e^{-\pi\xi_1}\left(\frac{\alpha}{\gamma}\right)^{i\xi_1}\left(\frac{\alpha+\delta}{\gamma}\right)^{-i\xi_2}F\left(1-i\xi_1,\ i\xi_2,\ 1, \frac{\alpha\delta-\beta\gamma}{\alpha(\gamma+\delta)}\right), \end{aligned} \tag{29.41}$$

where

$$\alpha = \tfrac{1}{2}(q^2+\lambda^2), \qquad \beta = \boldsymbol{p}_2\boldsymbol{q} - i\lambda p_2,$$
$$\gamma = \boldsymbol{p}_1\boldsymbol{q} + i\lambda p_1 - \alpha, \qquad \delta = p_1 p_2 - \boldsymbol{p}_1\boldsymbol{p}_2 - \beta.$$

In order to evaluate $\boldsymbol{D}$ it is, obviously, sufficient to differentiate $J$ with respect to $\lambda$ and $\boldsymbol{q}$ and then to set $\lambda = 0$, $\boldsymbol{q} = \boldsymbol{p}_1 - \boldsymbol{p}_2$:

$$\boldsymbol{D} = N_1 N_2 \frac{\partial^2 J}{\partial\lambda\partial\boldsymbol{q}}\Bigg|_{\substack{\lambda=0 \\ \boldsymbol{q}=\boldsymbol{p}_1-\boldsymbol{p}_2}}. \tag{29.42}$$

By utilizing this value of $\boldsymbol{D}$ we can obtain the following expression for the differential bremsstrahlung cross section integrated over the angles (187):

$$d\sigma_\omega = \frac{16\pi^2}{3}\frac{Z^2\alpha^3}{p_1^2}\frac{1}{(1-e^{-2\pi\xi_2})(e^{2\pi\xi_1}-1)}\frac{d\omega}{\omega}x_0\frac{d}{dx_0}|F(x_0)|^2, \tag{29.43}$$

where $F(x_0)$ is the hypergeometric function

$$F(x_0) \equiv F(i\xi_1,\ i\xi_2,\ 1,\ x_0)$$

and

$$x_0 = -\frac{4\xi_1\xi_2}{(\xi_1-\xi_2)^2}.$$

We investigate the expressions to which this formula reduces for $\xi_1 \ll 1$ and $\xi_2 \ll 1$.

By utilizing the relations (cf., for example, (119))

$$\frac{d}{dx_0}F(i\xi_1,\ i\xi_2,\ 1,\ x_0) = -\xi_1\xi_2 F(1+i\xi_1,\ 1+i\xi_2,\ 2,\ x_0),$$

$$F(a,\ b,\ c,\ z) = (1-z)^{-a}F\left(a,\ c-b,\ c,\frac{z}{z-1}\right),$$

we have

$$\frac{d}{dx_0}F(i\xi_1,\ i\xi_2,\ 1,\ x_0) = -\xi_1\xi_2(1-x_0)^{-1-i\xi_1}F\left(1+i\xi_1,\ 1-i\xi_2,\ 2,\frac{x_0}{x_0-1}\right).$$

If $\xi_1 \ll 1$, $\xi_2 \ll 1$, then

$$F(i\xi_1,\ i\xi_2,\ 1,\ x_0) \approx 1,$$

$$\frac{d}{dx_0}F(i\xi_1,\ i\xi_2,\ 1,\ x_0) \approx -\xi_1\xi_2\frac{1}{1-x_0}F\left(1,\ 1,\ 2,\frac{x_0}{x_0-1}\right).$$

Further, on noting that for $0 < z < 1$

$$F(1,\ 1,\ 2,\ -z) = \frac{1}{z}\ln(1+z),$$

and on setting $z = x_0/(x_0-1)$, we obtain

$$\frac{d}{dx_0}F(i\xi_1,\ i\xi_2,\ 1,\ x_0) \approx \xi_1\xi_2\frac{\ln(1-x_0)}{x_0}, \quad \text{where} \quad \xi_1 \ll 1,\ \xi_2 \ll 1.$$

Therefore, we have

$$\frac{d}{dx_0}|F(x_0)|^2 = 2F(x_0)\xi_1\xi_2\frac{\ln(1-x_0)}{x_0},$$

i.e.,

$$\frac{d}{dx_0}|F(x_0)|^2 = 2\xi_1\xi_2\frac{\ln(1-x_0)}{x_0}.$$

Thus, for $\xi_1 \ll 1$, $\xi_2 \ll 1$ the following formula holds:

$$d\sigma_\omega = \frac{16}{3}\frac{Z^2\alpha^3}{p_1^2}\frac{4\pi^2\xi_1\xi_2}{(e^{2\pi\xi_1}-1)(1-e^{-2\pi\xi_2})}\cdot\frac{d\omega}{\omega}\cdot\ln\left(\frac{p_1+p_2}{p_1-p_2}\right). \tag{29.44}$$

On replacing in this expression $(2\pi)^2\xi_1\xi_2/[\exp(2\pi\xi_1)-1]\,[1-\exp(-2\pi\xi_2)]$ by unity we obtain formula (29.24) for the bremsstrahlung cross section in the nonrelativistic domain in the Born approximation.

However, formula (29.44) has a broader region of validity than formula (29.24), viz., it is valid under the single condition $\xi_1 \ll 1$, and insofar as $\xi_2$ is concerned this quantity may be arbitrary. Indeed, if $\xi_1 \ll 1$, while $\xi_2 \gg 1$, then $x_0 = -4\xi_1/\xi_2$, i.e., $|x_0| \ll 1$, and, therefore

$$F(i\xi_1, i\xi_2, 1, x_0) \approx 1,$$

$$\frac{d}{dx_0}F(i\xi_1, i\xi_2, 1, x_0) \approx -\xi_1\xi_2.$$

Further, on noting that for $|x_0| \ll 1$ $\frac{1}{x_0}\ln(1-x_0) \approx -1$, we obtain

$$\frac{d}{dx}|F(x_0)|^2 \approx 2\xi_1\xi_2\frac{\ln(1-x_0)}{x_0}, \quad \text{where} \quad \xi_1 \ll 1,\ \xi_2 \approx 1.$$

On substituting this expression into (29.43) we obtain formula (29.44).

### 29.9. *Exact Theory of Bremsstrahlung in the Extreme Relativistic Domain*

We now show how the matrix element (27.3), which determines the bremsstrahlung in the extreme relativistic domain, may be evaluated (22).

This calculation is based on using the relativistic wave functions given in subsection 14.5

$$\psi = Ne^{i\mathbf{pr}}\left(1-\frac{i}{2\varepsilon}\boldsymbol{\alpha}\nabla\right)uF(i\xi, 1, ipr-i\mathbf{pr}), \tag{29.45}$$

where $F$ is the same hypergeometric function which appears in the nonrelativistic wave function (29.38), $u$ is the normalized spinor amplitude of the plane wave, $\boldsymbol{\alpha}$ are the Dirac matrices, and $N$ is a normalizing factor.

The basic property of the functions (29.45) is the fact that if they are expanded in terms of spherical waves, then for any value of the energy (not necessarily large) the terms of these expansions corresponding to the values $l \gg l_0$, where $l_0 = Z/137$, practically coincide with the terms of the expansions of the exact solutions of the Dirac equation for a Coulomb field. Thus, the functions (29.45) differ from the exact wave functions for a Coulomb field only for values of $l$ of the order of $l_0$.

This property of the functions (29.45) makes them extremely convenient for the evaluation of matrix elements of those processes in which small values of the angular momentum $l$ are not important.

In particular, bremsstrahlung in the nuclear Coulomb field in the domain of high energies is included among such processes (as well as pair production by a photon in the field of a nucleus, cf. § 32).

Indeed, for the bremsstrahlung process the important impact parameters are of the order of, or greater than, $B \sim 1/ma$, and since the effective values of $l$ are of the order of magnitude of $l \sim b/\lambda\!\!\!^{-}$ where $\lambda\!\!\!^{-}$ is the electron wave length, we have $l \sim bp \sim (1/(\varepsilon/a)m)$. If, therefore, the electron energy $\varepsilon$ is considerably greater than $m$, then large values of $l$ will be the important ones and the approximation in which the functions (29.45) are used as wave functions must yield good results.

Large values of $l$ correspond to scattering through small angles, which are prominent in bremsstrahlung in the relativistic domain. Thus, the approximation under consideration must give particularly good results in the domain of high energies and small scattering angles.

The matrix element determining bremsstrahlung (omitting the time factors) can be written as

$$M^{(1)}_{i\to f} = -\frac{2\pi ie}{\sqrt{2\omega}} \int \psi_2^*(r) e\alpha e^{-ikr} \psi_1(r)\, dr,$$

where $\psi_1$ and $\psi_2$ are the electron wave functions in the initial and final states, for which we take the functions (29.45); here, as has been explained in subsection 29.1, the wave function for the initial state must for $r \to \infty$ go over asymptotically into the sum of a plane wave and an outgoing spherical wave, while the wave function for the final state must go over into the sum of a plane wave and an incoming wave. The functions $\psi_1$ and $\psi_2$ are assumed to be normalized to unit amplitude at infinity.

In such a case $N_1 = e^{\frac{\pi\xi_1}{2}}\Gamma(1-i\xi_1)$ and $N_2 = e^{\frac{\pi\xi_2}{2}}\Gamma(1+i\xi_2)$. Thus we have

$$M^{(1)}_{i\to f} = -\frac{2\pi ie}{\sqrt{2\omega}}N_1 N_2^*$$

$$\times\int e^{-i\mathbf{p}_2\mathbf{r}}\left[u_2^* F_2^* + \frac{i}{2\varepsilon_2}\boldsymbol{\alpha}^*\nabla F_2^* u_2^*\right]$$

$$\times(\mathbf{e}\boldsymbol{\alpha})e^{-i\mathbf{k}\mathbf{r}}e^{i\mathbf{p}_1\mathbf{r}}\left[u_1 F_1 - \frac{i}{2\varepsilon_1}\boldsymbol{\alpha}\nabla F_1 u_1\right]d\mathbf{r}$$

$$= -\frac{2\pi ieN_1N_2^*}{\sqrt{2\omega}}\Big[(u_2^*\boldsymbol{\alpha}\mathbf{e}u_1)I_1$$

$$+\left(u_2^*(\mathbf{e}\boldsymbol{\alpha})(\boldsymbol{\alpha}\mathbf{I}_2)u_1\right) + \left(u_2^*(\boldsymbol{\alpha}\mathbf{I}_3)(\boldsymbol{\alpha}\mathbf{e})u_1\right)\Big], \qquad (29.46)$$

where

$$F_1 = F(i\xi_1,\ 1,\ ip_1r - i\mathbf{p}_1\mathbf{r}),$$

$$F_2 = F(-i\xi_2,\ 1,\ -ip_2r - i\mathbf{p}_2\mathbf{r}),$$

$$I_1 = \int e^{i\mathbf{q}\mathbf{r}}F_2^*F_1\,d\mathbf{r}.$$

$$\mathbf{I}_2 = -\frac{i}{2\varepsilon_1}\int e^{i\mathbf{q}\mathbf{r}}F_2^*\nabla F_1\,d\mathbf{r},$$

$$\mathbf{I}_3 = \frac{i}{2\varepsilon_2}\int e^{i\mathbf{q}\mathbf{r}}(\nabla F_2^*)F_1\,d\mathbf{r},$$

$$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}.$$

It might appear that in the high energy region, which we are investigating, there is no necessity to retain in functions of the form (29.45) the second term, which is inversely proportional to $\varepsilon$. However, this is not so, because the factor $u_2^*\mathbf{e}\boldsymbol{\alpha}u_1$ arising from the first main terms of (29.45) physically corresponds to the component of the electron velocity parallel to the photon polarization, i.e., perpendicular to its momentum, which in the relativistic domain is extremely small. For this reason the factor $u_2^*\mathbf{e}\boldsymbol{\alpha}u_1$ is of order of magnitude of $m/\varepsilon$, while at the same time the factor $u_2^*(\mathbf{e}\boldsymbol{\alpha})\boldsymbol{\alpha}u_1$ is of order of magnitude unity. As a result, both terms of (29.45) lead to quantities of the same order of magnitude in the matrix element.

The evaluation of the integrals appearing in (29.46) can be carried out with the aid of formula (29.41). Indeed,

$$
\begin{aligned}
I_1 &= -\left(\frac{\partial J}{\partial \lambda}\right)_{\lambda=0}, \\
I_2 &= i\frac{p_1}{2\varepsilon_1}(\nabla_{p_1} J)_{\lambda=0}, \qquad\qquad (29.47)\\
I_3 &= i\frac{p_2}{\varepsilon_2}(\nabla_{p_2} J)_{\lambda=0}.
\end{aligned}
$$

The differential bremsstrahlung cross section summed over the photon polarizations $\nu$ is related to $M^{(1)}_{i\to f}$ by the expression

$$d\sigma = 2\pi\frac{1}{2}\sum_{\mu_1,\mu_2,\nu} |M^{(1)}_{i\to f}|^2 \frac{dp_2\, dk}{(2\pi)^6}\,\delta(\varepsilon_1-\varepsilon_2-\omega).$$

We do not reproduce here the detailed calculations, but state only the final result (22)

$$d\sigma = d\sigma_0 \frac{V^2(x)+Z^2\alpha^2 y^2 W^2(x)}{V^2(1)}, \qquad (29.48)$$

where $d\sigma_0$ is the bremsstrahlung cross section evaluated in the Born approximation, and given by formula (29.16),

$$
\begin{aligned}
V(x) &= F(-i\xi, i\xi, 1, x) \\
&= 1+\frac{\xi^2}{(1!)^2}x+\frac{\xi^2(1+\xi^2)}{(2!)^2}x^2+\frac{\xi^2(1+\xi^2)(2^2+\xi^2)}{(3!)^2}x^3+\cdots,
\end{aligned}
$$

$$V(1) = \frac{\sinh \pi\xi}{\pi\xi},$$

$$W(x) = \frac{1}{\xi^2}\frac{dV(x)}{dx} = 1+\frac{1+\xi^2}{1!\,2!}x+\frac{(1+\xi^2)(2^2+\xi^2)}{2!\,3!}x^2+\cdots,$$

$$x = 1-y, \qquad y = \frac{\omega^2(\varepsilon_2-p_2\cos\theta_2)(\varepsilon_1-p_1\cos\theta_1)}{\varepsilon_1\varepsilon_2 q^2}$$

and $\theta_1$, $\theta_2$ are the angles between $k$ and $p_1$, $p_2$.

Over a considerable range of important angles the differential bremsstrahlung cross section (29.48) practically coincides with $d\sigma_0$. This is associated with the fact that the scattered waves in the initial and the final electron states in practice overlap very little.

The bremsstrahlung cross section integrated over the angles has the following form (145):

$$d\sigma(\omega) = d\sigma_0(\omega) - 4Z^2\alpha^3 \frac{1}{m^2\omega p_1^2}\left(\varepsilon_1^2+\varepsilon_2^2-\frac{2}{3}\varepsilon_1\varepsilon_2\right) f(Z)\, d\omega, \quad (29.49)$$

where $d\sigma_0(\omega)$ is the cross section evaluated in the Born approximation taking screening into account, and

$$f(Z) = (Z\alpha)^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+Z^2\alpha^2)}. \quad (29.49')$$

Thus, the bremsstrahlung cross section is always less than the value obtained in the Born approximation. Insofar as the form of the bremsstrahlung spectrum is concerned, it is essentially unaltered when the correction to the Born approximation is taken into account.

We note that for the evaluation of the bremsstrahlung cross section integrated over all the directions of the scattered electron we need not necessarily use for the final state the electron wave functions which have the asymptotic behavior (29.8); we can equally well use the functions with the asymptotic behavior (29.7) since such a replacement merely indicates a different choice of the complete system of functions describing the particle. This enables us to relate bremsstrahlung cross section integrated over the angles to the cross section for pair production (cf. subsection 32.2).

### 29.10. *Radiation Emitted in Electron-Electron and Electron-Positron Collisions*

We now investigate the process of photon emission in electron-electron collisions.

In contrast to the problem of bremsstrahlung in the collision of an electron with a nucleus, we cannot in this case replace the effect of one of the particles by an external field. Therefore, we have to investigate the elements of the scattering matrix which connect one photon state and four electron states (two initial and two final states). Such elements are contained in the third order scattering matrix $S^{(3)}$. The eight diagrams corresponding to them are shown in Fig. 30. In these diagrams $p_1$ and $p_1'$ denote the momenta of the initial states, $p_2$ and $p_2'$ the momenta of the final states, $k$ is the momentum of the emitted photon, $f_s$ are the momenta of the virtual electrons, and $q_s$ are the momenta of the virtual photons ($s = 1, 2, \ldots, 8$). The last four diagrams (*5, 6, 7, 8*) differ from the first four by the replacement of $p_2$ by $p_2'$ and of $p_2'$ by $p_2$. We shall refer to these diagrams as exchange diagrams (the terms corresponding to these diagrams appear in the matrix element with opposite sign).

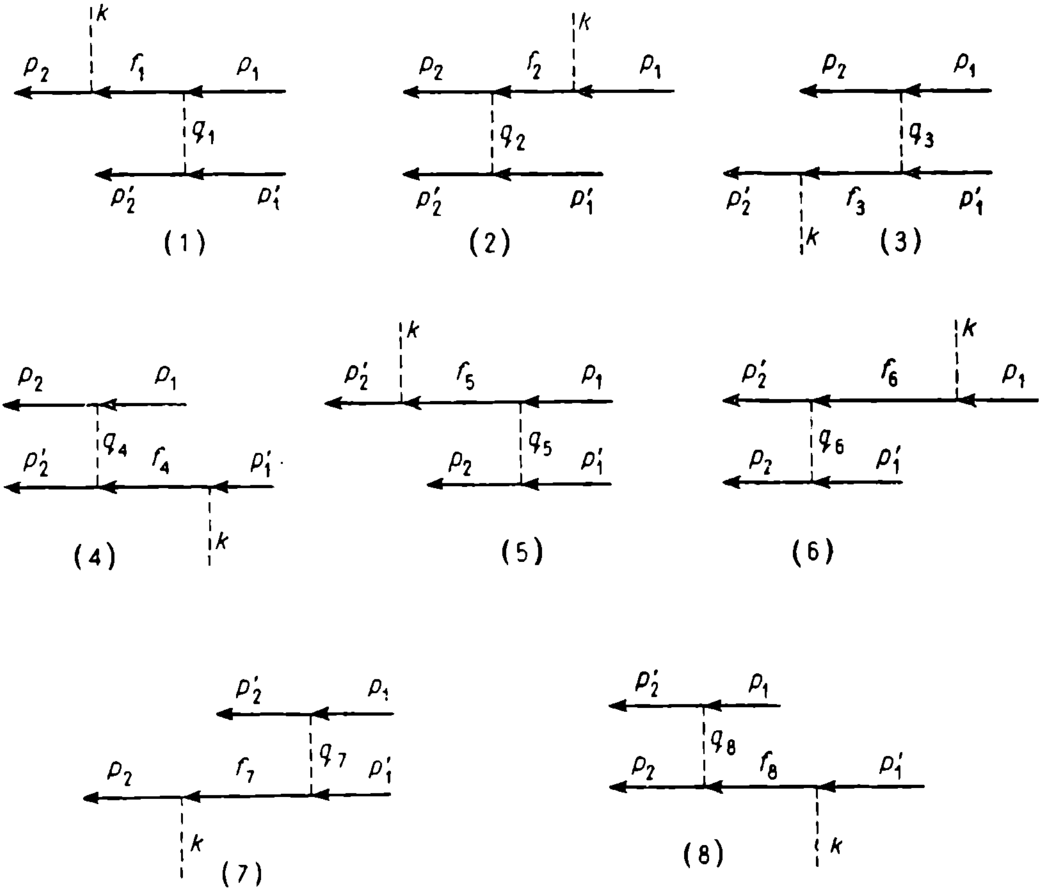


Fig. 30.

In diagrams *1* and *3* scattering occurs first, followed by emission of radiation, while in diagrams *2* and *4* conversely emission of radiation occurs first, followed by scattering. In diagrams *1* and *2* the radiation is emitted by the first electron, while in diagrams *3* and *4* it is emitted by the second electron.

By moving along the electron lines of the diagram from left towards right, and by replacing the lines and the vertices by the amplitudes and the matrices corresponding to them, we obtain in accordance with Feynman's rules the following expression for the matrix element:

$$
\begin{aligned}
S^{(3)}_{i\to f}=\frac{-e^3}{\sqrt{2\omega}}\Bigg\{&\left(\bar{u}_2\hat{e}\frac{i\hat{f}_1-m}{f_1^2+m^2}\gamma_\nu u_1\right)\frac{1}{q_1^2}(\bar{u}_2'\gamma_\nu u_1')\\
&+\left(\bar{u}_2\gamma_\nu\frac{i\hat{f}_2-m}{f_2^2+m^2}\hat{e}u_1\right)\frac{1}{q_2^2}(\bar{u}_2'\gamma_\nu u_1')+(\bar{u}_2\gamma_\nu u_1)\frac{1}{q_3^2}\left(\bar{u}_2'\hat{e}\frac{i\hat{f}_3-m}{f_3^2+m^2}\gamma_\nu u_1'\right)\\
&+(\bar{u}_2\gamma_\nu u_1)\frac{1}{q_4^2}\left(\bar{u}_2'\gamma_\nu\frac{i\hat{f}_4-m}{f_4^2+m^2}\hat{e}u_1'\right)-\left(\bar{u}_2'\hat{e}\frac{i\hat{f}_5-m}{f_5^2+m^2}\gamma_\nu u_1\right)\frac{1}{q_5^2}(\bar{u}_2\gamma_\nu u_1')\\
&-\left(\bar{u}_2'\gamma_\nu\frac{i\hat{f}_6-m}{f_6^2+m^2}\hat{e}u_1\right)\frac{1}{q_6^2}(\bar{u}_2\gamma_\nu u_1')-(\bar{u}_2'\gamma_\nu u_1)\frac{1}{q_7^2}\left(\bar{u}_2\hat{e}\frac{i\hat{f}_7-m}{f_7^2+m^2}\gamma_\nu u_1'\right)\\
&-(\bar{u}_2'\gamma_\nu u_1)\frac{1}{q_8^2}\left(\bar{u}_2\gamma_\nu\frac{i\hat{f}_8-m}{f_8^2+m^2}\hat{e}u_1'\right)\Bigg\}(2\pi)^4\delta(p_1+p_1'-p_2-p_2'-k). \qquad (29.50)
\end{aligned}
$$

In each term the first bracketed expression corresponds to the upper electron line, and the second to the lower one. The values of $f_s$ and $q_s$ are determined by the conservation laws at each vertex of the diagram:

$$
q_1=q_2=p_2'-p_1',\quad q_3=q_4=p_2-p_1,\quad f_1=p_2+\kappa,
$$

$$
f_2=p_1-k,\quad f_3=p_2'+k,\quad f_4=p_1'-k.
$$

The expression for the effective cross section corresponding to the matrix element (29.50) turns out to be very awkward, since after summation over the polarizations of the electrons and of the photon a large number of terms appears containing traces of products of six $\gamma_\mu$ matrices.

We shall give here only the results referring to two limiting cases: the nonrelativistic case $|p_1|\ll m$, $|p_1'|\ll m$ and the extreme relativistic case $|p_1|\gg m$.

Since the dipole moment of a system of two electrons is equal to zero, the radiation accompanying the collision of two nonrelativistic electrons is in the first approximation quadrupole radiation. The radiation cross section can therefore be evaluated not only by means of (29.50), but also in accordance with formula (27.14). In doing this we have to use the fact that in the nonrelativistic approximation the system of two particles may be replaced by a single particle moving in an external field, the Coulomb field describing the interaction between the two electrons. For the wave functions of the initial and the final states we can take expression (29.6′) in which $\psi_q$ has the form

$$\psi_q = \frac{1}{\sqrt{2}}(e^{i\mathbf{q}\mathbf{r}} \pm e^{-i\mathbf{q}\mathbf{r}})e^{-i\varepsilon t},$$

where $\mathbf{r}$ is the relative radius-vector of the particles, and $\mathbf{q}$ is the momentum of their relative motion (the plus sign corresponds to total spin equal to zero, and the minus sign corresponds to spin unity).

We note that in § 27 in investigating quadrupole radiation we had assumed that $\omega r \ll 1$, where the magnitude of $r$ defined the distances important for the problem. In the case of bound states $r$ has the meaning of atomic or nuclear dimensions, while in the present case of radiation emitted in a collision $r \sim v\tau$, where $v$ is the electron velocity and $\tau$ is the "collision time" which is of the order of magnitude of $\tau \sim 1/\omega$.

The expression for the differential cross section for the emission of radiation has the following form in the nonrelativistic case (125), (58):

$$\begin{aligned} d\sigma = \frac{16}{45}\alpha r_0^2 \Bigg\{ & \frac{3}{\mathbf{q}^4}\big(4(\mathbf{p}_1^2-\mathbf{p}_2^2)+3[\mathbf{p}_1\mathbf{p}_2]^2\big) \\ & + \frac{1}{\mathbf{q}^4\mathbf{P}^8}\big(144[\mathbf{p}_1\mathbf{p}_2]^6+264(\mathbf{p}_1^2-\mathbf{p}_2^2)^2[\mathbf{p}_1\mathbf{p}_2]^4+105(\mathbf{p}_1^2-\mathbf{p}_2^2)^4[\mathbf{p}_1\mathbf{p}_2]^2 \\ & + 12(\mathbf{p}_1^2-\mathbf{p}_2^2)^6\big) - \frac{1}{\mathbf{q}^4\mathbf{P}^4}\big(36[\mathbf{p}_1\mathbf{p}_2]^4+39(\mathbf{p}_1^2-\mathbf{p}_2^2)^2[\mathbf{p}_1\mathbf{p}_2]^2 \\ & + 12(\mathbf{p}_1^2-\mathbf{p}_2^2)^4\big)\Bigg\} \frac{d\mathbf{q}}{|\mathbf{p}_1|(\mathbf{p}_1^2-\mathbf{p}_2^2)}, \end{aligned} \tag{29.51}$$

where

$$\mathbf{q} = \mathbf{p}_1-\mathbf{p}_2, \qquad \mathbf{P} = \mathbf{p}_1+\mathbf{p}_2.$$

The angular distribution of the radiation in the center of mass system of the electrons is determined by its quadrupole nature, i.e., $d\sigma \sim |Y_2^{(0)}|^2$.

Integration of the cross section (29.51) over the angles, which must be carried out over a hemisphere, since the particles are identical, yields

$$\begin{aligned} d\sigma(\omega) = \frac{4}{15}\alpha r_0^2 \Bigg\{ 17 - \frac{3x^2}{(2-x)^2} + & \frac{12(2-x)^4-7(2-x)^2x^2-3x^4}{(2-x)^3\sqrt{1-x}} \\ & \times \ln\left[\frac{1}{\sqrt{x}}+\sqrt{\frac{1}{x}-1}\right]\Bigg\} \frac{\sqrt{1-x}}{x}\,dx, \end{aligned} \tag{29.52}$$

where $x = \omega/\varepsilon_1, \omega$ is the energy of the emitted photon given by $\omega = \varepsilon_1 - \varepsilon_2 = (p_1^2/m) - (p_2^2/m)$, and $\varepsilon_1$ and $\varepsilon_2$ are the initial and final electron energies in the center of mass system.

The cross section for the radiation energy loss in electron-electron collisions is given by the formula

$$\Phi = \frac{1}{\varepsilon_1} \int \omega d\sigma(\omega) \approx 8\alpha r_0^2. \tag{29.53}$$

It is of the same order of magnitude as the cross section for the radiation energy loss in the field of a nucleus for $Z = 1$ (cf. (29.34)). From this it follows that the cross section for the radiation energy loss in the collision of an electron with atomic shell electrons is approximately equal to $8\alpha r_0^2 Z$ where $Z$ is the number of electrons. In order of magnitude this cross section is smaller by a factor $Z$ than the cross section for the radiation energy loss in the field of a nucleus, which is proportional to $Z^2$.

In the extreme relativistic case the cross section for the emission of a photon in the collision of a fast electron with an electron at rest is determined by (72)

$$d\sigma(\omega) = 2\alpha r_0^2 \frac{\varepsilon_1}{\varepsilon_2} \frac{d\omega}{\omega} \left\{ \left( -\frac{2}{3} + \frac{\varepsilon_1^2 + \varepsilon_2^2}{\varepsilon_1 \varepsilon_2} \right) \left( 2 \ln \frac{2\varepsilon_1 \varepsilon_2}{\omega m} - 1 \right) - \left( 1.70 \frac{\omega^2}{p_1 p_2} + 2.14 \right) \right\}. \tag{29.54}$$

In this case the cross section for the energy loss by radiation is given by

$$\Phi = 4\alpha r_0^2 \left( \ln \frac{2\varepsilon_1}{m} - 1.23 \right). \tag{29.55}$$

We see that the cross section for the emission of radiation coincides up to a factor in the argument of the logarithmic term with the cross section for the emission of radiation by an electron in the field of a nucleus with $Z = 1$. Such a result can be easily understood on the basis of the following considerations. As we have seen (cf. (29.27)) at high energies the effective momentum transfer to the nucleus becomes very small, and, therefore, the value of the nuclear mass becomes less and less important.

We now consider the radiation emitted in the collision of an electron with a positron. In the nonrelativistic case this radiation, in contrast to the radiation emitted in electron-electron or positron-positron collisions, is of dipole nature.

If one of the particles is at rest before the collision, the radiation cross section is equal to

$$d\sigma(\omega) = \frac{32}{3} a r_0^2 \frac{m}{\varepsilon_1} \frac{d\omega}{\omega} \ln \frac{1 + \sqrt{1 - \frac{\omega}{\varepsilon_1}}}{1 - \sqrt{1 - \frac{\omega}{\varepsilon_1}}}. \tag{29.56}$$

In the c. m. system the radiation cross section coincides with the radiation cross section in the field of a nucleus with $Z = 1$ in the nonrelativistic approximation.

The cross section for the emission of a photon in a collision between relativistic electrons and positrons does not differ from the cross section for the emission of radiation in a collision of two relativistic electrons.

## § 30. Emission of Photons of Long Wavelength

### 30.1. "*The Infrared Catastrophe*"

In the preceding section we have seen that the probability of photon emission in the low energy range is inversely proportional to the frequency $dw \sim (d\omega/\omega)$, while the total radiation probability diverges logarithmically as $\omega \to 0$.

This divergence in the region of low photon energies is referred to as the "infrared catastrophe". It is brought about by the unjustified application of ordinary perturbation theory, based on the expansion of the scattering matrix into a series in powers of $e$, to those processes in which long wavelength photons participate. Indeed, it can be easily shown that if the probability $w_1$ for the emission of one long wavelength photon is proportional to $e^2 \ln (\varepsilon/\omega)$ where $\varepsilon$ is an energy of the order of magnitude of the electron energy, then the probability of emission of two photons $w_2$ will be proportional to $(e^2 \ln (\varepsilon/\omega))^2$. Therefore, the ratio of the probabilities is given in order magnitude by

$$\xi \equiv \frac{w_2}{w_1} \sim e^2 \ln \frac{\varepsilon}{\omega}, \quad \text{where} \quad \omega \to 0 . \tag{30.1}$$

It is this ratio, and not the quantity $e^2$ as we have been assuming until now, which provides the perturbation theory expansion parameter applicable to the processes of interaction between the electron and long wavelength photons. Since when $\omega \to 0$, $\xi$ is not small compared to unity, perturbation theory is, strictly speaking, inapplicable to such cases.

The inapplicability of the usual perturbation theory is associated with the fact that the number of photons emitted by an electron per unit energy tends to infinity as $\omega \to 0$, while in perturbation theory it is assumed that the emission of one photon is always more probable than the emission of two or a greater number of photons.

In order to verify this we note that if the energy and the momentum of the photon are considerably smaller than the kinetic energy and the change in the momentum of the electron, and the photon wavelength is considerably greater than the classical electron "radius", we may regard the motion of the electron as given and use classical electrodynamics. Assuming for the sake of simplicity that the electron velocity is small compared to the velocity of light, we can start with the following formula[1] for the intensity of dipole radiation $d\mathcal{E}_\omega$ in the frequency interval $d\omega$ (118)

$$d\mathcal{E}_\omega = \frac{8\pi}{3c^3} |\ddot{\boldsymbol{d}}_w|^2 \, d\omega,$$

where $\ddot{d}_\omega$ is the Fourier component of the second derivative with respect to time of the dipole moment

$$\ddot{\boldsymbol{d}}_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ddot{\boldsymbol{d}} e^{i\omega t} \, dt.$$

If $\omega \to 0$, then

$$(\ddot{\boldsymbol{d}}_\omega)_{w\to 0} = \frac{1}{2\pi} (\dot{\boldsymbol{d}}_2 - \dot{\boldsymbol{d}}_1)$$

($\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are the values of the dipole moment before and after emission). In the case of interest to us $\dot{\boldsymbol{d}} = e\boldsymbol{v}$ and therefore

$$(\ddot{\boldsymbol{d}}_\omega)_{\omega\to 0} = (e/2\pi)(\boldsymbol{v}_2 - \boldsymbol{v}_1),$$

---

[1] Both here and in subsequent formulas we explicitly introduce the constants $c$ and $h$.

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the electron velocities before and after emission. Thus we have

$$(d\mathcal{E}_\omega)_{\omega\to 0} = \frac{2}{3\pi}\frac{e^2}{c^2}(\boldsymbol{v}_2-\boldsymbol{v}_1)^2\, d\omega. \tag{30.2}$$

We see that the radiation intensity per unit frequency interval $d\mathcal{E}_\omega/d\omega$ tends to a finite limit different from zero as $\omega \to 0$. From this it follows that the average number of photons emitted which is equal to $(1/\hbar\omega)$ $(dE_\omega/d\omega)$ tends to infinity as $\omega \to 0$, as has been stated earlier.

Since the probability of transition of an electron from a state of momentum $p_1$ to a state of momentum $p_2$ is always finite, the probability of the simultaneous emission of an infinite number of photons of vanishingly low frequency is also finite, and different from zero. Therefore, the probability of emission of a single photon or a finite number of photons with $\omega \to 0$ is in fact equal to zero, and is not infinite as given by perturbation theory.

The quantity $(1/\hbar\omega)(d\varepsilon_\omega/d\omega)$ is the average number of photons of frequency $\omega$ emitted by the electron in the frequency interval $d\omega$. We now obtain the probability of the electron emitting an arbitrary number $n$ of long wavelength photons whose frequency lies within the interval $\omega_1 \leqslant \omega \leqslant \omega_2$.

On postulating, as before, that

$$\frac{\hbar\omega}{\varepsilon} \ll 1, \quad \frac{\hbar\omega}{c\Delta p} \ll 1, \quad \frac{e^2\omega}{mc^3} = \frac{r_0}{\lambda} \ll 1, \tag{30.3}$$

where $\varepsilon$ is the electron energy, $\Delta p$ is the change in its momentum, $\lambda$ is the photon wavelength and $r_0$ is the classical electron "radius" we can assume that the emission of photons does not affect the electron motion, i.e., we may regard the electron motion as given. Under these conditions the processes of succesive emission of photons will be statistically independent and, consequently, the probability of photon emission will be given by the Poisson formula:

$$w(n) = (\bar{n})^n \frac{e^{-\bar{n}}}{n!}, \tag{30.4}$$

where $\bar{n}$ is the average number of emitted photons whose frequency lies within the given integral $\omega_1 \leqslant \omega \leqslant \omega_2$.

If conditions (30.3) are satisfied the quantity $\bar{n}$ can be determined by means of classical electrodynamics. Indeed, if $dJ = I d\omega do$ is the classical radiation intensity in the frequency interval $(\omega, \omega+d\omega)$ and in the solid angle $do$, then we have

$$\bar{n} = \int\limits_{\omega=\omega_1}^{\omega=\omega_2} \frac{I}{\hbar\omega} d\omega\, do. \tag{30.4'}$$

We now demonstrate how $I$ can be determined.

It is well known that $dJ = c|\boldsymbol{H}_\omega|^2 R^2 d\omega\, do$, where $\boldsymbol{H}_\omega$ is the Fourier component of frequency $\omega$ of the magnetic field $\boldsymbol{H}$ at the point $\boldsymbol{R}$; the quantity $\boldsymbol{H}_\omega$ is related to the Fourier component of the vector potential $\boldsymbol{A}_\omega$ by the relation $\boldsymbol{H}_\omega = i[\boldsymbol{k}\boldsymbol{A}_\omega]$. By using the Lienard-Wiechert potentials (118) we can easily show that the Fourier components of the vector and the scalar potentials are given by

$$\boldsymbol{A}_\omega = \frac{e}{2\pi c R} \int\limits_{-\infty}^{\infty} \boldsymbol{v}(t) e^{i[\omega t - \boldsymbol{k}\boldsymbol{r}(t)]}\, dt, \qquad \varphi_\omega = \frac{e}{2\pi R} \int\limits_{-\infty}^{\infty} e^{i[\omega t - \boldsymbol{k}\boldsymbol{r}(t)]}\, dt,$$

where $\boldsymbol{r} = \boldsymbol{r}(t)$ is the equation of the electron trajectory and $\boldsymbol{v}(t)$ is the electron velocity. On noting that $\boldsymbol{k}\boldsymbol{A}_\omega - (1/c)\omega\varphi_\omega = 0$, we obtain

$$dJ = c|\boldsymbol{H}_\omega|^2 R^2 d\omega\, do = cR^2k^2(|\boldsymbol{A}_\omega|^2 - |\varphi_\omega|^2) d\omega\, do$$

$$= \frac{e^2 c}{4\pi^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \left(\frac{1}{c^2}\boldsymbol{v}(t)\,\boldsymbol{v}(t') - 1\right) e^{i\{\omega(t-t') - \boldsymbol{k}[\boldsymbol{r}(t) - \boldsymbol{r}(t')]\}}\, dt\, dt' k^2 d\omega\, do. \tag{30.5}$$

If the frequency $\omega$ satisfies the condition $\omega\tau \ll 1$, where $\tau$ determines the order of magnitude of the time during which the electron scattering takes place, then in carrying out the integration in (30.5) we can assume that $\boldsymbol{r}(t) \approx \boldsymbol{v}_1 t + \boldsymbol{a}$, where $-\infty \leqslant t \leqslant 0$ and $\boldsymbol{r}(t) \approx \boldsymbol{v}_2 t + \boldsymbol{b}$, where $0 \leqslant t \leqslant \infty$, ($\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the electron velocities before and after emission, $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants). On substituting these values of $\boldsymbol{r}(t)$ into (30.5) we obtain

$$dJ = \frac{e^2}{4\pi^2 c^3} \left\{ \left( \frac{\boldsymbol{v}_2}{1 - (\boldsymbol{n}\boldsymbol{v}_2/c)} - \frac{\boldsymbol{v}_1}{1 - (\boldsymbol{n}\boldsymbol{v}_1/c)} \right)^2 \right.$$

$$\left. - \left( \frac{\boldsymbol{n}\boldsymbol{v}_2}{1 - (\boldsymbol{n}\boldsymbol{v}_2/c)} - \frac{\boldsymbol{n}\boldsymbol{v}_1}{1 - (\boldsymbol{n}\boldsymbol{v}_1/c)} \right)^2 \right\} d\omega\, do \tag{30.6}$$

($\boldsymbol{n}$ is a unit vector in the direction of emission).

If $v_1$ and $v_2$ are considerably smaller than the velocity of light, then formula (30.6) reduses to formula (30.2).

We now determine the probability of the scattering of an electron by an external field accompanied by the emission of $n$ long wavelength photons. On assuming that the conditions (30.3) are satisfied, which means that the radiation reaction on the electron motion is extremely small, we can express the probability on the electron being scattered and emitting $n$ long wavelength photons in the form

$$dw = w(n)\,dw_s, \tag{30.7}$$

where $dw_s$ is the probability for the elastic scattering of an electron by an external field into the solid angle $do_s$ which was obtained in § 14. For the frequency interval $(\omega_1, \omega_2)$ we must obviously take the interval containing zero frequency. The average number of photons emitted into such an interval becomes infinite in accordance with (30.4′), and therefore $w(n) = 0$. In other words, the probability of the electron being scattered and emitted a finite number of long wavelength photons is equal to zero. In particular, the probability of purely elastic electron scattering is equal to zero. Since, on the other hand, $\lim\limits_{\bar{n}\to\infty} \sum\limits_{n=0}^{\infty} w(n) = 1$, it follows that

$$\sum_{n=0}^{\infty} dw = dw_s. \tag{30.7′}$$

Therefore, we can say that the probability $dw_s$ for the elastic scattering of an electron obtained in § 26, which does not take into account the interaction between the electron and the radiation field, is the total probability for electron scattering to occur, independently of the number of long wavelength photons emitted by the electron. As we have shown, this probability can be determined in the case of a sufficiently weak external field by perturbation theory, in contrast to the probability of scattering accompanied by the emission of a single photon or of any finite number of photons of frequency $\omega \to 0$: according to perturbation theory this probability is infinite, whereas in fact it is equal to zero.

The average energy of the long wavelength photons emitted in the frequency interval $(\omega, \omega + d\omega)$ and solid angle $do$, with the electron being scattered into the solid angle $do_s$, can in accordance with (30.4), be given as

$$\overline{d\varepsilon} = \sum_{n=0}^{\infty} n\hbar\omega e^{-\bar{n}} \frac{(\bar{n})^n}{n!} dw_s = \hbar\omega\bar{n}\, dw_s .$$

Since in this case $\bar{n} = dJ/\hbar\omega$, we have

$$\overline{d\varepsilon} = dJ\, dw_s . \tag{30.8}$$

As should have been expected, this quantity is equal to the product of the total probability $dw_s$ of the electron being scattered into the solid angle $do_s$, and the classical expression $dJ$ for the energy radiated in this process. The usual perturbation theory leads to the same result in the low frequency region. Thus, we see that the average radiated energy and the total probability of electron scattering (independently of the number of long wavelength photons emitted) can be determined by means of the usual perturbation theory in the domains of both high and low frequencies, although the probability of electron scattering accompanied by the emission of a finite number of photons of $\omega \to 0$ is not given correctly by perturbation theory.

The probability of emission of photons whose energy is not vanishingly small may be obtained by means of the usual perturbation theory, i.e., by the expansion of the matrix elements into a power series in the electron charge, but the probability of emission of a single photon determined in this manner should be interpreted not as the probability of just that one photon being emitted, but as the probability of the emission simultaneously with this photon of an arbitrary number of long wavelength photons of frequency $\omega \to 0$.

### 30.2. *Investigation of the Divergence in the Low Frequency Domain by means of the Scattering Matrix*

We now show that the Poisson distribution for the probability of emission of long wavelength photons can be obtained with the aid of the scattering matrix.

We assume that the conditions (30.3) are fulfilled, which means that the emission of radiation has only a small effect on the electron motion, so that the latter may be regarded as given. Under these conditions the current density can be regarded not as an operator, but as a $c$-number, and the expression (24.27) for the scattering matrix can be used.

On expanding the current density into a Fourier integral

$$j_\mu(x) = \frac{1}{(2\pi)^4} \int j_\mu(k) e^{ikx}\, d^4k, \quad \text{where} \quad j_\mu^*(k) = j_\mu(-k)$$

and on utilizing the expansion for the operators of the potentials of the electromagnetic field into plane waves we can express the scattering matrix S in the form

$$S = \exp\left\{-\frac{1}{2(2\pi)^4} \int j_\nu^*(k)\, D^c(k) j_\nu(k)\, d^4k\right\}$$
$$\times N \exp\left\{\frac{1}{\sqrt{\Omega}} \sum_{k,\lambda} \frac{1}{\sqrt{2\omega}} [j_\mu^*(k) c_{k\lambda} + j_\mu(k) c_{k\lambda}^+] e_{\lambda\mu}\right\},$$

where

$$D^c(k) = -\frac{i}{k^2}.$$

We now consider the integral appearing in the first exponential:

$$K = \frac{1}{2} \int j_\nu^*(k)\, D^c(k) j_\nu(k)\, d^4k = -\frac{i}{2} \int j_\nu^*(\boldsymbol{k}, k_0) j_\nu(\boldsymbol{k}, k_0) \frac{d\boldsymbol{k}\, dk_0}{\boldsymbol{k}^2 - k_0^2}.$$

By utilizing the rule for going around the poles in the integrand of this expression, i.e., by taking for the poles the points $k_0 = \pm\sqrt{\boldsymbol{k}^2 - i0}$, we obtain as a result of integrating over $k_0$

$$K = \pi \int j_\nu^*(\boldsymbol{k}) j_\nu(\boldsymbol{k}) \frac{d\boldsymbol{k}}{2\omega},$$

where $\omega = |\boldsymbol{k}|$ and $j_\nu(\boldsymbol{k}) \equiv j_\nu(\boldsymbol{k}, \omega)$.

Thus, the S-matrix can be written in the form

$$S = AN \exp\left\{\frac{i}{\sqrt{\Omega}} \sum_{k,\lambda} \frac{1}{\sqrt{2\omega}} [j_\nu^*(\boldsymbol{k}) c_{k\lambda} + j_\nu(\boldsymbol{k}) c_{k\lambda}^+] e_{\lambda\nu}\right\}, \tag{30.9}$$

where

$$A = \exp\left(-\frac{1}{(2\pi)^4} K\right) = \exp\left(-\frac{1}{2(2\pi)^3} \int j_\nu^*(\boldsymbol{k}) j_\nu(\boldsymbol{k}) \frac{d\boldsymbol{k}}{2\omega}\right).$$

We now evaluate the element of the S-matrix between the vacuum state and the state which contains $n$ photons of different kinds

$n=\sum_k n_k$, where $n_k$ is the number of photons of momentum $k$. In order to do this we investigate the $n$th term in the expansion of the S-matrix

$$\mathrm{S}^{(n)}=AN\left\{\frac{1}{n!}\left[\frac{i}{\sqrt{\Omega}}\sum_{k,\lambda}\frac{1}{\sqrt{2\omega}}(j_\mu^*(k)c_{k\lambda}+j_\mu(k)c_{k\lambda})e_{\lambda\mu}\right]^n\right\}$$

and obtain the matrix element

$$M_{i\to f}^{(n)}=(n|\mathrm{S}^{(n)}|0)=A\left(n\left|\frac{1}{n!}\left(\frac{i}{\sqrt{\Omega}}\sum_{k,\lambda}j_\mu(k)c_{k\lambda}^+e_{\lambda\mu}\frac{1}{\sqrt{2\omega}}\right)^n\right|0\right).\qquad(30.10)$$

We use the formula

$$(B_1+B_2+B_3+\cdots+B_r)^n=\sum_{n_1'+n_2'+\cdots+n_r'=n}B_1^{n_1'}B_2^{n_2'}\ldots B_r^{n_r'}\frac{n}{n_1'!n_2'!\ldots n_r'!}$$

and in it set

$$B_r=\frac{1}{\sqrt{2\omega}}j_\mu(k)c_{k\lambda}^+e_{\lambda\mu},\qquad r\equiv k,\lambda,\qquad n_r'\equiv n_{k\lambda}'.$$

We are interested in the final state containing $n_r\equiv n_{k\lambda}$ photons of the $r$th kind. Therefore, the contribution to $M_{i\to f}^{(n)}$ introduces only one term into this formula in which $n_1'=n_1$, $n_2'=n_2$, $\ldots$, $n_i'=n_i\equiv n_{k\lambda}$, i.e.,

$$M_{i\to f}^{(n)}=A\left(\frac{i}{\sqrt{\Omega}}\right)^n\prod_{k,\lambda}\left(n_{k\lambda}\left|\frac{\left(\frac{1}{\sqrt{2\omega}}j_\mu(k)c_{k\lambda}^+e_{\lambda\mu}\right)^{n_{k\lambda}}}{n_{k\lambda}!}\right|0\right).$$

On noting that in accordance with (16.18) $(n_{k\lambda}|(c_{k\lambda}^+)^{n_{k\lambda}}|0)=\sqrt{n_{k\lambda}!}$, we obtain finally

$$M_{i\to f}^{(n)}=A\left(\frac{i}{\sqrt{\Omega}}\right)^n\prod_{k,\lambda}\frac{1}{\sqrt{n_{k\lambda}!}}\left(\frac{1}{\sqrt{2\omega}}j_\mu(k)e_{\lambda\mu}\right)^{n_{k\lambda}}.\qquad(30.10')$$

We now obtain the probability of emission of $n=\sum_k n_k$ photons of different momenta. This probability, which we denote by $P_n$, is evidently equal to the square of the absolute value of $M_{i\to f}^{(n)}$ summed over $\lambda$ and $k$, with the sum $\sum_{\lambda=1}^{2} n_{k\lambda}$ being equal to a given number $n_k$. Since $j_\mu e_{\lambda\mu} = j_\lambda$, we have

$$P_n = \frac{A^2}{\Omega^n} \sum_{n_{k1}+n_{k2}=n_k} \prod_k \frac{\left(\frac{1}{2\omega}|j_1(k)|^2\right)^{n_{k1}}}{n_{k1}!} \frac{\left(\frac{1}{2\omega}|j_2(k)|^2\right)^{n_{k2}}}{n_{k2}!}$$

$$= \frac{A^2}{\Omega^n} \prod_k \left(\frac{1}{2\omega}\right)^{n_k} \frac{\left(|j_1(k)|^2+|j_2(k)|^2\right)^{n_k}}{n_k!}.$$

The sum of the squares of the absolute values of the transverse components of the current density $|j_1(k)|^2+|j_2(k)|^2$ is equal in virtue of the equation of continuity $k_\mu j_\mu(k) = 0$ to the square of the absolute value of the current density four-vector

$$|j_1(k)|^2+|j_2(k)|^2 = |\boldsymbol{j}(k)|^2 - \left|\frac{1}{\omega}\boldsymbol{k}\cdot\boldsymbol{j}(k)\right|^2 = |j_\mu(k)|^2.$$

Further, on going over in the expression for $K$ from integration to summation over $k$ in accordance with the formula

$$\int f(k)\frac{dk}{(2\pi)^3}\Omega = \sum_k f(k),$$

we obtain

$$A = \exp\left\{-\frac{1}{2\Omega}\sum_k \frac{|j_\mu(k)|^2}{2\omega}\right\},$$

and therefore $P_n$ can finally be written in the form

$$P_n = \prod_k \exp\left\{-\frac{|j_\mu(k)|^2}{2\omega\Omega}\right\} \prod_k \frac{\left(\frac{|j_\mu(k)|^2}{2\omega\Omega}\right)^{n_k}}{n_k!} = \prod_k e^{-\bar{n}_k}\frac{(\bar{n}_k)^{n_k}}{n_k!}, \quad (30.11)$$

where

$$\bar{n}_k = \frac{|j_\mu(k)|^2}{2\omega\Omega}.$$

Thus, we have obtained the Poisson distribution for the probability of emission of long wavelength photons. It is clear that $n_k$ is the average number of photons of momentum $k$, since according to (30.11)

$$\sum_{n_k} n_k P_n = \bar{n}_k.$$

It now remains to be shown that the resultant expression for the average number of photons emitted

$$\bar{n} = \sum_k \bar{n}_k = \int |j_\mu(k)|^2 \frac{dk}{2\omega(2\pi)^3} \tag{30.12}$$

coincides with the expression for $\bar{n}$ which was found earlier by the classical method (formula (30.4′)).

In order to do this we determine $j_\mu(k)$ by assuming that the current density has the following components

$$j_\mu(x) = \Big(ev\delta\big(\boldsymbol{r}-\boldsymbol{r}(t)\big) \text{ and } ie\delta\big(\boldsymbol{r}-\boldsymbol{r}(t)\big)\Big),$$

where $\boldsymbol{r} = \boldsymbol{r}(t)$ defines the trajectory of the moving particle

$$j_\mu(k) = \int j_\mu(x) e^{-ikx} d^4x = \left(e\int_{-\infty}^{\infty} \boldsymbol{v}(t) e^{i(\omega t - \boldsymbol{k}\boldsymbol{r}(t))}\, dt,\ ie\int_{-\infty}^{\infty} e^{i(\omega t - \boldsymbol{k}\boldsymbol{r}(t))}\, dt\right).$$

From this it follows that

$$|j_\mu(k)|^2 = e^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(\boldsymbol{v}(t)\boldsymbol{v}(t')-1\big)\, e^{i\{\omega(t-t')-\boldsymbol{k}[\boldsymbol{r}(t)-\boldsymbol{r}(t')]\}}\, dt\, dt'.$$

The average number of photons emitted of frequency lying in the range $\omega_1 \leqslant \omega \leqslant \omega_2$ is, according to (30.12), given by

$$\bar{n} = \int_{\omega=\omega_1}^{\omega=\omega_2} \frac{|j_\mu(k)|^2}{2\omega} \frac{dk}{(2\pi)^3}$$

$$= \int_{\omega=\omega_1}^{\omega=\omega_2} \left\{\frac{e^2}{2(2\pi)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(\boldsymbol{v}(t)\boldsymbol{v}(t')-1\big)\, e^{i\{\omega(t-t')-\boldsymbol{k}[\boldsymbol{r}(t)-\boldsymbol{r}(t')]\}}\, \frac{k^2\, dt\, dt'}{\omega}\right\} do\, d\omega$$

$$= \int_{\omega=\omega_1}^{\omega=\omega_2} \frac{I}{\omega}\, d\omega\, do. \tag{30.13}$$

This expression coincides with the classical formula (if we take into account the fact that in (30.13) the charge is expressed in Heaviside units).

### 30.3. *Relation between the Photon "Mass" and the Minimum Frequency*

Since perturbation theory gives an incorrect value for the probabilities of the different processes of interaction between the electron and the long wavelength photons, we shall in the future, in using perturbation theory, segregate the low frequency region, i.e., we shall assume that the photon frequency exceeds a certain minimum value $\omega_{\min}$. This quantity must later be eliminated by a separate investigation of the interaction between an electron and the long wavelength photons (cf., for example, the problem of the radiation corrections to electron scattering, and the radiation shift of atomic levels in Chapter VIII).

However, in practice, it is more convenient to use a somewhat different condition, which is equivalent to the condition $\omega > \omega_{\min}$, viz., without imposing any restrictions on the photon frequency, to assume that the photon has a certain very small mass $\lambda$ which is different from zero. The introduction of this mass, which is a relativistically invariant quantity, in contrast to the noninvariant quantity $\omega_{\min}$, simplifies the calculations considerably.

We now show how to obtain the relation between the photon "mass" $\lambda$ and the minimum frequency $\omega_{\min}$.

We consider the emission of a photon by an electron in some constant external field $A_\mu^{(e)}(x)$. According to (29.9) the matrix element which determines the emission of a photon of energy $k_0$, and with its polarization vector directed along the $\mu$-axis, is given by

$$S_{i\to f}^{(2)} = ie^2\bar{u}_2\left(\frac{\gamma_\mu}{\sqrt{2k_0}}\frac{i(\hat{p}_2+\hat{k})-m}{(p_2+k)^2+m^2}\hat{A}^{(e)}(q)+\hat{A}^{(e)}(q)\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\frac{\gamma_\mu}{\sqrt{2k_0}}\right)u_1, \tag{30.14}$$

where $p_1$ and $p_2$ are the four-momenta of the electron before and after scattering, and $k$ is the photon four-momentum. Since we are interested in the low frequency region we can neglect the quantity $\hat{k}$ in the numerators. Further, on introducing the photon "mass" $\lambda$ by means of the relation $k^2+\lambda^2 = 0$, we rewrite (31.14) in the form

$$S_{i\to f}^{(2)} = \frac{-ie^2}{\sqrt{2k_0}}\bar{u}_2\left(\gamma_\mu\frac{i\hat{p}_2-m}{\lambda^2-2p_2k}\hat{A}^{(e)}(q)+\hat{A}^{(e)}(q)\frac{i\hat{p}_1-m}{\lambda^2+2p_1k}\gamma_\mu\right)u_1,$$

where

$$pk = \mathbf{pk}-\varepsilon k_0, \quad \varepsilon = \sqrt{\mathbf{p}^2+m^2}, \quad k_0 = \sqrt{\mathbf{k}^2+\lambda^2}.$$

Since $(i\hat{p}_1+m)u_1 = 0$, we have

$$\bar{u}_2\hat{A}^{(e)}(q)(i\hat{p}_1-m)\gamma_\mu u_1 = i\bar{u}_2\hat{A}^{(e)}(q)(\hat{p}_1\gamma_\mu+\gamma_\mu\hat{p}_1)u_1 = 2i\bar{u}_2\hat{A}^{(e)}(q)u_1p_{1\mu}.$$

Similarly we have

$$\bar{u}_2\gamma_\mu(i\hat{p}_2-m)\hat{A}^{(e)}(q)u_1 = 2i\bar{u}_2\hat{A}^{(e)}(q)u_1p_{2\mu}.$$

Therefore

$$S^{(2)}_{i\to f} = \frac{e^2}{\sqrt{2k_0}}(\bar{u}_2\hat{A}^{(e)}(q)u_1)\left(\frac{p_{2\mu}}{(\lambda^2/2)-p_2k}+\frac{p_{1\mu}}{(\lambda^2/2)+p_1k}\right). \tag{30.15}$$

On assuming that $\varepsilon \gg \lambda$ we may neglect $\lambda^2$, since $pk$ contains the term $\varepsilon\sqrt{k^2+\lambda^2}$ which is considerably larger than $\lambda^2$.

We now obtain the cross section for electron scattering accompanied by the emission of a photon of energy not exceeding $\Delta\varepsilon$. With respect to the magnitude of $\Delta\varepsilon$ we assume that it is considerably smaller than the electron energy $\varepsilon$ and considerably greater than the photon mass $\lambda$: $\lambda \ll \Delta\varepsilon \ll \varepsilon$.

The differential cross section for such scattering, summed and averaged over the electron spin components in the initial and the final states, is given by

$$d\sigma' = \frac{2\pi}{v}\frac{e^4}{4}\sum_{\mu_1,\mu_2}|\bar{u}_2\hat{A}^{(e)}(q)u_1|^2\frac{B}{(2\pi)^3}\varrho_f\,do,$$

where

$$B = \sum_{\mu=1}^{4}\int\limits_{k_0\leqslant\Delta\varepsilon}\left(\frac{p_{2\mu}}{p_2k}-\frac{p_{1\mu}}{p_1k}\right)\frac{dk}{k_0}, \tag{30.16}$$

$do$ is the element of solid angle containing the electron momentum $\mathbf{p}_2$ after scattering, $v$ is the initial electron velocity and

$$\varrho_f = \frac{p_2^2\,dp_2}{(2\pi)^3\,d\varepsilon} = \frac{p_2\varepsilon}{(2\pi)^3}$$

is the number of final states of the electron per unit energy, and per unit solid angle.

In the case of the Coulomb field of a nucleus

$$\hat{A}^{(e)}(q) = -i\gamma_4\frac{Ze}{|\mathbf{q}|^2}$$

and

$$\sum_{\mu_1,\mu_2}|\bar{u}_2\hat{A}^{(e)}(q)u_1|^2=2\left(\frac{Ze}{q^2}\right)^2\left(1-v^2\sin^2\frac{\vartheta}{2}\right)=8\pi\alpha\frac{Z^2}{q^4}\left(1-v^2\sin^2\frac{\vartheta}{2}\right),$$

where $\vartheta$ is the scattering angle.

In order to establish the connection in which we are interested between the photon mass $\lambda$ and the minimum frequency $\omega_{\min}$ we evaluate the quantity $B$ appearing in (30.16) in two different ways: first by assuming that $\lambda\neq 0$ and that $k_0$ varies from $k_0=\lambda$ to $k_0=\Delta\varepsilon$, and then by assuming that $\lambda=0$ and that $k_0$ varies from $k_0=\omega_{\min}$ to $k_0=\Delta\varepsilon$. By comparing the two expressions for $B$, which we shall in future denote by $B_\lambda$ and $B_{\omega_{\min}}$, we obtain the relation between $\lambda$ and $\omega_{\min}$.

On noting that $p_{1,2}^2+m^2=0$ we obtain

$$B_\lambda=-\int\limits_{|k|=0}^{|k|=\Delta\varepsilon}\frac{k^2\,d|k|\,do_\gamma}{\sqrt{k^2+\lambda^2}}\left\{\frac{m^2}{(\varepsilon_2\sqrt{k^2+\lambda^2}-p_2k)^2}+\frac{m^2}{(\varepsilon_1\sqrt{k^2+\lambda^2}-p_1k)^2}\right.$$
$$\left.+\frac{2p_1p_2}{(\varepsilon_2\sqrt{k^2+\lambda^2}-p_2k)(\varepsilon_1\sqrt{k^2+\lambda^2}-p_1k)}\right\},\tag{30.17}$$

$$B_{\omega_{\min}}=-\int\limits_{|k|=\omega_{\min}}^{|k|=\Delta\varepsilon}|k|\,d|k|\,do_\gamma\left\{\frac{m^2}{(\varepsilon_2|k|-p_2k)^2}+\frac{m^2}{(\varepsilon_1|k|-p_1k)^2}\right.$$
$$\left.+\frac{2p_1p_2}{(\varepsilon_2|k|-p_2k)(\varepsilon_1|k|-p_1k)}\right\},$$

where $do_\gamma$ is the element of solid angle containing the photon momentum $k$.

We now use the identity

$$\frac{1}{(\varepsilon_2\sqrt{k^2+\lambda^2}-p_2k)(\varepsilon_1\sqrt{k^2+\lambda^2}-p_1k)}=\frac{1}{2}\int\limits_{-1}^{1}\frac{dz}{(\varepsilon_z\sqrt{k^2+\lambda^2}-p_zk)^2},$$

where

$$p_z=\tfrac{1}{2}(1+z)p_1+\tfrac{1}{2}(1-z)p_2,\qquad \varepsilon_z=\tfrac{1}{2}(1+z)\varepsilon_1+\tfrac{1}{2}(1-z)\varepsilon_2.$$

Then

$$B_\lambda = -\int\limits_{|k|=0}^{|k|=\Delta\varepsilon} \frac{k^2 d|k|\, do_\gamma}{\sqrt{k^2+\lambda^2}} \left\{ \frac{m^2}{(\varepsilon_1\sqrt{k^2+\lambda^2}-p_1k)^2} + \frac{m^2}{(\varepsilon_2\sqrt{k^2+\lambda^2}-p_2k)^2} \right.$$

$$\left. + p_1p_2 \int\limits_{-1}^{1} \frac{dz}{(\varepsilon_z\sqrt{k^2+\lambda^2}-p_zk)^2} \right\}.$$

Since

$$\int \frac{do_\gamma}{(\varepsilon\sqrt{k^2+\lambda^2}-pk)^2} = \frac{4\pi}{k^2(\varepsilon^2-p^2)+\lambda^2\varepsilon^2},$$

we have

$$B_\lambda = -4\pi \int\limits_0^{\Delta\varepsilon} \frac{k^2 d|k|}{\sqrt{k^2+\lambda^2}} \left\{ \frac{m^2}{k^2(\varepsilon_1^2-p_1^2+\lambda^2\varepsilon_1^2} + \frac{m^2}{k^2(\varepsilon_2^2-p_2^2)+\lambda^2\varepsilon_2^2} \right.$$

$$\left. + p_1p_2 \int\limits_{-1}^{1} \frac{dz}{k^2(\varepsilon_z^2-p_z^2)+\lambda^2\varepsilon_z^2} \right\}. \qquad (30.18)$$

It can be easily shown that

$$\int\limits_0^{\Delta\varepsilon} \frac{k^2 d|k|}{[k^2(\varepsilon^2-p^2)+\lambda^2\varepsilon^2]\sqrt{k^2+\lambda^2}} = \frac{1}{\varepsilon^2-p^2}\ln\frac{2\Delta\varepsilon}{\lambda} - \frac{\varepsilon}{2|p|}\frac{1}{\varepsilon^2-p^2}\ln\frac{\varepsilon+|p|}{\varepsilon-|p|}.$$

Therefore we have

$$B_\lambda = 4\pi \left\{ -\left[ 2 + (p_1p_2-\varepsilon_1\varepsilon_2) \int\limits_{-1}^{1} \frac{dz}{\varepsilon_z^2-p_z^2} \right] \ln\frac{2\Delta\varepsilon}{\lambda} + \frac{1}{2}\left[ \frac{\varepsilon_1}{|p_1|}\ln\frac{\varepsilon_1+|p_1|}{\varepsilon_1-|p_1|} \right.\right.$$

$$\left.\left. + \frac{\varepsilon_2}{|p_2|}\ln\frac{\varepsilon_2+|p_2|}{\varepsilon_2-|p_2|} + (p_1p_2-\varepsilon_1\varepsilon_2) \int\limits_{-1}^{1} \frac{dz}{\varepsilon_z^2-p_z^2}\frac{\varepsilon_z}{|p_z|}\ln\frac{\varepsilon_z+|p_z|}{\varepsilon_z-|p_z|} \right] \right\}. \qquad (30.19)$$

On introducing in place of $z$ the new variable of integration $\zeta$: $\zeta = \dfrac{|p_z|}{v\varepsilon_z} = \sqrt{\cos^2(\vartheta/2)+z^2\sin^2(\vartheta/2)}$, we write $B_\lambda$ in the form

$$B_\lambda = 4\pi\left\{2(2\Phi\coth 2\Phi-1)\ln\frac{2\Delta\varepsilon}{\lambda}+\frac{1}{v}\ln\frac{1+v}{1-v}\right.$$

$$\left.-\frac{1-v^2}{v\sin(\vartheta/2)}\cosh 2\Phi\int\limits_{\cos(\vartheta/2)}^{1}\ln\frac{1+v\zeta}{1-v\zeta}\,\frac{d\zeta}{(1-v^2\zeta^2)\sqrt{\zeta^2-\cos^2(\vartheta/2)}}\right\},\qquad(30.20)$$

where the quantity $\Phi$ is related to $q$ by the expression $q^2 = 4m^2\sinh^2\Phi$. Since $q_4 = 0$, we have $q^2 = \mathbf{q}^2 = 4p^2\sin^2(\vartheta/2)$ and

$$\sinh\Phi = \frac{v}{\sqrt{1-v^2}}\sin(\vartheta/2).$$

On setting $\lambda = 0$ in (30.18) and on integrating over $|\mathbf{k}|$ from $|\mathbf{k}| = \omega_{\min}$ to $|\mathbf{k}| = \Delta\varepsilon$ we obtain $B_{\omega_{\min}}$:

$$B_{\omega_{\min}} = -4\pi\left\{2+(p_1p_2-\varepsilon_1\varepsilon_2)\int\limits_{-1}^{1}\frac{dz}{\varepsilon_z^2-p_z^2}\right\}\int\limits_{\omega_{\min}}^{\Delta\varepsilon}\frac{d|\mathbf{k}|}{|\mathbf{k}|}.\qquad(30.21)$$

Since

$$\int\limits_{-1}^{1}\frac{dz}{\varepsilon_z^2-p_z^2} = \frac{4\Phi}{m^2\sinh 2\Phi}$$

and $p_1p_2-\varepsilon_1\varepsilon_2 = -m^2\cosh 2\Phi$, we obtain finally

$$B_{\omega_{\min}} = 8\pi(2\Phi\coth 2\Phi-1)\ln\frac{\Delta\varepsilon}{\omega_{\min}}.\qquad(30.22)$$

On comparing (30.20) and (30.22) we obtain the desired relation between $\lambda$ and $\omega_{\min}$:

$$2(1-2\Phi\coth 2\Phi)\ln\frac{2\omega_{\min}}{\lambda}$$

$$=\frac{1}{v}\ln\frac{1+v}{1-v}-\frac{(1-v^2)}{v\sin(\vartheta/2)}\cosh 2\Phi\int\limits_{\cos(\vartheta/2)}^{1}\ln\frac{1+v\zeta}{1-v\zeta}\,\frac{d\zeta}{(1-v^2\zeta^2)\sqrt{\zeta^2-\cos^2(\vartheta/2)}}.\qquad(30.23)$$

In the limiting case of low electron energies $v \ll 1$ this formula leads to

$$\ln 2\omega_{\min} = \ln\lambda+\tfrac{5}{6}.\qquad(30.24)$$

On substituting (30.20) into (30.16) we obtain the following expression for the differential cross section for electron scattering in the nuclear Coulomb field accompanied by the emission of a photon whose energy does not exceed $\Delta\varepsilon\,(\Delta\varepsilon \ll \varepsilon)$:

$$d\sigma' = \left(\frac{Z\alpha}{2mv^2\sin^2(\vartheta/2)}\right)^2 (1-v^2)\left(1-v^2\sin^2\frac{\vartheta}{2}\right)$$
$$\times\frac{\alpha}{\pi}\left\{2(2\Phi\coth 2\Phi-1)\ln\frac{2\Delta\varepsilon}{\lambda}+\frac{1}{v}\ln\frac{1+v}{1-v}\right.$$
$$\left.-\frac{1-v^2}{v\sin(\vartheta/2)}\cosh 2\Phi\, G(v,\vartheta)\right\}do, \qquad (30.25)$$

where

$$G(v,\vartheta) = \int\limits_{\cos\frac{\vartheta}{2}}^{1} \ln\frac{1+v\zeta}{1-v\zeta}\,\frac{d\zeta}{(1-v^2\zeta^2)\sqrt{\zeta^2-\cos^2(\vartheta/2)}} \qquad (30.25')$$

We now also evaluate the quantity $B_\lambda$ in the case when $p_1 = 0$. On setting $\varepsilon_2 = m\cosh 2y$, $p_2 = m\sinh 2y$, and on noting that $\varepsilon_z^2-|\boldsymbol{p}_z|^2 = \cosh^2 y - z^2\sinh^2 y$, we obtain

$$\int\limits_{-1}^{1}\frac{dz}{\varepsilon_z^2-|\boldsymbol{p}_z|^2} = \frac{4y}{m^2\sinh 2y}\,.$$

Further, we have

$$\frac{\varepsilon_1}{p_1}\ln\frac{\varepsilon_1+p_1}{\varepsilon_1-p_1} = 2, \qquad \frac{\varepsilon_2}{p_2}\ln\frac{\varepsilon_2+p_2}{\varepsilon_2-p_2} = 4y\coth 2y,$$
$$\varepsilon_z = \cosh^2 y - z\sinh^2 y, \qquad |\boldsymbol{p}_z| = (1-z)\sinh y\cosh y$$

and

$$J \equiv \int\limits_{-1}^{1}\frac{dz}{\varepsilon_z^2-|\boldsymbol{p}_z|^2}\cdot\frac{\varepsilon_z}{|\boldsymbol{p}_z|}\ln\frac{\varepsilon_z+|\boldsymbol{p}_z|}{\varepsilon_z-|\boldsymbol{p}_z|}$$
$$= \int\limits_{-1}^{1}\frac{dz}{\cosh^2 y - z^2\sinh^2 y}\,\frac{\cosh^2 y - z\sinh^2 y}{(1-z)\sinh y\,\mathrm{coch}\, y}\ln\frac{e^{2y}(\cosh y - z\sinh y)}{\cosh y + z\sinh y}\,.$$

On setting $z = \tanh v/\tanh y$, we obtain

$$J = 2 \int\limits_{-y}^{y} \frac{dv}{\cosh^2 y} \frac{\coth y - \tanh v}{\tanh y - \tanh v} (y-v)$$

$$= \frac{2}{\cosh^2 y \tanh y} \int\limits_{-y}^{y} dv \frac{\cosh(y-v)}{\sinh(y-v)} (y-v) = \frac{8y}{\sinh 2y} \cdot h(2y),$$

where

$$h(y) = \frac{1}{y} \int\limits_{0}^{y} u \coth u \, du.$$

The quantity $B_\lambda$ which is defined by (30.19) will finally be given for $p_1 = 0$ by

$$B_\lambda = 4\pi \left\{ 2(2y \coth 2y - 1) \left[ \ln \frac{2\Delta\varepsilon}{\lambda} - \frac{1}{2} \right] + 4y \coth 2y [1 - h(2y)] \right\}. \tag{30.26}$$

## § 31. Photoeffect

### 31.1. *Photoeffect in the Nonrelativistic Domain*

If an atom has incident on it a photon of energy greater than the ionization energy of the atom, then absorption of the photon can occur accompanied by the liberation of an atomic electron. This phenomenon is called the photoeffect.

The matrix element for the photoeffect can be written as

$$S^{(1)}_{i\to f} = -2\pi i U_{i\to f} \delta(\varepsilon_0 + \omega - \varepsilon),$$

where

$$U_{i\to f} = -\frac{e}{\sqrt{2\omega}} \int \bar{\psi}_f(\boldsymbol{r}) \gamma \boldsymbol{e} \, e^{i\boldsymbol{kr}} \psi_i(\boldsymbol{r}) \, d\boldsymbol{r} \equiv -\frac{e}{\sqrt{2\omega}} M_{i\to f}, \tag{31.1}$$

$\psi_i(\boldsymbol{r})$ and $\psi_f(\boldsymbol{r})$ are the electron wave functions in the initial and the final states, $\varepsilon_0$ and $\varepsilon$ are the energies corresponding to these states, and $\omega, \boldsymbol{k}, \boldsymbol{e}$ are the frequency, momentum, and polarization of the photon.

The cross section for the photoeffect is determined by

$$d\sigma = \frac{e^2}{(2\pi)^2\omega} \sum_{\mu_i,\mu_f} |M_{i\to f}|^2\, \delta(\varepsilon_0+\omega-\varepsilon)\, d\boldsymbol{p}, \tag{31.1'}$$

where the summation is taken over the different spin orientations of the electron in the initial and the final states; $\boldsymbol{p}$ is the momentum of the ejected electron.

We restrict ourselves here to the calculation of the photoeffect cross section only, for the $K$-shell of the atom (formula (31.1′) takes into account two electrons in the $K$-shell).

We shall, first of all, consider the nonrelativistic case, when the photon energy differs but little from the atomic ionization energy $I$. On assuming that the photon wavelength $\lambda$ is considerably larger than the atomic dimensions $a$, $\lambda \gg a$, we can replace exp $(i\boldsymbol{kr})$ in (31.1) by unity. Further, on noting that in the nonrelativistic region the matrix $\alpha$ corresponds to the electron velocity $\boldsymbol{v}$, we obtain

$$M_{i\to f} = (\boldsymbol{ev})_{i\to f} = -i\omega(\boldsymbol{er})_{i\to f}. \tag{31.2}$$

Thus, the problem is reduced to the evaluation of the matrix element of the component of the radius-vector of the electron along the photon polarization vector $\boldsymbol{e}$. We have encountered an analogous problem in the theory of bremsstrahlung in the nonrelativistic domain. The difference consists of the fact that in (29.37′) a wave function belonging to the continuous spectrum appears as the wave function for the initial state, while in (31.1) it is replaced by the wave function for the $K$-electron. This function is of the form $\psi_i(\boldsymbol{r}) = N_i e^{-\eta r}$, where $\eta = Z/a_0$, $a_0 = \hbar^2/me^2$ and $N_i = \sqrt{Z^3/\pi a_0^3}$.

Insofar as the wave function of the final state is concerned, we must take for it a wave function for the electron in the Coulomb field of the nucleus belonging to the continuous spectrum. Since an electron is liberated in the final state, then in accordance with subsection 29.1 the function $\psi_f(r)$ must at $r \to \infty$ have the form of a superposition of a plane wave and an incoming spherical wave. The function having this asymptotic behavior is of the form

$$\psi_f(\boldsymbol{r}) = e^{\frac{\pi}{2}\xi}\, \Gamma(1+i\xi) F[-i\xi,\quad 1,\quad -i(pr+\boldsymbol{pr})]\, e^{i\boldsymbol{pr}},$$

where $\xi = Ze^2/\hbar v$.

The substitution of the functions $\psi_i(\mathbf{r})$ and $\psi_f(\mathbf{r})$ into (31.1) yields

$$M_{i\to f} = i\omega e\mathbf{M}, \tag{31.2'}$$

where

$$\mathbf{M} = e^{\frac{\pi}{2}\xi}\Gamma(1-i\xi)N_i\int e^{-i\mathbf{p}\mathbf{r}-\eta r}\mathbf{r}F[i\xi,1,i(pr+\mathbf{p}\mathbf{r})]\,d\mathbf{r}$$

$$\equiv e^{\frac{\pi}{2}\xi}\Gamma(1-i\xi)N_i\mathbf{J}. \tag{31.3}$$

On noting that

$$\mathbf{J} = \left(i\frac{\partial}{\partial\mathbf{q}}\frac{\partial}{\partial\eta}\int e^{i\mathbf{q}\mathbf{r}-i\mathbf{p}\mathbf{r}-\eta r}F[i\xi,\ 1,\ i(pr+\mathbf{p}\mathbf{r})]\frac{d\mathbf{r}}{r}\right)_{\mathbf{q}=0}$$

and on utilizing

$$\int e^{i(\mathbf{q}-\mathbf{p})\mathbf{r}-\eta r}F[i\xi,1,i(pr+\mathbf{p}\mathbf{r})]\frac{d\mathbf{r}}{r}$$

$$= 2\pi\left(\frac{(\mathbf{q}-\mathbf{p})^2}{2}+\frac{\eta^2}{2}\right)^{i\xi-1}\left[\frac{(\mathbf{q}-\mathbf{p})^2}{2}+\frac{\eta^2}{2}+\mathbf{p}(\mathbf{q}-\mathbf{p})-i\eta p\right]^{-i\xi},$$

we obtain

$$\mathbf{J} = 16\pi i\eta(i\xi-1)\frac{\mathbf{p}}{(p^2+\eta^2)^3}\left(\frac{\eta+ip}{\eta-ip}\right)^{i\xi}.$$

Since for a purely imaginary $n = -i|n|$

$$\left(\frac{n-1}{n+1}\right)^n = e^{-2n\operatorname{arccot}|n|},$$

we finally obtain

$$\mathbf{J} = 16\pi i\frac{\mathbf{p}}{p^5}\frac{\xi(i\xi-1)}{(\xi^2+1)^3}e^{-2\xi\operatorname{arccot}\xi} \tag{31.3'}$$

and

$$M_{i\to f} = -16\sqrt{\pi}\,\omega e^{\frac{\pi}{2}\xi}\,\Gamma(1-i\xi)\frac{(\xi p)^{3/2}\xi(i\xi-1)}{(\xi^2+1)^3}e^{-2\xi\operatorname{arccot}\xi}\frac{\mathbf{e}\mathbf{p}}{p^5}. \tag{31.4}$$

On substituting this expression into (31.1′) we obtain the differential cross section for the photoeffect due to an unpolarized photon:

$$d\sigma = 2^7\frac{e^2\omega}{\xi^4}\left(\frac{\xi^2}{1+\xi^2}\right)^5\frac{e^{-4\xi\operatorname{arccot}\xi}}{1-e^{-2\pi\xi}}\frac{1}{2}\sum_\nu\frac{(\mathbf{p}\mathbf{e}_\nu)^2}{p^7}\delta(\varepsilon_0-\varepsilon+\omega)\,d\mathbf{p},$$

where the summation is taken over the photon polarizations. It can be carried out with the aid of

$$\sum_{\nu} (pe_\nu)^2 = p^2 - \frac{1}{k^2}(pk^2) = p^2(1-\cos^2\vartheta),$$

where $\vartheta$ is the angle between $\boldsymbol{p}$ and $\boldsymbol{k}$.

On eliminating the $\delta$-function by integration over the energy of the emerging electron we obtain:

$$d\sigma = 2^4 \frac{e^2\omega}{m\xi^4}\left(\frac{\xi^2}{\xi^2+1}\right)^5 \frac{e^{-4\xi \operatorname{arccot}\xi}}{1-e^{-2\pi\xi}} \frac{1}{(I-\omega)^2}(1-\cos^2\vartheta)\,do, \qquad (31.5)$$

where $do$ is the element of solid angle into which the electron is emitted.

In order to obtain the cross section for the photoeffect in the case of polarized photons we must carry out in this formula the substitution $\sin^2\vartheta \to \cos^2\theta = \sin^2\vartheta\cos^2\varphi$, where $\theta$ is the angle between the photon polarization $\boldsymbol{e}$ and the momentum of the emerging electron $\boldsymbol{p}$, while $\varphi$ is the angle between the $(\boldsymbol{p}, \boldsymbol{k})$ and $(\boldsymbol{k}, \boldsymbol{e})$ planes, and we must omit the factor $\frac{1}{2}$ in front of $\Sigma_\nu$ which corresponds to averaging over unpolarized photons.

Thus, the angular dependence of the photoeffect in the nonrelativistic domain is determined by the factor $\cos^2\theta = \sin^2\vartheta\cos^2\varphi$. We see that the majority of the photoelectrons are emitted in the direction of polarization of the incident photon.

On integrating (31.5) with respect to $do$, we obtain the total cross section for the photoeffect in the $K$-shell:

$$\sigma_K = 2^7 \frac{\pi e^2}{3m}\frac{I^3}{\omega^4}\frac{e^{-4\xi \operatorname{arccot}\xi}}{1-e^{-2\pi\xi}}. \qquad (31.6)$$

In the Born approximation, when $\xi \ll 1$, this formula assumes the form

$$\sigma_K = \frac{2^6}{3}\frac{e^2}{mI}\left(\frac{I}{\omega}\right)^{7/2}. \qquad (31.7)$$

On multiplying $\sigma_K$ by the number of atoms $n$ per cm$^3$, we obtain the absorption coefficient $\tau_K$ at the frequency $\omega$

$$\tau_K = n\sigma_K = 2^7 \frac{\pi e^2}{3m}\frac{nI^3}{\omega^4}\frac{e^{-4\xi \operatorname{arccot}\xi}}{1-e^{-2\pi\xi}}. \qquad (31.6')$$

Figure 31 shows the dependence of $\ln(\sigma_K/\sigma_0)$ on the photon frequency.

We now consider the photoeffect in the $L$-shell. The photon absorption coefficient due to the electrons of the $L$-shell is given by the following formulas (16). The absorption coefficient due to the two $2s$-electrons ($L_{\mathrm{I}}$-subshell) is given by

$$\tau_{L_{\mathrm{I}}} = 2^{10}\frac{\pi e^2}{3m}\frac{nI_2^3}{\omega^4}\left(1+3\frac{I_2}{\omega}\right)\frac{e^{-4\xi \operatorname{arc\,cot}\frac{1}{2}\xi}}{1-e^{-2\pi\xi}} \tag{31.8}$$

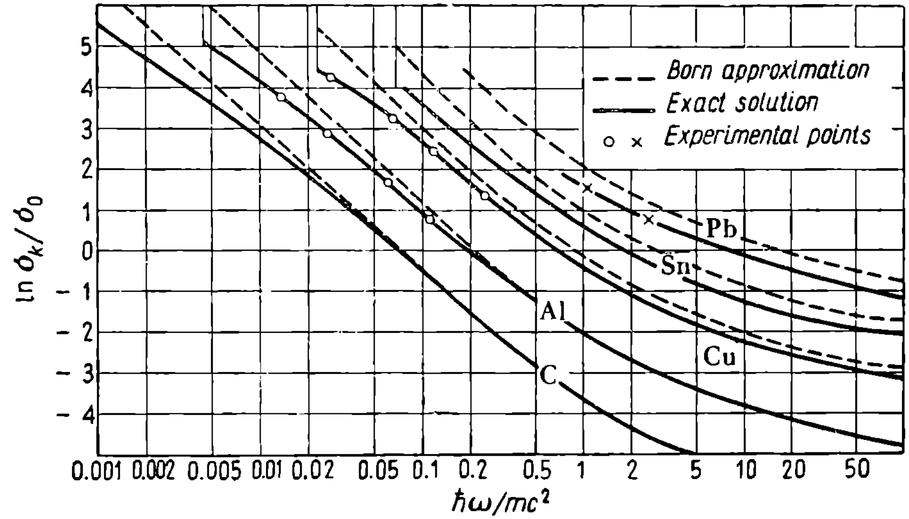


Fig. 31.

and the absorption coefficient due to the six $2p$-electrons (the $L_{\mathrm{II}}$ and $L_{\mathrm{III}}$ subshells) is given by

$$\tau_{L_{\mathrm{II}}}+\tau_{L_{\mathrm{III}}} = 2^{11}\frac{\pi e^2}{3m}\frac{nI_2^4}{\omega^5}\left(3+8\frac{I_2}{\omega}\right)\frac{e^{-4\xi \operatorname{arccot}\frac{1}{2}\xi}}{1-e^{-2\pi\xi}}, \tag{31.9}$$

where $I_2$ is the ionization potential for an $L$-electron.

### 31.2. *Photoeffect in the Relativistic Domain*

We now investigate the photoeffect in the relativistic domain, when the photon energy is large compared to the energy of a $K$-electron. In this case we must use for the electron wave functions the solutions of the Dirac equation corresponding to the nuclear Coulomb field. Insofar as the function $\psi_f(r)$ is concerned, we take for it the wave function introduced in subsection 14.5:

$$\overline{\psi}_f(r) = N_f\bar{u}(p)e^{-ipr}\left(\gamma_4-\frac{1}{2\varepsilon}\gamma\nabla\right)F[i\xi,\, 1,\, i(pr+pr)], \tag{31.10}$$

where

$$N_f = \left(\frac{2\pi\xi}{1-e^{-2\pi\xi}}\right)^{1/2}, \quad \xi = \frac{aZ}{v}.$$

The electron wave functions describing discrete states in the nuclear Coulomb field can, in accordance with (12.4), be written in the form

$$\psi_{jlm} = \begin{pmatrix} ig_\varkappa(r)\Omega_{jlm}(\boldsymbol{n}) \\ -f_\varkappa(r)\Omega_{jl'm}(\boldsymbol{n}) \end{pmatrix},$$

where $\boldsymbol{n} = \boldsymbol{r}/r$, $l' = 2j-l$, $\Omega_{jl'm} = -(\boldsymbol{\sigma n})\Omega_{jlm}$. In the case of the $K$-shell $j=\frac{1}{2}$, $l=0$, $\varkappa=-1$ and the function $\psi_{jlm}$ assumes the form

$$\psi_k = \begin{pmatrix} ig_\varkappa(\mathrm{r})u \\ -f_\varkappa(r)\boldsymbol{\sigma n}u \end{pmatrix},$$

where $u$ is a constant spinor satisfying the normalization condition $u^*u = 1/4\pi$.

In the case of light nuclei where $aZ \ll 1$, we must in formulas (13.5) and (13.5′) defining the functions $g_\varkappa(r)$ and $f_\varkappa(r)$ set $N=1$, $n=1$, $n'=0$, $\gamma=\sqrt{1-a^2Z^2}\approx 1$, and $\varepsilon \approx m(1+\frac{1}{2}a^2Z^2)$.

As a result of this we obtain $g_\varkappa(r) = 2\eta^{3/2}e^{-\eta r}$, $f_\varkappa(r) = -2\eta^{3/2}(aZ/2)e^{-\eta r}$, $\eta = \dfrac{z}{a_0}$,

$$\psi_k(\boldsymbol{r}) = N_i \begin{bmatrix} 1 \\ 0 \\ i\dfrac{aZ}{2}\cos\vartheta \\ i\dfrac{aZ}{2}\sin\vartheta e^{i\varphi} \end{bmatrix} e^{-\eta r} \qquad N_i = \sqrt{\frac{z^3}{\pi a_0^3}}$$

if the spin is directed along $\boldsymbol{n}$, and

$$\psi_k(\boldsymbol{r}) = N_i \begin{bmatrix} 0 \\ 1 \\ i\dfrac{aZ}{2}\sin\vartheta e^{-i\varphi} \\ -i\dfrac{aZ}{2}\cos\vartheta \end{bmatrix} e^{-\eta r},$$

if the spin is oriented opposite to $\boldsymbol{n}$.

On introducing the matrices $\gamma_i$ and the constant bispinor $u_0$ with components

$$u_{0_{\sigma n=\frac{1}{2}}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$u_{0_{\sigma n=-\frac{1}{2}}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

we finally express the $K$-electron wave function for $aZ \ll 1$ in the form

$$\psi_k(\boldsymbol{r}) = N_i(1-\tfrac{1}{2}aZ\gamma_4\boldsymbol{\gamma}\boldsymbol{n})e^{-\eta r}u_0, \quad \text{where} \quad aZ \ll 1. \tag{31.11}$$

We shall use this function for the evaluation of the matrix element $M_{i\to f}$.

On substituting into (31.1) the expressions (31.10) and (31.11) we obtain

$$M_{i\to f} = N_i N_f^* \bar{u}(p)\{(\boldsymbol{e}\boldsymbol{\gamma})J_0+(\boldsymbol{e}\boldsymbol{\gamma})\gamma_4(\boldsymbol{\gamma}\boldsymbol{J}_1)+(\boldsymbol{\gamma}\boldsymbol{J}_2)\gamma_4(\boldsymbol{e}\boldsymbol{\gamma})\}u_0, \tag{31.12}$$

where

$$J_0 = \int e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{r}-\eta r}F[i\xi,\, 1,\, i(pr+\boldsymbol{p}\boldsymbol{r})]d\boldsymbol{r},$$

$$\boldsymbol{J}_1 = -\frac{1}{2}aZ\int e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{r}-\eta r}\frac{\boldsymbol{r}}{r}F[i\xi,\, 1,\, i(pr+\boldsymbol{p}\boldsymbol{r})]d\boldsymbol{r},$$

$$\boldsymbol{J}_2 = -\frac{1}{2\varepsilon}\int e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{r}-\eta r}\nabla F[i\xi,\, 1,\, i(pr+\boldsymbol{p}\boldsymbol{r})]d\boldsymbol{r}.$$

The quantity $\sum\limits_{\mu_i,\mu_f}|M_{i\to f}|^2$ can be evaluated by means of formula (26.12):

$$\sum_{\mu_i,\mu_f}|M_{i\to f}|^2 = \frac{1}{4\varepsilon\varepsilon_0}\mathrm{Sp}\{A(m-i\hat{p}_0)\bar{A}(m-ip)\}, \quad p_0 = (0,\, i\varepsilon_0), \tag{31.12'}$$

where

$$A = (\boldsymbol{e}\boldsymbol{\gamma})J_0+(\boldsymbol{e}\boldsymbol{\gamma})\gamma_4(\boldsymbol{J}_1\boldsymbol{\gamma})+(\boldsymbol{J}_2\boldsymbol{\gamma})\gamma_4(\boldsymbol{e}\boldsymbol{\gamma}),$$

$$\bar{A} = \gamma_4A^+\gamma_4 = -(\boldsymbol{e}\boldsymbol{\gamma})J_0^*+(\boldsymbol{J}_1^*\boldsymbol{\gamma})\gamma_4(\boldsymbol{e}\boldsymbol{\gamma})+(\boldsymbol{e}\boldsymbol{\gamma})\gamma_4(\boldsymbol{J}_2\boldsymbol{\gamma}).$$

The evaluation of the trace of the matrix product in (31.12) yields

$$\sum_{\mu_i,\mu_f} |M_{i\to f}|^2 = \frac{\varepsilon-m}{\varepsilon}|J_0|^2+\frac{\varepsilon+m}{\varepsilon}(\mathbf{J}_1^2+\mathbf{J}_2^2)$$

$$-\frac{i}{\varepsilon}\{J_0(\mathbf{p}\mathbf{J}_1^*)-J_0(\mathbf{p}\mathbf{J}_2^*)-2J_0(\mathbf{e}\mathbf{p})(\mathbf{e}\mathbf{J}_1^*)-\text{compl. conj.}$$

$$+\frac{m+\varepsilon}{\varepsilon}\{2(\mathbf{e}\mathbf{J}_1)(\mathbf{e}\mathbf{J}_2)-\mathbf{J}_1\mathbf{J}_2+\text{compl. conj.} \qquad (31.13)$$

The integrals $J_0$, $\mathbf{J}_1$, $\mathbf{J}_2$ can be evaluated with the aid of

$$B\equiv \int e^{i(\mathbf{k}-\mathbf{p})\mathbf{r}-\eta r}F[i\xi,\, 1,\, i(pr+\mathbf{p}\mathbf{r})]\frac{d\mathbf{r}}{r} = 4\pi\,\frac{e^{-i\xi}}{(a+b)^{1-i\xi}},$$

where

$$a = \mathbf{k}^2+(\eta-ip)^2,$$
$$a+b = (\mathbf{k}-\mathbf{p})^2+\eta^2.$$

We can easily show that

$$J_0 = -\frac{\partial B}{\partial \eta} = \left\{\frac{i\xi}{a}(2\eta-ip)+2\frac{1-i\xi}{a+b}\right\}B,$$

$$\mathbf{J}_1 = \frac{1}{2}i\alpha Z\frac{\partial}{\partial \mathbf{k}}B = -i\alpha Z\left\{\mathbf{k}\frac{i\xi}{a}+(\mathbf{k}-\mathbf{p})\,\frac{1-i\xi}{a+b}\right\}B, \qquad (31.14)$$

$$\mathbf{J}_2 = i\frac{\alpha Z}{2}\,\frac{m}{\varepsilon}\left\{\frac{\partial}{\partial \mathbf{k}}-\frac{\mathbf{k}-\mathbf{p}}{\eta}\,\frac{\partial}{\partial \eta}\right\}B = -\frac{\alpha Z}{a}\,\frac{m}{\varepsilon}\xi\left[\frac{ip}{\eta}(\mathbf{k}-\mathbf{p})-\mathbf{p}\right]B.$$

These formulas become considerably simplified if $v \sim 1$ and $Z\alpha \ll 1$. In this case $\xi \ll 1$ and

$$J_0 = -\frac{\alpha Z v}{m(1-\sqrt{1-v^2})}\,\frac{v-\cos\vartheta}{1-v\cos\vartheta}B,$$

$$\mathbf{J}_1 = -\frac{i\alpha Z}{2m^2}\,\frac{1-v^2}{1-\sqrt{1-v^2}}\,\frac{\mathbf{k}-\mathbf{p}}{1-v\cos\vartheta}B,$$

$$\mathbf{J}_2 = -\frac{i\alpha Z}{2m^2}\,\frac{\sqrt{1-v^2}}{1-\sqrt{1-v^2}}(\mathbf{k}-\mathbf{p})B, \qquad (31.14')$$

$$|B|^2 = \frac{4\pi^2}{m^4}\left(\frac{1-v^2}{(1-\sqrt{1-v^2})(1-v\cos\vartheta)}\right)^2$$

($\vartheta$ is the angle between $\mathbf{p}$ and $\mathbf{k}$).

On making use of these formulas we finally obtain the following expression for the differential cross section for the photoeffect which is valid for $\alpha Z \ll 1$, $v \sim 1$, (169), (187):

$$d\sigma = \frac{2\alpha^6 Z^5}{m^2} A \frac{\sin^2\vartheta}{(1-v\cos\vartheta)^4}\left\{\frac{(1-\sqrt{1-v^2})^2}{(1-v^2)^{3/2}}(1-v\cos\vartheta)\right.$$
$$\left. -\frac{2}{1-v^2}(1-\sqrt{1-v^2})(1-v\cos\vartheta)\cos^2\varphi+4\cos^2\varphi\right\} do, \qquad (31.15)$$

where

$$A = \frac{v^3(1-v^2)^3}{(1-\sqrt{1-v^2})^5}, \quad \sqrt{1-v^2} = \frac{m}{\omega+m}$$

($v$ is the velocity of the emitted electrons, $\varphi$ is the angle between the ($\boldsymbol{p}$, $\boldsymbol{k}$) and the ($\boldsymbol{k}$, $\boldsymbol{e}$) planes).

The integrated cross section for the photoeffect in the $K$-shell is given by (169), (187)

$$\sigma = 4\pi\alpha^4 Z^5 r_0^2 \left(\frac{m}{\omega}\right)^5 (\gamma^2-1)^{3/2}$$
$$\times\left\{\frac{4}{3}+\frac{\gamma(\gamma-2)}{\gamma+1}\left(1-\frac{1}{2\gamma\sqrt{\gamma^2-1}}\ln\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}}\right)\right\}, \quad \omega \gg I, \quad \alpha Z \ll 1, \qquad (31.16)$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{\omega+m}{m}, \qquad r_0 = \frac{e^2}{4\pi m}.$$

In the extreme relativistic case $\omega \gg m$ this formula assumes the form

$$\sigma = 4\pi r_0^2 \alpha^4 Z^5 \frac{m}{\omega}, \qquad \omega \gg m. \qquad (31.17)$$

For large values of $Z$, when these formulas are inapplicable, a numerical calculation of the photoeffect cross section was made (93). The following table lists the values of the photoeffect cross section $\sigma$ obtained in this manner for a series of values of $Z$ and $\omega$.

TABLE 12

| $\omega/m$ | 0.69 | 1 | 2.2 | 5 |
|---|---|---|---|---|
| Al | 22.3 | 8.1 | 1.24 | 0.35 |
| Fe | 17.8 | 6.5 | 1.05 | 0.30 |
| Sn | 12.3 | 4.5 | 0.79 | 0.24 |
| Pb | 7.9 | 3.2 | 0.60 | 0.19 |

## § 32. Production of Electron-Positron Pairs

### 32.1. *Production of an Electron-Positron Pair by a Photon in the Field of a Nucleus*

We now proceed to investigate the production of electron-positron pairs.

In order for such a process to be possible, obviously an energy is required which is not less than $2m$. However, a single photon having sufficient energy cannot produce a pair, since according to subsection 27.1 the laws of conservation of energy and momentum cannot be simultaneously satisfied. Therefore, for the production of a pair by single photon the presence of an additional particle, for example a nucleus, is required.

Electron-positron pairs can also be produced in a collision of two charged particles or photons possessing sufficient energy. However, such processes play a less important role than pair production by a photon in the field of a nucleus. We shall, therefore, first of all consider this latter process.

Pair production by a photon in the field of a nucleus corresponds to the two diagrams shown in Fig. 32. In these diagrams $p_-$ and $p_+$ denote the four-momenta of the electron and the positron, while $k$ denotes the four-momentum of the photon. Topologically these diagrams are, obviously, equivalent to the diagrams representing the process of bremsstrahlung in the field of a nucleus (cf. Fig. 25).

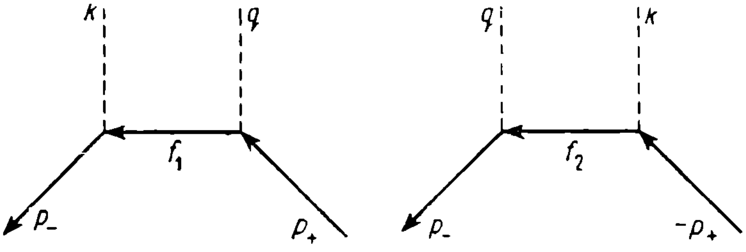


Fig. 32.

The matrix element for the process of pair production can be written in accordance with Feynman's rules in the form

$$S_{i\to f}^{(2)}=\frac{ie^2}{\sqrt{2\omega}}\,\bar{u}(p_-)\left\{\hat{e}\,\frac{i\hat{f}_1-m}{f_1^2+m^2}\,\hat{A}^{(e)}(q)+\hat{A}^{(e)}(q)\,\frac{i\hat{f}_2-m}{f_2^2+m^2}\,\hat{e}\right\}v(-p_+), \tag{32.1}$$

where $\hat{A}_{\mu}^{(e)}(q)$ is the Fourier component of the nuclear field, $u(p_-)$ and $v(-p_+)$ are the electron and positron spinor amplitudes and $f_1 = p_- - k = -p_+ + q$, $f_2 = -p_+ + k = p_- - q$.

A comparison of the matrix element (32.1) with the matrix element (29.10) for the case of bremsstrahlung shows that the former may be obtained from the latter if the following substitution is carried out in the latter: $p_1 \to -p_+$, $p_2 \to p_-$, $k \to -k$, $\bar{u}_1(p_1) \to v(-p_+)$, $\bar{u}_2(p_2) \to \bar{u}(p_-)$ (the replacement of $k$ by $-k$ corresponds to the fact that in contrast to bremsstrahlung the photon is now absorbed). Therefore, we do not need to calculate the cross section for pair production, but can utilize the already available results for the bremsstrahlung cross section. We must only keep in mind that in contrast to the bremsstrahlung process, for which the density of incident particles is equal to the electron velocity, the density of the incident particles—photons—is now equal to the velocity of light. Moreover, the number of final states is altered and is now given by $\varrho_f = d\mathbf{p}_- d\mathbf{p}_+/(2\pi)^6$.

On taking these changes into account we obtain the following expression for the differential cross section for pair production by a photon in the nuclear Coulomb field (21):

$$d\sigma = \frac{Z^2 a^3}{(2\pi)^2} \frac{|\mathbf{p}_+||\mathbf{p}_-|}{\omega^3} \frac{d\varepsilon_+ do_+ do_-}{\mathbf{q}^4} \frac{4}{m^4} \left\{ -4\left[\mathbf{k}, \frac{\varepsilon_+ \mathbf{p}_-}{\varkappa_1} + \frac{\varepsilon_- \mathbf{p}_+}{\varkappa_2}\right]^2 \right.$$
$$\left. + \mathbf{q}^2 \left[\mathbf{k}, \frac{\mathbf{p}_-}{\varkappa_1} - \frac{\mathbf{p}_+}{\varkappa_2}\right]^2 + \frac{2\omega^2}{\varkappa_1 \varkappa_2} [\mathbf{k}, \mathbf{p}_- + \mathbf{p}_+]^2 \right\}, \qquad (32.2)$$

where

$$m^2 \varkappa_1 = f_1^2 + m^2 = -2p_- k; \qquad m^2 \varkappa_2 = f_2^2 + m^2 = -2p_+ k,$$

$do_+$ and $do_-$ are elements of solid angles containing the momenta $\mathbf{p}_+$ and $\mathbf{p}_-$ of the positron and the electron, $\varepsilon_+$ and $\varepsilon_-$ are the energies of the two particles.

They are, obviously, related by $\varepsilon_+ + \varepsilon_- = \omega$.

Formula (32.2) can also be rewritten in the form

$$d\sigma = -\frac{Z^2 a^3}{2\pi} \frac{p_+ p_- d\varepsilon_+}{\omega^3} \frac{\sin\theta_+ \sin\theta_- d\theta_+ d\theta_- d\varphi_+}{\mathbf{q}^4}$$
$$\times \left\{ \frac{p_+^2 \sin^2\theta_+}{(\varepsilon_+ - p_+ \cos\theta_+)^2} (4\varepsilon^2 - q^2) + \frac{p_-^2 \sin^2\theta_-}{(\varepsilon_- - p_- \cos\theta_-)^2} (4\varepsilon_+^2 - q^2) + \right.$$

$$+\frac{2p_+p_-\sin\theta_+\sin\theta_-\cos\varphi_+}{(\varepsilon_--p_-\cos\theta_-)(\varepsilon_+-p_+\cos\theta_+)}(4\varepsilon_+\varepsilon_-+q^2-2\omega^2)$$
$$-2\omega^2\frac{p^2_+\sin^2\theta_++p^2_-\sin^2\theta_-}{(\varepsilon_--p_-\cos\theta_-)(\varepsilon_+-p_+\cos\theta_+)}\Bigg\},\tag{32.3}$$

where $\theta_\pm$ are the angles between $\mathbf{p}_\pm$ and $\mathbf{k}$, $\varphi_+$ is the angle between the $(\mathbf{k}, \mathbf{p}_+)$ and the $(\mathbf{k}, \mathbf{p}_-)$ planes and

$$q^2=(k-p_--p_+)^2.$$

The angular distribution determined by this formula is of a fairly complicated nature. It simplifies considerably only in the extreme relativistic domain when the principal role is played by small angles $\theta_\pm$. In this case the electron and the positron are emitted predominantly forward, i.e., within a narrow cone surrounding the direction of motion of the photon; the effective angular aperture of this cone is given in order of magnitude by $\theta_\pm \sim m/\omega$.

On replacing in (32.3) $\sin\theta_\pm$ by $\theta_\pm$ and $\cos\theta_\pm$ by $1-\frac{1}{2}\theta^2_\pm$, we obtain the following angular distribution for small angles:

$$d\sigma=4\frac{Z^2\alpha^3}{\pi m^2}\frac{\varepsilon^2_-\varepsilon^2_+}{\omega^3q^4}[\omega^2(u^2+v^2)\xi\eta-2\varepsilon_-\varepsilon_+(u^2\xi^2+v^2\eta^2)$$
$$+2(\varepsilon^2_-+\varepsilon^2_+)uv\xi\eta\cos\varphi_+]d\varepsilon_+\theta_-d\theta_-\theta_+d\theta_+d\varphi_+,\tag{32.4}$$

where

$$u=\frac{p_-\theta_-}{m},\quad v=\frac{p_+\theta_+}{m},\quad \xi=\frac{1}{1+u^2},\quad \eta=\frac{1}{1+v^2}.$$

Integration of (32.3) over the angles yields the differential cross section for pair production with the positron energy lying in the range $\varepsilon_+$ to $\varepsilon_++d\varepsilon_+$:

$$d\sigma=\bar{\Phi}\frac{|\mathbf{p}_+||\mathbf{p}_-|}{\omega^3}d\varepsilon_+\Bigg\{-\frac{4}{3}-2\varepsilon_+\varepsilon_-\frac{\mathbf{p}^2_++\mathbf{p}^2_-}{\mathbf{p}^2_+\mathbf{p}^2_-}$$
$$+m^2\left(\frac{\varepsilon_+\zeta_-}{|\mathbf{p}_-|^3}+\frac{\varepsilon_-\zeta_+}{|\mathbf{p}_+|^3}-\frac{\zeta_+\zeta_-}{|\mathbf{p}_+||\mathbf{p}_-|}\right)$$
$$+L\Bigg[\frac{\omega^3}{|\mathbf{p}_+|^3|\mathbf{p}_-|^3}(\varepsilon^2_+\varepsilon^2_-+\mathbf{p}^2_+\mathbf{p}^2_-)-\frac{8}{3}\frac{\varepsilon_+\varepsilon_-}{|\mathbf{p}_+||\mathbf{p}_-|}$$
$$-\frac{m^2\omega}{2|\mathbf{p}_+||\mathbf{p}_-|}\left(\frac{\varepsilon_+\varepsilon_--\mathbf{p}^2_-}{|\mathbf{p}_-|^3}\zeta_-+\frac{\varepsilon_+\varepsilon_--\mathbf{p}^2_+}{|\mathbf{p}_+|^3}\zeta_++\frac{2\omega\varepsilon_+\varepsilon_-}{\mathbf{p}^2_+\mathbf{p}^2_-}\right)\Bigg]\Bigg\},\tag{32.5}$$

where

$$\bar{\Phi} = r_0^2 Z^2 \alpha, \qquad \zeta_{\pm} = 2 \ln \frac{\varepsilon_{\pm} + |p|_{\pm}}{m},$$

$$L = 2 \ln \frac{\varepsilon_+ \varepsilon_- + |p_+||p_-| + m^2}{m\omega}.$$

In the extreme relativistic domain when all the energies are considerably greater than $m (\omega, \varepsilon_{\pm} \gg m)$ this expression simplifies considerably:

$$d\sigma = 4\bar{\Phi} \frac{d\varepsilon_+}{\omega^3} \left( \varepsilon_+^2 + \varepsilon_-^2 + \frac{2}{3} \varepsilon_+ \varepsilon_- \right) \left( \ln \frac{2\varepsilon_+ \varepsilon_-}{\omega m} - \frac{1}{2} \right). \qquad (32.6)$$

Formulas (32.5), (32.6) do not take into account the screening of the nuclear field. Therefore, in accordance with (29.27) they are valid if $(2\varepsilon_+ \varepsilon_- / m\omega) \ll 137 Z^{-1/3}$.

Screening can be taken into account in the same manner as in the case of bremsstrahlung. In the extreme relativistic domain $(\omega, \varepsilon_{\pm} \gg m)$ we obtain for the cross section for pair production

$$d\sigma = \bar{\Phi} \frac{d\varepsilon_+}{\omega^3} \left\{ (\varepsilon_-^2 + \varepsilon_+^2) \left( \Phi_1(\zeta) - \frac{4}{3} \ln Z \right) \right.$$

$$\left. + \frac{2}{3} \varepsilon_- \varepsilon_- \left( \Phi_2(\zeta) - \frac{4}{3} \ln Z \right) \right\}, \quad \zeta = \frac{100 m\omega}{Z^{1/3} \varepsilon_- \varepsilon_+}, \qquad (32.7)$$

where $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ are the same functions which appear in (29.29).

In the case of complete screening when $(2\varepsilon_+ \varepsilon_- / m\omega) \gg 137 Z^{-1/3}$, formula (32.7) assumes the form

$$d\sigma = 4\bar{\Phi} \frac{d\varepsilon_+}{\omega^3} \left\{ \left( \varepsilon_+^2 + \varepsilon_-^2 + \frac{2}{3} \varepsilon_+ \varepsilon_- \right) \ln (183 Z^{-1/3}) - \frac{\varepsilon_+ \varepsilon_-}{9} \right\}. \qquad (32.8)$$

In order to obtain the total cross section for pair production we must integrate (32.5) over the positron energy. However, it is impossible to integrate this expression directly. Such an integration can be carried out only in the extreme relativistic case by utilizing formula (32.6). On integrating this formula we obtain the following expression for the total cross section in the extreme relativistic case without taking screening into account:

$$\sigma = \bar{\Phi} \left( \frac{28}{9} \ln \frac{2\omega}{m} - \frac{218}{27} \right), \quad \text{where} \quad \omega \gg m. \qquad (32.9)$$

On integrating (32.8) over $\varepsilon_+$ we obtain the total cross section for pair production in the case of complete screening

$$\sigma = \bar{\Phi}\left(\frac{28}{9}\ln 183Z^{-\frac{1}{3}} - \frac{2}{27}\right). \tag{32.10}$$

In § 55 we shall establish the connection between the total cross section for pair production by a photon and the cross section for the forward scattering of a photon in the field of a nucleus. By utilizing this relation we can show that the total cross section for pair production without taking screening into account is determined by the following formula (160):

$$\sigma(\omega) = \bar{\Phi}\frac{1}{\eta^2}\left\{4\int_1^\eta \frac{L(x)}{x}dx - 2L(\eta) + \frac{2}{27}\left[-(64+109\eta^2)E\left(\sqrt{1-\frac{1}{\eta^2}}\right)\right.\right.$$
$$\left.\left. + \left(125+\frac{6}{\eta^2}+42\eta^2\right)F\left(\sqrt{1-\frac{1}{\eta^2}}\right)\right]\right\}, \tag{32.11}$$

where

$$\eta = \frac{\omega}{2m},$$

$$L(x) = \int_1^{x^2} \ln\frac{x+\sqrt{x^2-y}}{x-\sqrt{x^2-y}}\,\frac{dy}{\sqrt{y(y-1)}}$$

and $F(x)$ and $E(x)$ are elliptic integrals:

$$F(x) = \int_0^{\pi/2}(1-x^2\sin^2\varphi)^{-\frac{1}{2}}d\varphi, \qquad E(x) = \int_0^{\pi/2}(1-x^2\sin^2\varphi)^{\frac{1}{2}}d\varphi.$$

In the extreme relativistic domain this formula reduces to (32.9).

Figure 33 shows the energy distribution of the particles of the pair determined by formulas (32.5) and (32.7). Along the horizontal axis we have plotted the ratio of the kinetic energy of the positron $\varepsilon_+ - m$ to the total kinetic energy of the pair $\omega - 2m$, while along the vertical axis we have plotted the quantity $[(\omega-2m)/\bar{\Phi}]\,(d\sigma/d\varepsilon_+)$. The different curves refer to the different values of $\omega/m$ which are shown next to the curves. The curves corresponding to $\omega/m = 6$ and $\omega/m = 10$ are valid for all $Z$ since for these values of the photon energy screening can still be neglected. The other curves take screening into account, and refer to lead and aluminum. The curve corresponding to $\omega/m = \infty$ refers to complete screening.

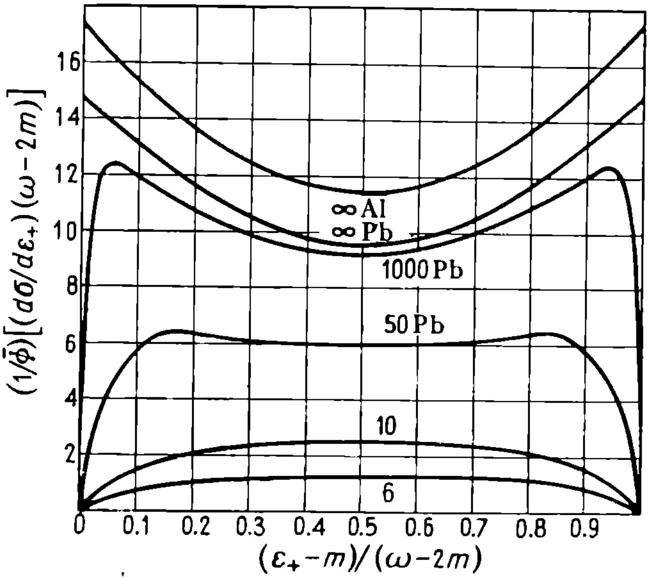


Fig. 33.

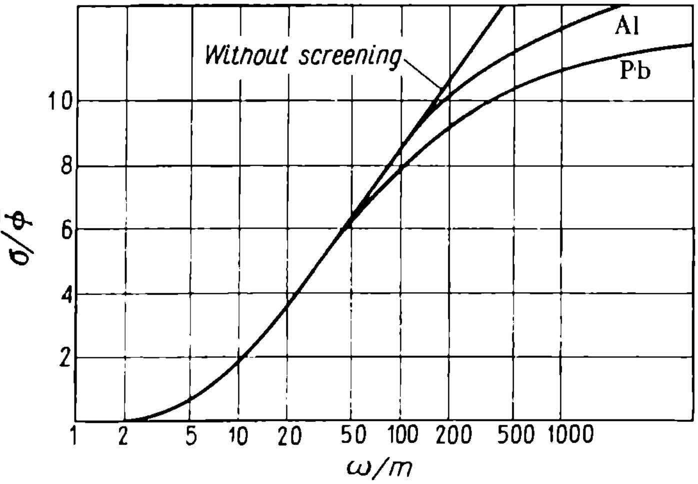


Fig. 34.

For low photon energies the curves have only a single flat maximum corresponding to the electron and positron having the same energy.

At large energies the curves have two identical symmetrically situated maxima of which one corresponds to maximum positron energy and minimum electron energy, while the other conversely corresponds to minimum positron energy and maximum electron energy.

Figure 34 shows the dependence of the total cross section for pair production on photon energy in the case of lead, copper, and aluminum.

### 32.2 *Exact Theory of Pair Production by a Photon in the Field of a Nucleus in the Nonrelativistic and the Extreme Relativistic Cases*

The foregoing formulas for the pair production cross sections are valid in the Born approximation, when $\xi_\pm \equiv (Ze^2/\hbar v_\pm) \ll 1$ ($v_-$ and $v_+$ are the electron and positron velocities).

However, in the nonrelativistic ($v_\pm \ll 1$) and the extreme relativistic ($1-v_\pm \ll 1$) cases it is possible to evaluate the pair production cross sections exactly by utilizing the exact electron and positron wave functions in the Coulomb field in the former case, and the functions introduced in subsection 14.5 in the latter case.

This calculation is carried out in a manner analogous to the evaluation of the bremsstrahlung cross section.

The matrix element determining pair production can be written in the form

$$S^{(1)}_{i\to f} = i\frac{e}{\sqrt{2\omega}}\int \psi_-^* e\boldsymbol{\alpha}\psi_+ e^{i\boldsymbol{kr}-i\omega t}\, d\boldsymbol{r}\, dt \equiv 2\pi M_\pm\, \delta(\omega-\varepsilon_+-\varepsilon_-), \qquad (32.12)$$

where $\psi_-(\boldsymbol{r})$ and $\psi_+(\boldsymbol{r})$ are the exact electron and positron wave functions in the nuclear Coulomb field. Since both these particles are created, the asymptotic behavior of the functions $\psi_-(\boldsymbol{r})$ and $\psi_+(\boldsymbol{r})$ for large values of $r$ must be the same, viz., the functions $\psi_-(\boldsymbol{r})$ and $\psi_+(\boldsymbol{r})$ must at $r \to \infty$ have the form of a sum of a plane wave and an incoming spherical wave. This asymptotic behavior represents the difference between the matrix elements for pair production and for bremsstrahlung: in the latter process only the wave function of the final state has such asymptotic behavior, while the wave function for the initial state has at $\mathrm{r} \to \infty$ the form of a sum of a plane wave and an outgoing spherical wave.

Let us evaluate first of all the cross section for pair production by a photon in the extreme relativistic case. In order to do this we use as $\psi_-$ and $\psi_+$ the functions

$$\begin{aligned}\psi_- &= N_- e^{i\mathbf{p}_-\cdot\mathbf{r}}\left[1-\frac{i}{2\varepsilon_-}\boldsymbol{\alpha}\nabla\right]uF(-i\xi_-,\,1,\,-ip_-r-i\mathbf{p}_-\mathbf{r}),\\ \psi_+ &= N_+ e^{-i\mathbf{p}_+\cdot\mathbf{r}}\left[1+\frac{i}{2\varepsilon_+}\boldsymbol{\alpha}\nabla\right]vF(-i\xi_+,\,1,\,ip_+r+i\mathbf{p}_+\mathbf{r}),\end{aligned}\tag{32.13}$$

where $u$ and $v$ are the unit spinor amplitudes for the electron and the positron, and $N_-$ and $N_+$ are normalizing factors which we choose equal to $N_- = e^{\frac{\pi}{2}\xi_-}\Gamma(1+i\xi_-)$, $N_+ = e^{-\frac{\pi}{2}\xi_+}\Gamma(1+i\xi_+)$, corresponding to unit amplitudes for the wave functions at infinity. (The subscripts $+$ and $-$ both here and later denote quantities which refer respectively to the positron and the electron.)

After substitution of formulas (32.13) into (32.12) the matrix element $M_\pm$ assumes the form

$$\begin{aligned}M_\pm &= -\frac{ie^2}{4\pi\sqrt{2\omega}}N_+N_-^*\int e^{i(\mathbf{k}-\mathbf{p}_+-\mathbf{p}_-)\mathbf{r}}\left[1+\frac{i}{2\varepsilon_-}\boldsymbol{\alpha}\nabla\right]u^*\\ &\quad\times F(i\xi_-,\,1,\,ip_-r+i\mathbf{p}_-\mathbf{r})\,\mathbf{e}\boldsymbol{\alpha}\left[1+\frac{i}{2\varepsilon_+}\boldsymbol{\alpha}\nabla\right]\\ &\quad\times v^*F(-i\xi_+,\,1,\,ip_+r+i\mathbf{p}_+\mathbf{r})\,d\mathbf{r}\\ &= C\{u^*\mathbf{e}\boldsymbol{\alpha}v\,I_1+u^*(\mathbf{e}\boldsymbol{\alpha})(\boldsymbol{\alpha}\mathbf{I}_2)v+u^*(\boldsymbol{\alpha}\mathbf{I}_3)(\mathbf{e}\boldsymbol{\alpha})v\},\end{aligned}\tag{32.14}$$

where

$$\begin{aligned}I_1 &= \int e^{i\mathbf{q}\mathbf{r}}F_1F_2\,d\mathbf{r},\\ \mathbf{I}_2 &= \frac{i}{2\varepsilon_+}\int e^{i\mathbf{q}\mathbf{r}}F_1\nabla F_2\,d\mathbf{r},\\ \mathbf{I}_3 &= \frac{i}{2\varepsilon_-}\int e^{i\mathbf{q}\mathbf{r}}F_2\nabla F_1\,d\mathbf{r},\\ F_1 &\equiv F(i\xi_-,\,1,\,ip_-r+i\mathbf{p}_-\mathbf{r}),\\ F_2 &\equiv F(-i\xi_+,\,1,\,ip_+r+i\mathbf{p}_+\mathbf{r}),\\ \mathbf{q} &= \mathbf{k}-\mathbf{p}_+-\mathbf{p}_-,\\ C &= -\frac{ie^2}{2\pi}\frac{N_+N_-^*}{\sqrt{2\omega}}.\end{aligned}$$

The integrals $I_1, I_2, I_3$ can be evaluated with the aid of formulas (29.47).

The pair production cross section is related to the matrix element $M_\pm$ by the expression

$$d\sigma_\pm = 2\pi\frac{1}{2}\sum_{\mu_+\mu_-,\nu}|M_\pm|^2\frac{p_+^2\,dp_+p_-\varepsilon_-\,do_+\,do_-}{(2\pi)^6},$$

where $\sum\limits_{\mu_+,\mu_-,\nu}$ denotes summation over the components of the particle spins.

The result of these calculations leads to the following formula for the cross section for pair production by a photon:

$$\begin{aligned}
d\sigma = &-a\left(\frac{\pi a}{\sinh\pi a}\right)^2\frac{a^2}{2\pi}\frac{1}{m^2}\frac{p_-p_+}{\omega^3}d\varepsilon_+\sin\theta_-\sin\theta_+\,d\theta_+\,d\theta_-\,d\varphi\\
&\times\left\{\frac{V^2(x)}{q^4}\left[\frac{p_-^2\sin^2\theta_-(4\varepsilon_+^2-q^2)}{(\varepsilon_--p_-\cos\theta_-)^2}+\frac{p_+^2\sin^2\theta_+(4\varepsilon_-^2-q^2)}{(\varepsilon_+-p_+\cos\theta_+)^2}\right.\right.\\
&+\frac{(4\varepsilon_-\varepsilon_++q^2-2\omega^2)2p_-p_+\sin\theta_-\sin\theta_+\cos\varphi}{(\varepsilon_--p_-\cos\theta_-)(\varepsilon_+-p_+\cos\theta_+)}\\
&\left.-\frac{2\omega^2(p_-^2\sin^2\theta_-+p_+^2\sin^2\theta_+)}{(\varepsilon_--p_-\cos\theta_-)(\varepsilon_+-p_+\cos\theta_+)}\right]+\frac{a^2[\omega^2-(p_-+p_+)^2]^2\,W^2(x)}{[4\omega^2(\varepsilon_--p_-\cos\theta_-)(\varepsilon_+-p_+\cos\theta_+)]^2}\\
&\times\left[\frac{p_-^2\sin^2\theta_-(4\varepsilon_+^2-q^2)}{(\varepsilon_--p_-\cos\theta_-)^2}+\frac{p_+^2\sin^2\theta_+(4\varepsilon_-^2-q^2)}{(\varepsilon_+-p_+\cos\theta_+)^2}\right.\\
&-\frac{(4\varepsilon_+\varepsilon_-+q^2-2\omega^2)\,2p_-p_+\sin\theta_-\sin\theta_+\cos\varphi}{(\varepsilon_--p_-\cos\theta_-)(\varepsilon_+-p_+\cos\theta_+)}\\
&\left.\left.+\frac{2\omega^2(p_-^2\sin^2\theta_-+p_+^2\sin^2\theta_+)}{(\varepsilon_+-p_+\cos\theta_+)(\varepsilon_--p_-\cos\theta_-)}-4\omega^2(\varepsilon_-\varepsilon_++p_-p_+\cos\theta_-\cos\theta_+\right]\right\},
\end{aligned}\tag{32.15}$$

where the same notation has been used as in formula (32.3). The functions $V(x)$ and $W(x)$ are defined by formulas (29.48′), $a = Za$, and

$$x = 1-\frac{q^2[\omega^2-(p_+-p_-)^2]}{4\omega^2(\varepsilon_+-p_+\cos\theta_+)(\varepsilon_--p_-\cos\theta_-)}.$$

In the particularly interesting case of small angles $\theta_-\sim\theta_+\sim m/\varepsilon$ the cross section is of the form

$$d\sigma = 8a\left(\frac{\pi a}{\sinh \pi a}\right)^2 \frac{a^2}{2\pi}\frac{1}{m^2}\frac{\varepsilon_-^2\varepsilon_+^2}{\omega^3}\,d\varepsilon_+\,\theta_+\,d\theta_+\,\theta_-\,d\theta_-\,d\varphi$$
$$\times\{q^{-4}V^2(x)[\omega^2(u^2+v^2)\xi\eta-2\varepsilon_-\varepsilon_+(u^2\xi^2+v^2\eta^2)$$
$$+2(\varepsilon_-^2+\varepsilon_+^2)uv\xi\eta\cos\varphi]+a^2W^2(x)\xi^2\eta^2[\omega^2(1-(u^2+v^2)\eta\xi)$$
$$-2\varepsilon_-\varepsilon_+(u^2\xi^2+v^2\eta^2)-2(\varepsilon_-^2+\varepsilon_+^2)uv\eta\xi\cos\varphi]\}, \qquad (32.16)$$

where $u=(p_+\theta_+/m)$, $v=(p_-\theta_-/m)$, $\xi=(1/1+u^2)$ and $\eta=(1/1+v^2)$.

The difference between the results given by formula (32.15) and by formula (32.3) which is valid in the Born approximation is greater than in the case of bremsstrahlung. This is associated with the fact that the scattered waves for the two particles overlap in the case of pair production to a greater extent than in the case of bremsstrahlung.

The pair production cross section integrated over the angles is of the form

$$d\sigma = 2\frac{Z^2a^3}{m^2}\frac{d\varepsilon_+}{\omega^3}\left(\varepsilon_+^2+\varepsilon_-^2+\frac{2}{3}\varepsilon_\pm\varepsilon_-\right)\left[2\ln\frac{2\varepsilon_+\varepsilon_-}{m\omega}-1-2f(Z)\right], \qquad (32.17)$$

where $f(Z)$ is defined by formula (29.49′).

If screening is taken into account the cross section is given by

$$d\sigma = \frac{Z^2a^3}{m^2}\frac{d\varepsilon_+}{\omega^3}\left\{\left(\varepsilon_+^2+\varepsilon_-^2\right)\left[\Phi_1(\gamma)-\frac{4}{3}\ln Z-4f(Z)\right]\right.$$
$$\left.+\frac{2}{3}\varepsilon_+\varepsilon_-\left[\Phi_2(\gamma)-\frac{4}{3}\ln Z-4f(Z)\right]\right\}, \qquad (32.18)$$

where $\Phi_1(\gamma)$ and $\Phi_2(\gamma)$ are the same functions which appear in (29.30), $\gamma = 100(m\omega/\varepsilon_+\varepsilon_-)Z^{-1/3}$.

The total pair production cross section without taking screening into account is equal to

$$\sigma = \frac{28}{9}\frac{Z^2r_0^2}{137}\left[\ln\frac{2\omega}{m}-\frac{109}{42}-f(Z)\right]. \qquad (32.19)$$

In the case of complete screening we have

$$\sigma = \frac{28}{9}\frac{Z^2r_0^2}{137}\left[\ln(183Z^{-1/3})-\frac{1}{42}-f(Z)\right]. \qquad (32.20)$$

We see that the correction due to the Coulomb field is always negative and is equal to

$$\Delta\sigma = \frac{28}{9}\frac{Z^2r_0^2}{137}f(Z). \qquad (32.20')$$

In the case of lead the relative correction amounts to approximately 10% (it is assumed that $\ln(2\omega/m)$ is of order of magnitude 3–5).

We now obtain the cross section for pair production by a photon near threshold, $p_+ \ll m$. The wave functions $\Psi_\pm$ are in this case determined by formulas (32.13) in which the expressions in square brackets are to be replaced by unity. Therefore the matrix element $M_\pm$ has the form

$$M_\pm = Cu^* e\alpha v I_1 \tag{32.14'}$$

A simple calculation leads to the following expression for the energy distribution of the positrons in the nonrelativistic case:

$$\begin{aligned} d\sigma = \frac{\alpha Z^2}{12\pi} r_0^2 & \frac{4\pi^2 \xi_- \xi_+}{(1-e^{-2\pi\xi_-})(e^{2\pi\xi_+}-1)} \\ & \times [\omega - 2m + (\alpha Z)^2 (\omega - m)] \\ & \times [(\varepsilon_+ - m)\{\omega - 2m - (\varepsilon_+ - m)\}]^{1/2} \frac{d\varepsilon_+}{m^3} \end{aligned} \tag{32.21}$$

### 32.3. *Pair Production by Two Photons*

The process of pair production by two photons corresponds to the two diagrams shown in Fig. 35. According to Feynman's rules the matrix element determining this process is equal to

$$\begin{aligned} S^{(2)}_{i\to f} = \frac{ie^2}{2\sqrt{\omega\omega'}} \bar{u}(p_-) & \left\{ \hat{e}' \frac{if_1 - m}{m^2 \varkappa_1} \hat{e} + \hat{e} \frac{i\hat{f}_2 - m}{m^2 \varkappa_2} \hat{e}' \right\} \\ & \times v(-p_+)(2\pi)^4 \delta(k + k' - p_+ - p_-), \end{aligned} \tag{32.22}$$

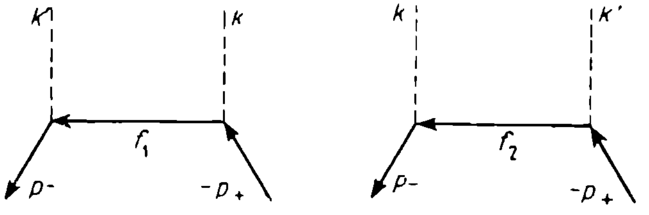


Fig. 35.

where

$$f_1 = -p_+ + k = p_- - k', \qquad f_2 = -p_+ + k' = p_- - k,$$
$$m^2 \varkappa_1 = f_1^2 + m^2 = -2p_+ k = -2p_- k',$$
$$m^2 \varkappa_2 = f_2^2 + m^2 = -2p_+ k' = -2p_- k,$$

$k$ and $k'$ are the four-momenta of the photons, $\hat{e}$ and $\hat{e}'$ are their polarizations, $p_-$ and $p_+$ are the four-momenta of the electron and the positron.

The differential cross section for pair production is related to $S^{(2)}_{i\to f}$ by

$$d\sigma = \frac{e^4}{4\omega\omega'}|\bar{u}Qv|^2 \frac{dp_- dp_+}{J(2\pi)^2}\,\delta(k+k'-p_+-p_-)\,\delta(\omega+\omega'-\varepsilon_+-\varepsilon_-), \tag{32.23}$$

where

$$Q = \hat{e}'\frac{i\hat{f}_1-m}{m^2\varkappa_1}\hat{e}+\hat{e}\frac{i\hat{f}_2-m}{m^2\varkappa_2}\hat{e}' \tag{32.23'}$$

and $J$ is the photon flux density.

The matrix $Q$ has the same structure as the analogous matrix in the case of the Compton effect. On setting in the latter $p_1 = -p_+$, $k_1 = k$, $e_1 = e$, $p_2 = p_-$, $k_2 = -k'$, and $e_2 = e'$, we obtain $Q$. Therefore, we can use the results already available for the Compton effect, and write the following expression for the quantity $|\bar{u}Qv|^2$ summed over the spins of the electron and the positron, and averaged over the photon polarizations:

$$\begin{aligned}\overline{\sum|\bar{u}Qv|^2} &= -\frac{1}{16\varepsilon_+\varepsilon_-}\operatorname{Sp}F\\ &= -\frac{1}{2\varepsilon_+\varepsilon_-}\left\{4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2-4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)-\left(\frac{\varkappa_2}{\varkappa_1}+\frac{\varkappa_1}{\varkappa_2}\right)\right\},\end{aligned} \tag{32.24}$$

where

$$\operatorname{Sp}F = \operatorname{Sp}\{Q(i\hat{p}_1-m)\gamma_4 Q^+\gamma_4(i\hat{p}_2-m)\},$$

and the following substitution has been performed in $Q$: $\hat{e}\to\gamma_\nu$ and $\hat{e}'\to\gamma_\lambda$.

It is convenient to carry out further calculations in the center of mass system of the colliding photons, in which $k' = -k$, $p_+ = -p_-$, $\omega = \omega' = \omega_0$ and $\varepsilon_+ = \varepsilon_- = \omega_0$.

On eliminating the $\delta$-functions by integrations over $dp_-$ and $d\varepsilon_+$:

$$\delta(k+k'-p_+-p_-)\,dp_- \to 1,$$

$$\delta(\omega+\omega'-\varepsilon_+-\varepsilon_-)\,dp_+ \to \frac{p_+\varepsilon_+ do_+}{2},$$

where $do_+$ is the element of solid angle containing the direction of positron motion, and on noting that the photon flux is given in the center of mass system by $J = 2c$, we obtain the following expression for the cross section for pair production by two photons:

$$d\sigma = -\frac{r_0^2}{8} m^2 \frac{\sqrt{\omega_0^2-m^2}}{\omega_0^3}\, do_+ \left\{4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2 - 4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right) - \left(\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right)\right\}. \qquad (32.25)$$

On substituting into this expression

$$m^2\varkappa_1 = -2p_+k = -2p_+k + 2\varepsilon_+\omega = 2\omega_0\left(\omega_0 - \sqrt{(\omega_0^2-m^2)}\cos\theta\right),$$

$$m^2\varkappa_2 = -2p_-k = -2p_-k + 2\varepsilon_-\omega$$

$$= 2p_+k + 2\varepsilon_-\omega = 2\omega_0\left(\omega_0 + \sqrt{\omega_0^2-m^2}\cos\theta\right),$$

where $\theta$ is the angle between the vectors $k$ and $p_+$, we obtain

$$d\sigma = \frac{r_0^2}{4} \frac{m^2\sqrt{\omega_0^2-m^2}}{\omega_0^3} \left\{\frac{2\omega_0^2 - m^2 + (\omega_0^2-m^2)\sin^2\theta}{m^2\cos^2\theta + \omega_0^2\sin^2\theta} - \frac{2(\omega_0^2-m^2)^2\sin^4\theta}{(m^2\cos^2\theta+\omega_0^2\sin^2\theta)^2}\right\} 2\pi \sin\theta\, d\theta. \qquad (32.26)$$

The total cross section for pair production by two photons is equal to

$$\sigma = \frac{\pi r_0^2 m^2}{\omega^2}\left\{\left(2 + \frac{2m^2}{\omega_0^2} - \frac{m^4}{\omega_0^4}\right) \ln\left|\frac{\omega_0}{m} + \sqrt{\frac{\omega_0^2}{m^2}-1}\right| - \sqrt{1-\frac{m^2}{\omega_0^2}}\left(1+\frac{m^2}{\omega_0^2}\right)\right\}. \qquad (32.27)$$

In accordance with the arguments of subsection 28.3 this cross section is relativistically invariant. In an arbitrary coordinate system in which the photon $k'$ is moving in the direction opposite to the photon $k$ the relation obviously holds $\omega \cdot \omega' = \omega_0^2 (k \| -k')$. Therefore, in such systems the cross section is defined by formula (32.26) in which we must set $\omega_0 = \sqrt{\omega\omega'}$.

On introducing the quantity $x = (1 - m^2/\omega\omega')^{1/2}$ we can rewrite the total cross section in the form

$$\sigma = \frac{\pi r_0^2}{2}(1-x^2)\left\{(3-x^4)\ln\left|\frac{1+x}{1-x}\right| + 2x(x^2-2)\right\}. \qquad (32.27')$$

### 32.4. *Pair Production in a Photon-Electron Collision*

For a pair to be produced in a collision of a photon with an electron at rest the photon energy must exceed $4m$. Indeed, from the conservation laws written in four-dimensional form $k+p = p'+p''+p'''$, where $k$ and $p$ are the four-momenta of the photon and the electron in the initial state while $p'$, $p''$, $p'''$ are the momenta of the particles (electrons and positrons) in the final state, it follows that $(k+p)^2 = -m^2-2m\omega = (p'+p''+p''')^2$. But the square of the four-momenta of three particles of equal mass cannot be smaller in absolute value than $9m^2$; therefore we have $-m^2-2m\omega \leqslant -9m^2$, from which it follows that $\omega \geqslant 4m$.

The process of pair production in a collision between a photon and an electron differs from the process of pair production in a collision of a photon with a nucleus by the fact that the effect of the electron cannot, generally speaking, be replaced by an external Coulomb field, as we have done in discussing pair production by a photon in the field of a nucleus. In this sense the process is analogous to the process of emission of radiation in a collision of two electrons, and like the latter process it is described by elements of the third order scattering matrix $S^{(3)}$ connecting one photon state and three electron states (more accurately speaking, two electron states and one positron state). The corresponding eight diagrams are similar to the diagrams of Fig. 30 representing the process of emission of radiation in an electron-electron collision. Two of these diagrams are shown in Fig. 36. The others can be obtained by interchanging the momenta of the real and the virtual photons, and also the momenta of the electrons in the final state $p_2$ and $p_2'$ ($p_1$ is the electron momentum in the initial state, $p_+$ is the positron momentum).

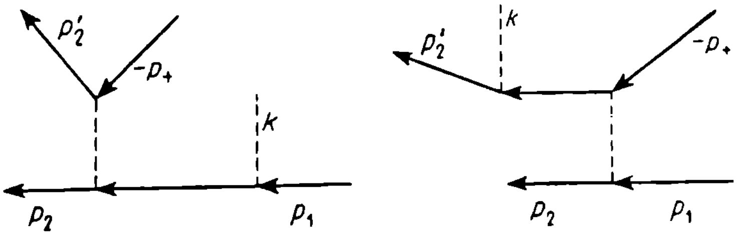


Fig. 36.

In order to obtain the matrix element $S^{(3)}_{i\to f}$ corresponding to the process under consideration it is, obviously, sufficient to perform the substitutions $p_2' \to -p_+$, $k \to -k$, and $u' \to v(-p_+)$ in the matrix element (29.50) which determined the emission of radiation in a collision of two electrons.

We give here only the results for the total pair production cross section. Near the pair production threshold the cross section is equal to (140)

$$\sigma = ar_0^2 \frac{\pi\sqrt{3}}{4\cdot 3^5}\left(\frac{\omega}{m}-4\right)^2. \tag{32.28}$$

In the extreme relativistic case (200) we have

$$\sigma = ar_0^2\left(\frac{28}{9}\ln\frac{2\omega}{m}-10.8\right). \tag{32.29}$$

This expression differs from the cross section for pair production by a photon in the field of a nucleus for $Z = 1$ only by the coefficient in the argument of the logarithm.

### 32.5. *Pair Production in a Collision of Two Fast Charged Particles*

We now consider pair production in a collision of two fast charged particles, for example, nuclei of charges $Z_1 e$ and $Z_2 e$. This process corresponds to the diagram given in Fig. 37 in which the dotted lines represent the external potentials associated with the two charges.

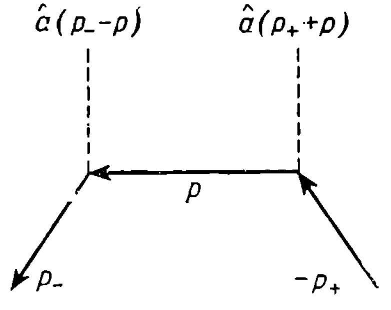


Fig. 37.

The matrix element for this process is given by

$$S^{(2)}_{i\to f} = i(2\pi)^4 e^2 \bar{u}(p_-)\int d^4p\,\frac{\hat{a}(p_- - p)}{(2\pi)^4}\,\frac{i\hat{p}-m}{p^2+m^2}\,\frac{\hat{a}(p_+ + p)}{(2\pi)^4}\,v(-p_+),$$

where $a(q)$ is the Fourier component of the total potential due to the two charges.

The probability of this process turns out to be independent of the time. After summing over the electron and positron spin components it can be written in the following form:

$$dw = \sum_{\mu_+,\mu_-} |S^{(2)}_{i\to f}|^2 \frac{dp_+\,dp_-}{(2\pi)^6}$$

$$= \frac{e^4}{4(2\pi)^8\varepsilon_+\varepsilon_-}\,\mathrm{Sp}\left\{\int d^4p\,\hat{a}(p_- - p)\frac{i\hat{p}-m}{p^2+m^2}\hat{a}(p_+ + p)(i\hat{p}_+ + m)\right.$$

$$\left.\times \int d^4p'\,\hat{a}(p_+ + p')\frac{i\hat{p}'-m}{p'^2+m^2}\hat{a}(p_- - p')(i\hat{p}_- - m)\right\}\frac{dp_+\,dp_-}{(2\pi)^6}. \quad (32.30)$$

We now determine the field produced by the charges. The potential $(\boldsymbol{A}, A_0)$ produced by a particle moving with uniform velocity $\boldsymbol{v}$ satisfies the equations

$$\Box \boldsymbol{A} = -Ze\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{v}t-\boldsymbol{r}_0),$$

$$\Box A_0 = -Ze\delta(\boldsymbol{r}-\boldsymbol{v}t-\boldsymbol{r}_0).$$

On expanding $A_\mu$ into the Fourier integral.

$$A_\mu(x) = \frac{1}{(2\pi)^4}\int A_\mu(q)e^{iqx}\,dq,$$

we obtain

$$A_\mu(q) = \frac{2\pi Ze}{q^2}u_\mu\delta(uq)e^{-iqx_0},$$

where $u_\mu([v/\sqrt{1-v^2}], [i/\sqrt{1-v^2}])$ is the four-velocity of the particle. Therefore, the Fourier component of the total potential produced by both particles will be equal to

$$a_\mu = A_{1\mu}+A_{2\mu} = \frac{2\pi eZ_1}{q^2}u^{(1)}_\mu\,\delta(u^{(1)}q)+\frac{2\pi eZ_2}{q^2}u^{(2)}_\mu\,\delta(u^{(2)}q)e^{-iqr_0}, \quad (32.31)$$

where the indices 1 and 2 denote the two particles and $r_0 = r_0^{(1)}-r_0^{(2)}$.

We now note that

$$\delta[(p_- - p)u^{(1)}]\delta[(p+p_+)u^{(1)}] = 0,$$

$$\delta[(p_- - p)u^{(2)}]\delta[(p+p_+)u^{(2)}] = 0,$$

$$\delta[(p'+p_+)u^{(1)}]\delta[(p'-p_-)u^{(1)}] = 0,$$

$$\delta[(p'+p_+)u^{(2)}]\delta[(p_- - p')u^{(2)}] = 0.$$

Indeed, since $uq$ is an invariant, we can use the coordinate system in which $u_\mu$ has only a fourth component, and in such a system

$$\delta[(p_- - p)u]\,\delta[(p+p_+)u] = \delta(\varepsilon_- - p_0)\,\delta(\varepsilon_+ + p_0) = \delta(\varepsilon_- + \varepsilon_+)\,\delta(\varepsilon_- - p_0) = 0,$$

where $\varepsilon_\pm$ and $p_0$ are the fourth components of the vectors $p_\pm$ and $p$.

Therefore, the substitution of (32.31) into (32.30) yields

$$dw = \frac{Z_1^2 Z_2^2}{(2\pi)^4}\,\frac{e^8}{\varepsilon_+ \varepsilon_-}\,\mathrm{Sp}\,G\,\frac{dp_+\,dp_-}{(2\pi)^4}, \tag{32.32}$$

where

$$\mathrm{Sp}\,G = \tfrac{1}{4}\mathrm{Sp}\,\{[\hat{u}^{(1)}(i\hat{I} - I_0)\,\hat{u}^{(2)} + \hat{u}^{(2)}(i\hat{I}' - I_0')\,\hat{u}^{(1)}]$$
$$\times (i\hat{p}_+ - m)\,[\hat{u}^{(1)}(i\hat{I}' - I_0')\hat{u}^{(2)} + \hat{u}^{(2)}(i\hat{I} - I_0)\,\hat{u}^{(1)}]\,(i\hat{p}_- - m)\},$$

$$I_\mu = \int d^4p\,\frac{p_\mu e^{-i(p+p_+)r_0}\,\delta[(p_- - p)u^{(1)})]\,\delta[(p+p_+)\,u^{(2)}]}{(p^2+m^2)\,(p_- - p)^2\,(p+p_+)^2},$$

$$I'_\mu = \int d^4p\,\frac{p_\mu e^{-i(p_- - p)r_0}\,\delta[(p_- - p)\,u^{(2)}]\,\delta[(p+p_+)\,u^{(1)}]}{(p^2+m^2)\,(p_- - p)^2\,(p+p_+)^2}.$$

In order to obtain the differential cross section for pair production we must integrate $dw$ over $2\pi r_0\,dr_0$. We give here only the result for the cross section integrated over the angles of emission of the electron and the positron (117) on the assumption that $m \ll \varepsilon_\pm \ll (m/\sqrt{1-v^2})$, where $v$ is the relative velocity of the colliding particles:

$$d\sigma = \frac{8}{\pi} r_0^2 (Z_1 Z_2 \alpha)^2 \frac{\varepsilon_+^2 + \varepsilon_-^2 + \dfrac{2}{3}\varepsilon_+\varepsilon_-}{(\varepsilon_+ + \varepsilon_-)^4} \ln \frac{\varepsilon_+ \varepsilon_-}{m(\varepsilon_+ + \varepsilon_-)}$$
$$\times \ln \frac{m}{(\varepsilon_+ + \varepsilon_-)\sqrt{1-v^2}}\,d\varepsilon_+\,d\varepsilon_-. \tag{32.33}$$

The total cross section is equal to

$$\sigma = \frac{28}{27}\,\frac{1}{\pi}(Z_1 Z_2 \alpha)^2 r_0^2 \left(\ln \frac{1}{\sqrt{1-v^2}}\right)^3. \tag{32.34}$$

This formula is valid if $\ln (1/\sqrt{1-v^2}) \gg 1$.

In § 34 we shall obtain this formula by utilizing the method of equivalent photons.

In the foregoing calculation we have replaced the effect of the colliding particles by an external field. Such a replacement is valid in the case of very fast particles whose state of motion is practically unaltered as a result of pair production. But in the general case the effect of the colliding particles cannot be reduced to that of an external field, and the process of pair production in a collision of two charged particles is shown not by the diagram of Fig. 37, but by the two diagrams of Fig. 38. In these diagrams the upper two solid lines correspond to the colliding particles, while the lower ones to the electron and the positron of the created pair.

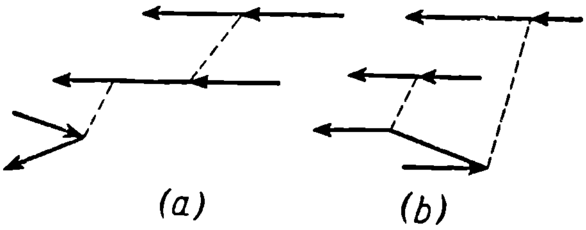


Fig. 38.

If a pair is formed in a collision of two electrons then in addition to the diagrams shown here we must also take into account the exchange diagrams (cf. subsection 29.10).

Thus, the production of a pair in a collision of two charged particles is, generally speaking, a fourth order process.

We shall not reproduce here the calculation of the cross section for pair production in a collision of two electrons, but shall confine ourselves to indicating the order of magnitude of the cross section. Since this process is a fourth order process, the cross section for the process amounts to approximately $\alpha = 1/137$ of the cross section for the emission of radiation in a collision of two electrons. If one of the electrons was at rest before the collision, then from the conservation laws it follows that the second electron must have an energy exceeding $7m$.

In the case when one of the colliding particles is fast, so that its state of motion is practically unaltered as a result of pair production, the effect of this particle may be replaced by an external field. In this case we may remove the upper line from the diagrams of Fig. 38, and the diagrams representing the process of pair production will assume the form shown in Fig. 39 (the external photon lines in this case correspond to an external field).

We see that in this case the pair production process is a third order effect. Such a case occurs, for example, in the case of pair production in the field of a nucleus. The effective cross section for pair production by an electron in the field of a nucleus in the extreme relativistic case $\varepsilon \gg m$ ($\varepsilon$ is the initial electron energy) is given in order of magnitude by the formula (23a, 143)

$$\sigma \approx r_0^2 Z^2 \frac{\alpha^2}{\pi}\left(\ln \frac{\varepsilon}{m}\right)^3, \tag{32.35}$$

where $Ze$ is the nuclear charge, $r_0$ is the classical electron radius. On setting $Z = 1$ in this formula we obtain the order of magnitude of the cross section for pair production in the collision of two relativistic electrons.

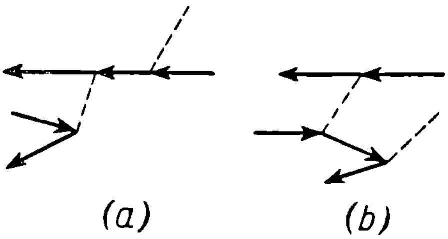

(a) (b)

Fig. 39.

Formula (32.35) actually corresponds to formula (32.34) if we set $Z_1 = 1$ in the latter, and assume that one of the particles is initially at rest.

We note that in comparison with the bremsstrahlung cross section the cross section (32.35) contains the coefficient $\alpha/4\pi$, but also a higher power of the logarithm (in formula (32.35) screening is not taken into account).

The diagrams of Fig. 39 also represent the process of pair production in a collision of two heavy particles moving at nonrelativistic velocities, since the nonrelativistic system of two particles is equivalent to a single particle situated in the external field. The pair production cross section is in this case approximately equal to (88)

$$\sigma \approx \alpha^2 r_0^2 \frac{m^2 Z_1^2 Z_2^2}{M_2 T_2}\left(\frac{Z_1 M_2 - Z_2 M_1}{M_1}\right)^2, \tag{32.36}$$

where $M_1$ and $M_2$ are the particle masses, $T_2$ is the kinetic energy of the particle of mass $M_2$, and it is assumed that the particle of mass $M_1$ is initially at rest, and that $T_2 \gg 2m$.

In conclusion we note that every process of radioactive decay with a sufficiently large release of energy may be accompanied by pair production. This phenomenon can be regarded as pair production by an external field with frequencies in excess of $2m$. In the disintegration of the nucleus into two parts the probability of pair production is determined by the expression (136):

$$w = \frac{2Z'\alpha}{9\pi^2} v^2 \ln\frac{2\varepsilon}{M}\left(\ln\frac{2\varepsilon}{m} - \frac{10}{3}\right), \tag{32.37}$$

where $Z' = Z_1(1 - Z_2A_2/Z_1A_1)$, $A_1$ and $A_2$ are the atomic mass numbers of the fragments, $Z_1e$, $Z_2e$ are their charges, $\varepsilon$ and $v$ are the energy and the velocity of the smaller fragment, $M$ is its mass, $m$ is the electron mass.

The probability of pair production in $\beta$-decay amounts to approximately $10^{-6}-10^{-7}$ (196).

## § 33. Annihilation of Electron-Positron Pairs into Photons

### 33.1. *Annihilation of a Pair into Two Photons*

The process of the production of electron-positron pairs by photons has its counterpart in the process of a pair disappearing and giving rise to photons. Such a process is known as pair annihilation. If the electron and the positron are free then annihilation with emission of a single photon is impossible since this process is not allowed by the laws of conservation of energy and momentum. For the annihilation of a free pair into photons at least two photons are needed, but if a positron collides with a bound electron then single photon annihilation becomes possible.

The most important process is the one in which the colliding free positron and free electron are converted into two photons. Therefore we start by investigating this process.

Two-photon pair annihilation corresponds to the two diagrams shown in Fig. 40. They coincide with the diagrams representing pair creation by two photons—a process which is the inverse of two-photon annihilation.

The matrix element which determined two-photon annihilation can be expressed in accordance with the rules of subsection 25.5 in the form

$$S^{(2)}_{i\to f} = \frac{ie^2}{2\sqrt{\omega\omega'}}\,\bar{v}(-p_+)\left\{\hat{e}\,\frac{i\hat{f}_1 - m}{m^2\varkappa_1}\,\hat{e}' + \hat{e}'\,\frac{i\hat{f}_2 - m}{m^2\varkappa_2}\,\hat{e}\right\} \times u(p_-)(2\pi)^4\,\delta(p_+ + p_- - k - k'), \quad (33.1)$$

where the same notation has been used as in (32.22).

The differential annihilation cross section is equal to

$$d\sigma = \frac{e^4}{4\omega\omega'}\,|\bar{v}Qu|^2\,\frac{dk\,dk'}{J(2\pi)^2}\,\delta(p_+ + p_- - k - k')\,\delta(\varepsilon_+ + \varepsilon_- - \omega - \omega'), \quad (33.2)$$

where $J$ is the flux density of the colliding positrons and electrons and

$$Q = \hat{e}\,\frac{i\hat{f}_1 - m}{m^2\varkappa_1}\,\hat{e}' + \hat{e}'\,\frac{i\hat{f}_2 - m}{m^2\varkappa_2}\,\hat{e}.$$

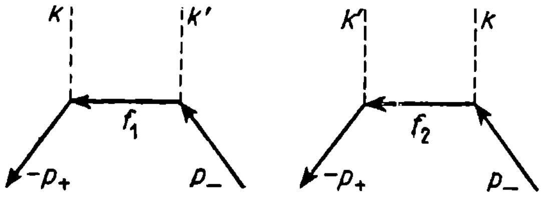


Fig. 40.

The structure of the matrix $Q$ is obviously identical with the structure of the corresponding matrix for the Compton effect: on carrying out in the latter the substitution $p_1 = p_-$, $k_1 = -k'$, $e_1 = e'$, $p_2 = -p_+$, $k_2 = k$, and $e_2 = e$, we obtain $Q$. Therefore, we can utilize the already available results of § 28 and write the following expression for the quantity $\overline{\sum|\bar{v}Qu|^2}$ summed and averaged over the particle polarizations:

$$\overline{\sum|\bar{v}Qu|^2} = -\frac{1}{2\varepsilon_-\varepsilon_+}\left\{4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2 - 4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right) - \left(\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right)\right\}. \quad (33.2')$$

It is convenient to carry out further calculations in the system associated with the center of mass of the pair in which $p_+ = -p_- = -p_0$, $\varepsilon_+ = \varepsilon_- = \varepsilon_0$, $k' = -k$, $\omega = \omega'$ and the flux density is equal to $J = 2(p_0/\varepsilon_0) = 2v_0$.

On eliminating the $\delta$-functions by integrating over $dk'$ and $d\omega$:

$$\delta(p_+ + p_- - k - k')\,dk' \to 1,$$

$$\delta(\varepsilon_+ + \varepsilon_- - \omega - \omega')\,dk \to \frac{\omega^2 do}{2},$$

where $do$ is the element of solid angle containing the photon momentum $\boldsymbol{k}$, we obtain

$$d\sigma = -\frac{r_0^2 m^2}{8p_0\varepsilon_0}\left\{4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2 - 4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right) - \left(\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right)\right\}do, \qquad (33.3)$$

where

$$m^2\varkappa_1 = -2p_+k = 2\boldsymbol{p}_0\boldsymbol{k} + 2\varepsilon_0\omega = 2\varepsilon_0(\varepsilon_0 + p_0\cos\theta),$$

$$m^2\varkappa_2 = -2p_-k = -2\boldsymbol{p}_0\boldsymbol{k} + 2\varepsilon_0\omega = 2\varepsilon_0(\varepsilon_0 - p_0\cos\theta)$$

and $\theta$ is the angle between $\boldsymbol{p}_0$ and $\boldsymbol{k}$.

Thus we have

$$d\sigma = r_0^2\frac{m^2}{4p_0\varepsilon_0}\left(\frac{\varepsilon_0^2+p_0^2+p_0^2\sin^2\theta}{\varepsilon_0^2-p_0^2\cos^2\theta} - \frac{2p_0^4\sin^4\theta}{(\varepsilon_0^2-p_0^2\cos^2\theta)^2}\right)\sin\theta\,d\theta\,d\varphi$$

$$= \frac{1}{4}r_0^2\frac{m^2}{v_0\varepsilon_0^2}\,[1+v_0^2\sin^2 2\theta - v_0^4(1-\sin^4\theta)]\frac{do}{(1-v_0^2\cos^2\theta)^2}. \qquad (33.3')$$

In order to obtain the total annihilation cross section we must integrate $d\sigma$ over $\theta$ from 0 to $\pi$ and over $\varphi$ from 0 to $\pi$ (these limits of integration correspond to two indistinguishable particles—photons in the final state). As a result of this we obtain

$$\sigma = \int\limits_{\theta=0}^{\theta=\pi}\int\limits_{\varphi=0}^{\varphi=\pi} d\sigma = \pi r_0^2\frac{m^2}{4v_0\varepsilon_0^2}\left\{\frac{3-v_0^4}{v_0}\ln\left|\frac{1+v_0}{1-v_0}\right| + 2(v_0^2-2)\right\}. \qquad (33.4)$$

Since the cross section is relativistically invariant, then in order to transform to another system it is sufficient to express $\varepsilon_0$ and $p_0$ in terms of the particle energies in that system.

We consider the especially interesting case when anihilation occurs as the result of a collision of a positron with an electron at rest.

The positron energy in the rest system of the electron is, obviously, equal to

$$\varepsilon_+^0 = \frac{\varepsilon_0 + v_0 p_0}{\sqrt{1-v_0^2}} = \frac{\varepsilon_0^2+p_0^2}{\sqrt{\varepsilon_0^2-p_0^2}} = \frac{2\varepsilon_0^2-m^2}{m^2},$$

where $v_0 = p_0/\varepsilon_0$ is the velocity of the electron rest with system respect to the center of mass system. From this we obtain

$$v_0 = \frac{p_0}{\varepsilon_0} = \sqrt{\frac{\varepsilon_+^0 - m}{\varepsilon_+^2 + m}}.$$

On substituting this value of $v_0$ into formula (33.4) we obtain the cross section for the annihilation of a pair into two photons in the electron rest system (48), (193):

$$\sigma = \frac{\pi r_0^2}{\gamma+1}\left\{\frac{\gamma^2+4\gamma+1}{\gamma^2-1}\ln(\gamma+\sqrt{\gamma^2-1}) - \frac{\gamma+3}{\sqrt{\gamma^2-1}}\right\}, \qquad (33.5)$$

where $\gamma = \varepsilon_+^0/m$.

For low positron energies the cross section is inversely proportional to the positron velocity:

$$\sigma = \frac{\pi r_0^2}{v_+}, \qquad (33.6)$$

while the annihilation probability $w$ is independent of the velocity:

$$w = Znv_+\sigma = Zn\pi r_0^2 \sec^{-1}, \qquad (33.6')$$

where $n$ is the number of atoms per unit volume. The lifetime of a slow positron is equal to $\tau \sim 1/w$; in the case of lead $\tau \sim 10^{-10}$ sec.

For large positron energies ($\varepsilon_+^0 \gg m$)

$$\sigma = \pi r_0^2 \frac{m}{\varepsilon_+^0}\left(\ln\frac{2\varepsilon_+^0}{m},\right). \qquad (33.7)$$

### 33.2 *Polarization Effects in the Two-Photon Annihilation of a Pair*

In the preceding subsection we have given the expressions determining the cross section for the two-photon annihilation of a pair averaged over the polarizations of the electrons and the photons. The annihilation cross section for polarized particles, and the polarization of the photons produced in this process can be investigated by the general methods presented in §§ 2, 10, 26 and applied in §§ 28 and 29. We confine ourselves here to stating some of the results.

1) *Correlation of photon polarizations.* The differential annihilation cross section for unpolarized particles with the production of two plane-polarized photons has the form

$$d\sigma = \frac{r_0^2}{8}\frac{m^2}{v_0\varepsilon_0^2}\left\{\frac{1}{1-v_0^2\cos^2\theta}\left[1-(1-v_0^2\cos^2\theta)(\boldsymbol{e}_1\boldsymbol{e}_2)^2\right.\right.$$

$$\left.\left.+4(\boldsymbol{v}_0\boldsymbol{e}_1)(\boldsymbol{v}_0\boldsymbol{e}_2)(\boldsymbol{e}_1\boldsymbol{e}_2)\right]-\frac{4(\boldsymbol{v}_0\boldsymbol{e}_1)^2(\boldsymbol{v}_0\boldsymbol{e}_2)^2}{(1-\boldsymbol{v}_0^2\cos^2\theta)^2}\right\}do, \qquad (33.8)$$

where $\varepsilon_0$ and $\boldsymbol{v}_0$ are the positron energy and velocity, and $\theta$ is the angle between the directions of the positron and photon momenta in the center of mass system.

At low velocities it follows from (33.8) that

$$d\sigma^{(1\ 1)} = d\sigma^{(2,2)} = 0, \qquad d\sigma^{(1,2)} = \frac{r_0^2}{8v_0}\,d\sigma, \qquad (33.9)$$

where $d\sigma^{(i,k)}$ is the cross section corresponding to the production of photons polarized in the direction $\boldsymbol{e}_i$, $\boldsymbol{e}_k$, with $\boldsymbol{e}_1$ lying in the plane $(\boldsymbol{p}_0, \boldsymbol{k})$, while $\boldsymbol{e}_2$ is perpendicular to this plane.

We see that for $v_0 \to 0$ photons are produced which are polarized perpendicular to each other. This result has a simple meaning. It follows from the law of conservation of parity. In the low-velocity limit annihilation occurs principally in an $S$-state which is odd. But a system of two photons is odd if their polarizations are mutually perpendicular.

2) *Annihilation of a longitudinally-polarized pair.* If the polarization vectors $\zeta^{(-)}$ for the electron and $\zeta^{(+)}$ for the positron have nonvanishing components only in the direction of their relative motion, the differential annihilation cross section has the form (147)

$$d\sigma = d\sigma_0(1+\zeta^{(+}{}^{(}\zeta^{)-)}f), \qquad (33.10)$$

where $d\sigma_0$ is the annihilation cross section for an unpolarized pair (cf. 33.3)) and

$$f = \frac{v_0(1-\cos^4\theta)-1+v_0^4-v_0^2(1-v_0^2)\sin^2\theta}{v_0^2(1-\cos^4\theta)+1-v_0^4+v_0^2(1-v_0^2)\sin^2\theta}$$

($v_0$ and $\theta$ are, as before, the velocity and the angle between the momenta of the positron and the photon in the center of mass system).

On integrating (33.10) over the angles we obtain

$$\sigma = \sigma_0(1+\zeta^{(-)}\zeta^{(+)}F), \qquad (33.11)$$

where $\sigma_0$ is the total annihilation cross section for unpolarized particles (cf. (33.5)) and

$$F = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2},$$

$$\sigma_1 = \frac{3+v_0^4}{2v_0^3} \ln \frac{1+v_0}{1-v_0} - \frac{3+v_0^2}{v_0^2},$$

$$\sigma_2 = \frac{1}{v_0^2}(3-3v_0^2+2v_0^4) + \frac{1-v_0^2}{2v_0^3}(2v_0^4+3v_0^2-3)\ln\frac{1+v_0}{1-v_0}.$$

Fig. 41 shows the dependence of $F$ on the energy in the laboratory system of coordinates (i.e., in the system in which the electron is at rest).

If only one of the particles undergoing annihilation has a longitudinal polarization equal to $\zeta$, then $d\sigma = d\sigma_0$; in this case the photons are circularly polarized:

$$\xi_2 = \zeta \frac{2v_0 - 2v_0^2 \dfrac{\varepsilon_0^2}{m^2} \sin^2\theta \cos\theta}{1+v_0^2+v_0^2 \dfrac{\varepsilon_0^2}{m^2} \sin^2\theta (1+\cos^2\theta) + v_0^2 \sin^2\theta}. \tag{33.12}$$

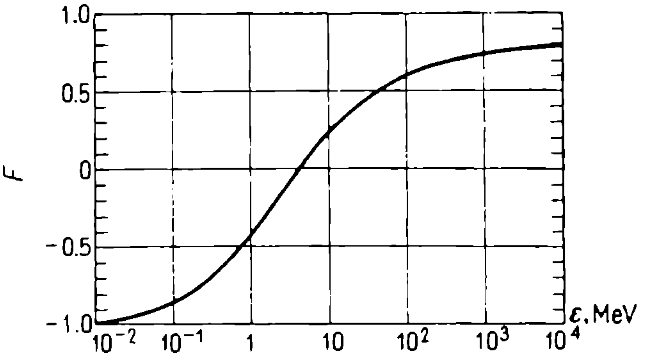


Fig. 41.

On integrating the numerator and the denominator of this expression separately with respect to $\theta$ between the limits $\pi/2$ and $\pi$, we obtain the circular polarization $x$ of that one of the two photons which has the greater energy in the laboratory system in the case of annihilation of polarized positrons in an unpolarized target. But if we integrate the numerator and the denominator of (33.12) with respect to $\theta$ between the limits 0 and $\pi/2$, we shall obtain the circular polarization of the photon, again having the greater energy in the laboratory system, in the case when annihilation of unpolarized positrons takes place in a polarized target. The quantities $x$ and $y$ are shown in Fig. 42 as functions of the kinetic energy of the positron in the laboratory system.

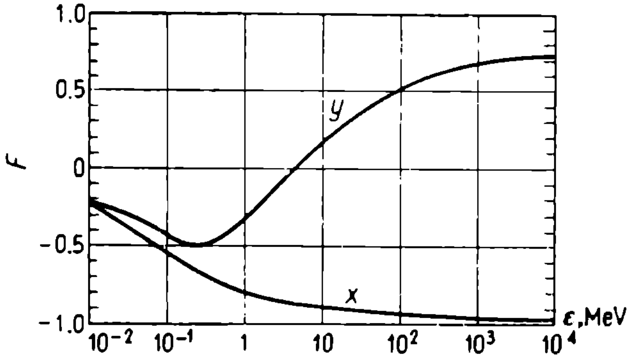


Fig. 42.

### 33.3 *Annihilation of a Pair into One Photon*

If a positron collides with an electron which is not free, both particles may be converted into a single photon (60), (92), (142), (18), (95). Such a process is determined by the first order matrix element

$$S^{(1)}_{i\to f} = -\frac{e}{\sqrt{2\omega}} \int \overline{\psi}_+ \hat{e}\, e^{ikx} \psi_- \, d^4x, \qquad (33.13)$$

where $\psi_-$ and $\psi_+$ are the electron and the positron wave functions.

If the electron is an atomic $K$-electron, then we have

$$\psi_- = \psi_k = \sqrt{\frac{1}{\pi a^3}}\, e^{-\frac{r}{a}} e^{-i\varepsilon_k t}, \qquad (33.14)$$

where $\psi_k$ is the wave function of the $K$-state, $\varepsilon_k$ is the energy of the $K$-electron and $a = \hbar^2/me^2Z$.

Insofar as the wave function of the positron is concerned, we must take for it the exact wave function belonging to the continuous spectrum in the Coulomb field of the nucleus. This function, as has been explained in subsection 29.1, must at large $r$ have the asymptotic form of the sum of a plane wave and an outgoing spherical wave.

On comparing the matrix element (33.13) with the matrix element for the photoeffect (31.1), it can be easily shown that the process of one-photon pair annihilation in which we are interested may be regarded as the inverse photoeffect for the $K$-electron accompanied by the emission of a photon of energy

$$\omega = m - I + \sqrt{p_+^2 + m^2}, \tag{33.15}$$

where $I$ is the ionization energy for the $K$-electron, and $p_+$ is the positron momentum.

In contrast to the photoeffect, the final state in this case has a negative energy and momentum $\boldsymbol{p} = -\boldsymbol{p}_+$. Moreover, the density of final states will now be given not by $p\varepsilon\, do_e/(2\pi)^3$, as in the case of the photoeffect, but by $\omega^2\, do_\gamma/(2\pi)^3$, where $do_e$ and $do_\gamma$ are the elements of solid angles containing the momenta of the electron and the photon; finally, the density of the incident particles will now be equal to the positron velocity $v_+$ instead of the velocity of light, as in the case of the photoeffect. On taking these changes into account and on carrying out in (31.16) the substitution $\gamma \to -\varepsilon_+/m$, $\omega \to \varepsilon_+ + m$, we obtain the following formula for the cross section for single-photon annihilation of a positron in the $K$-shell (normalized to two $K$-electrons):

$$\sigma = 4\pi r_0^2 \frac{Z^5}{(137)^4} \frac{m^3}{p_+(\varepsilon_+ + m)^2} \left[ \frac{\varepsilon_+^2}{m^2} + \frac{2}{3}\frac{\varepsilon_+}{m} + \frac{4}{3} + \frac{\varepsilon_+ + 2m}{p_+} \ln \frac{\varepsilon_+ + p_+}{m} \right]. \tag{33.16}$$

This formula, like formula (31.16) for the cross section for the photoeffect, is valid if $Z\alpha/v_+ \ll 1$.

In the nonrelativistic and the extreme relativistic cases formula (33.15) assumes the form

$$\begin{aligned} \sigma &= \frac{4\pi}{3} r_0^2 \frac{Z^5}{(137)^4} \frac{p_+}{m}, \quad \text{where} \quad v_+ \ll 1, \\ \sigma &= 4\pi r_0^2 \frac{Z^5}{(137)^4} \frac{m}{\varepsilon_+}, \quad \text{where} \quad \varepsilon_+ \gg m. \end{aligned} \tag{33.17}$$

We see that the cross section for single-photon annihilation of a positron is proportional at low energies to the positron velocity, in contrast to the cross section for two-photon annihilation, which is inversely proportional to $v_+$. Therefore, at low positron energies single-photon annihilation is much less probable than two-photon annihilation. The ratio of the cross sections for single-photon and two-photon annihilation has a maximum at $\varepsilon_+ \sim 10m$, and has the value of approximately 0.2 for Pb.

At low positron velocities and large values of $Z$ formula (33.15) ceases to be applicable.

### 33.4 *Positronium Decay*

The foregoing results referring to the annihilation of a pair into two photons can be applied to the problem of the decay of positronium, i.e., a bound system consisting of an electron and a positron.

Obviously, the ground state of positronium is an $S$-state. Since the spins of the electron and the positron are each equal to $\frac{1}{2}$, we must distinguish between the singlet $S$-state, in which the total spin of the two particles is equal to zero, and the triplet state in which their total spin is equal to unity. We denote these states respectively by $^1S$ and $^3S$. Positronium in the $^1S$ state is called parapositronium, while positronium in the $^3S$ state is called orthopositronium.

It was shown in § 7 that a system consisting of two photons cannot have states of angular momentum equal to unity. Therefore, orthopositronium cannot decay into two photons. Moreover, since the $^3S$ state of positronium is charge-odd, we can assert that, generally speaking, the decay of orthopositronium into any even number of photons is impossible.

On the other hand, parapositronium is a charge-even system and, therefore, it cannot decay into any arbitrary odd number of photons.

The main processes which determine the positronium lifetime are two-photon annihilation in the case of parapositronium, and three-photon annihilation in the case of orthopositronium.

The probability of positronium decay can be related to the probability of annihilation of a free pair. In order to do this we consider the wave function for the $S$-state of positronium

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}, \tag{33.18}$$

where $a$ is the positronium radius equal to twice the radius $a_0$ of the first Bohr orbit of hydrogen: $a = 2a_0 = (2\hbar^2/me^2)$ (the doubling of the hydrogen radius is associated with the fact that the reduced mass of positronium is equal to the electron mass divided by two). The Fourier component of the wave function $\psi(\mathrm{r})$

$$\psi_p = \int \psi(r) e^{-ipr} dr = \frac{8\sqrt{\pi a^3}}{(1+a^2p^2)^2} \tag{33.18'}$$

obviously determines the probability amplitude for the electron in the positronium having the momentum $\boldsymbol{p}$, while the positron has the momentum $-\boldsymbol{p}$. It follows from (33.18′) that the wave function $\psi$ $(r)$ involves momenta primarily of order of magnitude $p \sim 1/a$ which are considerably smaller than $m$. Therefore, the decay of positronium can be treated approximately as the annihilation of a free positron and a free electron of momenta equal to zero, but of definite spin orientations.

We denote by $w_1$ and $w_0$ the probabilities of decay of ortho and parapositronium. Further, we denote by $\bar{w}$ the probability of decay of a free pair of particles with zero momenta averaged over the orientations of their spins, and evaluated on the assumption that the particle density is equal to the particle density in positronium, i.e., $|\psi(0)|^2 = 1/\pi a^3$ (and not $1/\Omega$, where $\Omega$ is the normalizing volume, as is assumed in evaluating the annihilation probability for a free pair).

Since the relative weights of states of spins 1 and 0 are respectively equal to 3/4 and 1/4, we obviously have

$$\bar{w} = \tfrac{3}{4} w_1 + \tfrac{1}{4} w_0. \tag{33.19}$$

The quantity $\bar{w}$ can be written in the form

$$\bar{w} = \bar{w}_{2_\gamma} + \bar{w}_{3_\gamma} + \ldots, \tag{33.20}$$

where $\bar{w}_{n_\gamma}$ is the probability of the annihilation of a pair into $n$ photons, averaged over all spin orientations. Since parapositronium can decay only into an even, and orthopositronium only into an odd number of photons, it follows from (33.19) that

$$\tfrac{1}{4} w_0 = \bar{w}_{2_\gamma} + \bar{w}_{4_\gamma} + \ldots, \qquad \tfrac{3}{4} w_1 = \bar{w}_3 + \bar{w}_5 + \ldots,$$

i.e.,

$$w_0 \approx 4\bar{w}_{2_\gamma}, \qquad w_1 \approx \tfrac{4}{3} w_{3_\gamma}. \tag{33.21}$$

By utilizing formula (33.6) for the cross section for two-photon pair annihilation for low positron velocities we obtain from the preceding result:

$$w_0 = 4|\psi(0)|^2 (v_+ \sigma)_{v^+ \to 0} = \tfrac{1}{2}\alpha^5 m = 0.8 \cdot 10^{10} \mathrm{sec}^{-1}. \tag{33.22}$$

### 33.5 *Three-Photon Decay of Orthopositronium*

We now proceed to the determination of the probability of three-photon decay of orthopositronium (146).

Three-photon pair annihilation is a third order perturbation theory process. It corresponds to the six diagrams shown in Fig. 43. The matrix element for this process can, in accordance with the rules of subsection 25.5, be written in the form

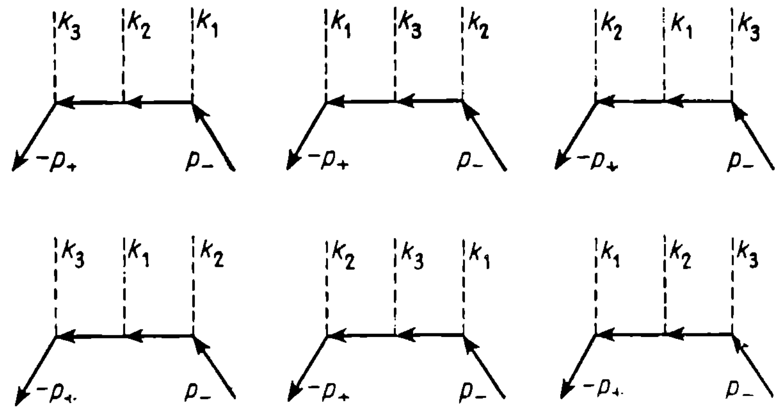


Fig. 43.

$$S^{(3)}_{i\to f}=\frac{e^3}{\sqrt{8\omega_1\omega_2\omega_3}}\bar{v}\left\{\hat{e}_3\frac{i\hat{f}_1'-m}{m^2\varkappa_1'}\hat{e}_2\frac{i\hat{f}_1-m}{m^2\varkappa_1}\hat{e}_1+\hat{e}_1\frac{i\hat{f}_2'-m}{m^2\varkappa_2'}\hat{e}_3\frac{i\hat{f}_2-m}{m^2\varkappa_2}\hat{e}_2\right.$$

$$+\hat{e}_2\frac{i\hat{f}_3'-m}{m^2\varkappa_3'}\hat{e}_1\frac{i\hat{f}_3-m}{m^2\varkappa_3}\hat{e}_3+\hat{e}_3\frac{i\hat{f}_4'-m}{m^2\varkappa_4'}\hat{e}_1\frac{i\hat{f}_4-m}{m^2\varkappa_4}\hat{e}_2+\hat{e}_2\frac{i\hat{f}_5'-m}{m^2\varkappa_5'}\hat{e}_3\frac{i\hat{f}_5-m}{m^2\varkappa_5}\hat{e}_1$$

$$\left.+\hat{e}_1\frac{i\hat{f}_6'-m}{m^2\varkappa_6'}\hat{e}_2\frac{i\hat{f}_6-m}{m^2\varkappa_6}\hat{e}_3\right\}u(2\pi)^4\,\delta(p_++p_--k_1-k_2-k_3),\quad(33.23)$$

where $k_i$ and $\omega_i$ are the momenta and the frequencies of the photons,

$$f_1=f_5=p_--k_1,\quad f_1'=f_4'=k_3-p_+,$$

$$f_2=f_4=p_--k_2,\quad f_2'=f_6'=k_1-p_+,$$

$$f_3=f_6=p_--k_3,\quad f_3'=f_5'=k_2-p_+,$$

$$\varkappa_\mu=\frac{f_\mu^2+m^2}{m^2},\quad \varkappa_\mu'=\frac{f_\mu'^2+m^2}{m^2},\quad \mu=1,2,3,4,5,6.$$

In the region of vanishingly small particle velocities $|p_+| = |p_-| = 0$, $f_1' = -f_3$, $f_2' = -f_1$, $f_3' = -f_2$ and the differential probability for three-photon pair annihilation is in this case equal to

$$\overline{dw}_{3\gamma} = \frac{e^6}{8\omega_1\omega_2\omega_3}\,\frac{dk_1\,dk_2\,dk_3}{(2\pi)^5\Omega}\,|\bar{v}Qu|\;\delta(k_1+k_2+k_3)\,\delta(\omega_1+\omega_2+\omega_3-2m), \tag{33.24}$$

where $1/\Omega = |\psi(0)|^2 = 1/\pi a^3$ and

$$\begin{aligned} Q = {} & \hat{e}_1\frac{i\hat{f}_2'-m}{m^2\varkappa_2'}\left\{\hat{e}_3\frac{i\hat{f}_2-m}{m^2\varkappa_2}\hat{e}_2+\hat{e}_2\frac{i\hat{f}_6-m}{m^2\varkappa_6}\hat{e}_3\right\} \\ & +\hat{e}_2\frac{i\hat{f}_3'-m}{m^2\varkappa_3'}\left\{\hat{e}_1\frac{i\hat{f}_3'-m}{m^2\varkappa_3}\hat{e}_3+\hat{e}_3\frac{i\hat{f}_5-m}{m^2\varkappa_5}\hat{e}_1\right\} \\ & +\hat{e}_3\frac{i\hat{f}_1'-m}{m^2\varkappa_1'}\left\{\hat{e}_2\frac{i\hat{f}_1-m}{m^2\varkappa_1}\hat{e}_1+\hat{e}_1\frac{i\hat{f}_4-m}{m^2\varkappa_4}\hat{e}_2\right\} \\ = {} & -\sum_{\mu}\hat{e}_\mu\frac{i\hat{f}_\mu+m}{m^2\varkappa_\mu}\left\{\hat{e}_{\mu-1}\frac{i\hat{f}_{\mu-2}-m}{m^2\varkappa_{\mu-2}}\hat{e}_{\mu+1}+\hat{e}_{\mu+1}\frac{i\hat{f}_{\mu-1}-m}{m^2\varkappa_{\mu-1}}\hat{e}_{\mu-1}\right\}, \end{aligned}$$

$$\mu = 1, 2, 3.$$

$$(e_0 = e_3, \quad e_4 = e_1, \quad f_{-1} = f_2, \quad f_0 = f_3).$$

The quantity $|\bar{v}Qu|^2$ averaged over the spins of the electron and the positron, and summed over the photon polarizations, can be written in the form

$$\frac{1}{4}\sum_{\mu_1,\mu_2}\sum_{\nu_1,\nu_2,\nu_3}|\bar{v}Qu|^2 = \frac{1}{16m^2}\,Sp\,F,$$

where

$$\begin{aligned} F = \sum_{\mu}\sum_{\mu'}\gamma_\mu\frac{i\hat{f}_\mu+m}{m^2\varkappa_\mu} & \left\{\gamma_{\mu-1}\frac{i\hat{f}_{\mu-2}-m}{m^2\varkappa_{\mu-2}}\gamma_{\mu+1}\right. \\ & \left.+\gamma_{\mu+1}\frac{i\hat{f}_{\mu-1}-m}{m^2\varkappa_{\mu-1}}\gamma_{\mu-1}\right\}(i\hat{p}-m)\left\{\gamma_{\mu'+1}\frac{i\hat{f}'_{\mu-2}-m}{m^2\varkappa_{\mu'-2}}\gamma_{\mu'-1}\right. \\ & \left.+\gamma_{\mu'-1}\frac{i\hat{f}_{\mu'-1}-m}{m^2\varkappa_{\mu'-1}}\gamma_{\mu'+1}\right\}\frac{i\hat{f}_{\mu'}+m}{m^2\varkappa_{\mu'}}\gamma_{\mu'}(i\hat{p}+m). \end{aligned}$$

$$(\mu,\ \mu' = 1,\ 2,\ 3).$$

We shall not reproduce here the calculations of $\mathrm{Sp} F$, but shall directly give the results for the differential probability for three-photon pair annihilation:

$$dw_{3\gamma} = \frac{\alpha^3}{4\omega_1\omega_2\omega_3}\,\frac{d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3}{(2\pi)^3 m^4 a_0^3}\,\{(1-\boldsymbol{n}_1\boldsymbol{n}_2)^2+(1-\boldsymbol{n}_2\boldsymbol{n}_3)^2+(1-\boldsymbol{n}_1\boldsymbol{n}_3)^3\}$$
$$\times\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,\delta(\omega_1+\omega_2+\omega_3-2m), \qquad (33.25)$$

where $\boldsymbol{n}_i$ is a unit vector in the direction of the photon momentum $\boldsymbol{k}_i$.

The total probability of three-photon pair annihilation is equal to

$$\bar{w}_{3\gamma} = \frac{1}{6}\,\frac{3\alpha^3}{8\pi m^4 a_0^3}\int_0^\infty d\omega_1\int_0^\infty d\omega_2\int_0^\pi \sin\vartheta_{12}\,d\vartheta_{12}\,\frac{\omega_1\omega_2}{\omega_3}$$
$$\times\,(1-\cos\vartheta_{12})^2\,\delta(\omega_1+\omega_2+\omega_3-2m), \qquad (33.26)$$

where $\vartheta_{12}$ is the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ (here the factor 1/6 takes into account identical states which differ by an interchange of photon momenta). We now evaluate the integral appearing in (33.26)

$$K = \int_0^\infty d\omega_1\int_0^\infty d\omega_2\int_{-1}^1 \frac{\omega_1\omega_2}{\omega_3}\,(1-x)^2\,dx\,\delta(\omega_1+\omega_2+\omega_3-2m),$$

where $x = \cos\vartheta_{12}$.

In eliminating the $\delta$-function by integration over $x$ we must keep in mind that $\omega_3$ is a function of $x$ so that $\omega_3 = \sqrt{\omega_1^2+\omega_2^2+2\omega_1\omega_2 x}$. Therefore

$$\int_{-1}^1 f(x)\,\delta[\omega(x)+\omega_1+\omega_2-2m]\,dx = \frac{f(x_0)}{|d\omega_3(x)/dx|_{x=x_0}},$$

where $x_0$ is the root of the equation $\omega_3(x) = 2m-\omega_1-\omega_2$:

$$x_0 = 1-\frac{2m}{\omega_1\omega_2}(\omega_1+\omega_2-m).$$

Further, on noting that $(d\omega_3/dx) = (\omega_1\omega_2/\omega_3)$ and that the absolute value of the quantity $x_0$ cannot be less than unity we obtain

$$K = \int_0^m d\omega_1\int_{m-\omega_1}^m (\omega_1+\omega_2-m)^2\,\frac{d\omega_2}{\omega_1^2\omega_2^2}.$$

On integrating over $\omega_2$ we obtain the spectrum of the decay photons:

$$\bar{w}_{3\gamma} = \int_0^m F(\omega_1)\,d\omega_1, \tag{33.27}$$

where

$$F(\omega_1) = \frac{4}{9\pi}\left(\frac{r_0}{a_0}\right)^3 \left\{\frac{\omega_1(m-\omega_1)}{(2m-\omega_1)^2} - 2m\frac{(m-\omega_1)^2}{(2m-\omega_1)^3}\ln\frac{m-\omega_1}{m} + \frac{2m-\omega_1}{\omega_1} + \frac{2(m-\omega_1)m}{\omega_1^2}\ln\frac{m-\omega_1}{m}\right\}.$$

The shape of this function is shown in Fig. 44.

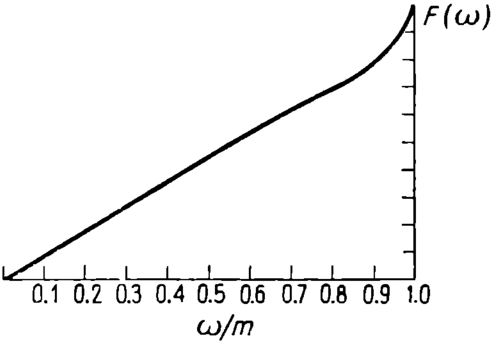


Fig. 44.

On carrying out the integration over $\omega_1$ we obtain the total probability of three-photon pair annihilation

$$\bar{w}_{3\gamma} = \frac{1}{6\pi}(\pi^2 - 9)\alpha\frac{r_0^2}{a_0^3}. \tag{33.27'}$$

Further, by utilizing expressions (33.21) we obtain the probability for the decay of orthopositronium

$$w_1 = \frac{2}{9\pi}(\pi^2 - 9)\alpha\frac{r_0^2}{a_0^3} = 0.71 \cdot 10^7 \text{ sec}^{-1}. \tag{33.28}$$

### 33.6. *Multiple Photon Production Accompanying the Annihilation of a Pair*

In the preceding subsections we have determined the cross section for the annihilation of a pair into two photons. This is the smallest number of photons which can be formed in the annihilation of a free electron-positron pair. The annihilation of a pair into a larger number of photons, having an energy of the same order of magnitude, will be considerably less probable than the annihilation of a pair into two photons.

However, if in addition to two photons sharing almost the total energy of the pair there is also emitted a number of long wavelength photons whose total energy is small compared to the energy of the first two photons, then the probability of such a process may be comparable to the probability of two-photon annihilation.

We are here dealing with a process analogous to the radiation by an electron of long wavelength photons for which, as has been shown in § 30, the ratio of the probability of radiation of two photons to the probability of radiation of a single photon is determined not by the constant $\alpha$ but by the quantity $\xi = \alpha \ln(\varepsilon/\omega)$, where $\omega$ is the photon energy, while $\varepsilon$ is the electron energy. Since for $\omega \ll \varepsilon$ this quantity is not small, the probability of emission of a number of long wavelength photons can be comparable with the probability of emission of a single photon.

For the same reasons, in the case of pair annihilation the probability of the annihilation of a pair into two hard photons and a number of soft photons can be comparable with the probability of the fundamental process—the annihilation of a pair into two hard photons.

We now give an estimate of the order of magnitude of the probability for such a multiple process of photon production aceompanying pair annihilation (97).

As a preliminary remark we recall that in accordance with (30.15) the matrix element for the emission by an electron of a long wavelength photon is determined by

$$S^{(2)}_{i\to f} = \frac{-e}{\sqrt{2\omega}} S^{(1)}_{i\to f}\left(\frac{p_2 e}{p_2 k} - \frac{p_1 e}{p_1 k}\right),$$

where $S^{(1)}_{i\to f}$ is the matrix element for the fundamental process (the scattering of an electron by an external field), $p_1$ and $p_2$ are the initial and the final momenta of the electron, $k$ is the momentum and $e$ is the polarization of the photon.

On repeating the arguments leading to (30.15) we can easily show that in the case of pair annihilation in which we are interested, the matrix element $M_{n+2}$ for the annihilation of a pair into two hard and $n$ soft photons is related to the matrix element for the two-photon annihilation $M_2$ by

$$M_{n+2}=\left(\prod_s \frac{e_s}{\sqrt{2\omega_s}}\left(\frac{e_s p_+}{p_+ q_s}-\frac{e_s p_-}{p_- q_s}\right)\right)M_2, \tag{33.29}$$

where $q_s$, $\omega_s$, $e_s$ are the four-momentum, the energy, and the polarization of the soft photon and $p_+$ and $p_-$ are the positron and electron momenta.

By utilizing this expression we can obtain the cross section for the annihilation of a pair into $n+2$ photons, which turns out to be given by

$$\sigma_{n+2}=\sigma_2\frac{\left(\alpha C \ln \frac{\varepsilon_2}{\varepsilon_1}\right)^n}{n!}, \tag{33.30}$$

where $\sigma_2$ is the cross section for two-photon annihilation, $m\varepsilon_2$, $m\varepsilon_1$ are the upper and the lower limits between which lie the energies of the soft photon, and $C \approx (2/\pi)(\ln 2\gamma_+ - 1)$. ($\gamma_+ m$ is the positron energy in the laboratory system. The factor $1/n!$ arises due to the indistinguishability of the soft photons.)

The quantity $\varepsilon_1$ is determined by the resolving power of the apparatus, while the quantity $\varepsilon_2$ may be roughly estimated as $\varepsilon_2 \sim \gamma_+$. With such an estimate we obtain

$$\sigma_n=\sigma_2\frac{1}{(n-2)!}\left[\frac{2\alpha}{\pi}\ln\frac{\gamma_+}{\varepsilon_1}(\ln 2\gamma_+ - 1)\right]^{n-2}$$
$$\left(\gamma_+ \gg 1, \ln\frac{\gamma_+}{\varepsilon_1} \gg 1\right). \tag{33.30'}$$

In accordance with the results of § 30 this formula shows that the emission of soft photons obeys the Poisson distribution. The average number of soft photons emitted is evidently equal to

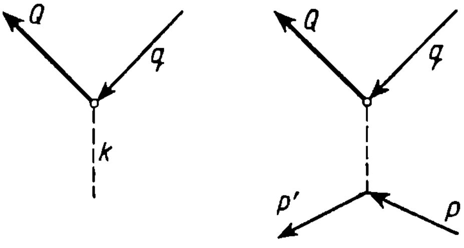
$$\bar{n}=\frac{2\alpha}{\pi}\ln\frac{\gamma_+}{\varepsilon_1}(\ln 2\gamma_+ - 1). \tag{33.31}$$

## § 34. The Method of Equivalent Photons

### 34.1. *The Number of Equivalent Photons*

We have seen earlier that the matrix elements for bremsstrahlung and for Compton scattering have a similar structure. However, a deeper connection exists between these processes due to the fact that the electromagnetic field of a fast uniformly moving charged particle has properties which are very close to those of the field of a light wave.

We compare the two processes represented by the diagrams of Fig. 45. The diagram of Fig. 45*a* corresponds to the collision of particle of momentum $q$ (mass $m$) with a photon (momentum $k$), as a result of which a set of particles of momentum $Q$ is produced. The diagram of Fig. 45*b* corresponds to the process of the scattering of a particle $q$ by another particle (momentum $p$, mass $M$), as a result of which the same set of particles $Q$ is produced while the particle acquires the momentum $p'$. The second process is equivalent to the collision of the particle $q$ with a virtual photon. If the square of the photon momentum $k^2$ is small, so that the difference between the virtual and the real photons is small, the cross section for the process ($b$) can be expressed in terms of the cross section for the process ($a$).



(a) (b)

Fig. 45.

On denoting the invariant amplitudes for these processes respectively by $A_r$ and $A$ we can, in accordance with (26.6), write the differential cross section which we denote by $d\sigma_r$ and $d\sigma$ in the form

$$d\sigma_r = |A_r|^2 \frac{m^2}{|kq|} d\varrho_Q$$

$$d\sigma = |A|^2 \frac{M^2 m^2}{\sqrt{(pq)^2 - m^2 M^2}} d\varrho_Q \frac{d^4 p' \delta(p'^2 + M^2)}{(2\pi)^3}$$

where

$$d\varrho_Q = (2\pi)^4 \prod_{a=1}^{N} \frac{1}{(2\pi)^3} d^4 q_a \delta(q_a^2 m_a^2),$$

$N$ is the number of particles in the set $Q$, $q_a$ are the momenta, $m_a$ are the masses of the particles produced. We now show that under certain conditions the amplitude $A$ can be expressed in terms of $A_\gamma$ (180).

From the diagrams of Fig. 45 it follows that $A_\gamma = e_\mu J_\mu(0)$ and $A = j_\mu J_\mu(k^2)$. Here $e_\mu$ is the photon polarization vector, $J_\mu(k^2)$ describes the set $q+k \to Q$, with the same vector $J_\mu$ appearing in the expressions for both $A_\gamma$ and $A$, but taken for different values of $k^2$ respectively equal to 0 and $(p-p')^2$. $j_\mu$ describes the vertex for the process of emission of a photon by the particle $p$ $(p \to p'+k)$; if the particle $p$ is an electron, then $j_\mu = \bar{u}(p')\gamma_\mu u(p)$.

The quantities $j_\mu$ and $J_\mu$ satisfy the condition of transversality $j_\mu k_\mu = 0$ and $J_\mu k_\mu = 0$. We consider the process in the rest system of the particle $q$ $(\boldsymbol{q} = 0)$.

If the velocity of the particle $p$ is large $(|\boldsymbol{p}| \gg M)$ and $p'$ differs but little from $p$, then we have $j_\mu = 2Zeu_\mu$, where $Ze$ is the charge of the particle, while $u_\mu = p_\mu/M$ is the velocity four-vector

$$u_0 = \frac{1}{\sqrt{1-v^2}}, \quad \boldsymbol{u} = \frac{\boldsymbol{v}}{\sqrt{1-v^2}} \quad \left(\frac{1}{\sqrt{1-v^2}} \gg 1\right).$$

From the transversality conditions we can express the frequency of the virtual photon $\omega$ in terms of its momentum $\boldsymbol{k}$: $\omega = \boldsymbol{kv} = k_{\|} v$. Also $k^2 = (1-v^2)\,\omega^2 + k_\perp^2$ and $\boldsymbol{k}_\perp = \boldsymbol{k} - \boldsymbol{k}_{\|}$.

We require that $|k^2|$ be small, $|k^2| \ll m^2$. Then we have $k_\perp \ll m$ and $\omega \ll (m/\sqrt{1-v^2})$.

We see that under these conditions the case is possible for which

$$k_\perp \sim \omega\sqrt{1-v^2} \ll \omega, \qquad k_\perp \gg \omega(1-v^2), \tag{34.1}$$

and which we now discuss. On expressing by means of the transversality conditions $J_0$ in terms of $\boldsymbol{J}$, $J_0 = (1/\omega)\,\boldsymbol{Jk}$, we obtain

$$J_\mu j_\mu = \frac{2Ze}{\omega\sqrt{1-v^2}} \left(J_\perp k_\perp + J_{\|}\omega(1-v^2)\right).$$

On taking (34.1) into account we can neglect the second term in this expression. Moreover, since $k_\perp \ll k_\parallel$, $J_\perp$ differs from $\boldsymbol{J}\boldsymbol{e}$, where $\boldsymbol{e}$ is the unit vector perpendicular to $\boldsymbol{k}$ ($\boldsymbol{e}\boldsymbol{k} = 0$), by a quantity of the order of magnitude $\sqrt{1-v^2}$. Thus, for small $k^2$ we have

$$J_\mu j_\mu = \frac{2Ze}{\sqrt{1-v^2}} \frac{k_\perp}{\omega} \boldsymbol{J}(0)\boldsymbol{e}.$$

On substituting this into the expression for $d\sigma$ and on comparing it with the expression for $d\sigma_r$, we obtain

$$d\sigma = d\sigma_r = \frac{4Z^2e^2k_\perp^2 M}{\sqrt{1-v^2}\,\omega k^4} \frac{d^4p'\delta(p'^2+M^2)}{(2\pi)^3}. \tag{34.2}$$

Thus, we express $d\sigma$ in terms of $d\sigma_r$. Formula (34.2) is valid if the conditions (34.1) are satisfied.

We can express the quantities appearing in (34.2) solely in terms of invariants characterizing the process. In order to do this we utilize the relations

$$k^2 = k_\perp^2 + (1-v^2)\omega^2,$$

$$Q^2 = (q+k)^2 = k^2 - m^2 - 2\omega m; \qquad \frac{1}{\sqrt{1-v^2}} = \frac{E}{M} = -\frac{pq}{Mm},$$

$$\int d^4p'\delta(p'^2+m^2) = \frac{\pi}{4\sqrt{(pq)^2 - M^2m^2}} \int dk^2 dQ^2$$

(the last relation can be easily obtained, for example, by going over to the coordinate system in which $\boldsymbol{p} = 0$) and obtain

$$d\sigma = d\sigma_r \frac{2Z^2\alpha}{\pi} \frac{k^2 - [(Q^2+m^2)^2 M^2/4(qp)^2]}{|Q^2+m^2|\,k^4}\, dk^2\, dQ^2. \tag{34.3}$$

The conditions for the validity of this formula are

$$k^2 \ll m^2; \qquad |Q^2| \ll |qp|; \qquad |Q^2+m^2| \gg k^2.$$

We obtain a formula which is simple and convenient for applications by choosing for our variables $k_\perp$ and $\omega$. We then have

$$dQ^2 dk^2 = 4mk_\perp d\omega dk_\perp$$

and

$$d\sigma = d\sigma_r \frac{2Z^2\alpha}{\pi} \frac{d\omega}{\omega} \frac{k_\perp^3 \, dk_\perp}{[k_\perp^2 + \omega^2(1-v^2)]^2}. \tag{34.4}$$

We denote by $d\sigma'$ the cross section for the process integrated over all $\omega$ and $k_\perp$ (i.e., over the recoil momenta $\boldsymbol{p}'$). It can be written as

$$d\sigma' = \int n(\omega) \, d\sigma_r(\omega),$$

where

$$n(\omega) d\omega = \frac{Z^2\alpha}{\pi} \frac{d\omega}{\omega} \int \frac{k_\perp^3 \, dk_\perp}{[k_\perp^2 + \omega^2(1-v^2)]^2} \tag{34.5}$$

We can interpret this formula in the following manner. In order to obtain the cross section of some process due to a fast particle acting on a particle at rest ($\boldsymbol{q} = 0$ in our coordinate system) we can replace the fast particle by the spectrum of equivalent photons $n(\omega)d\omega$, and if we know the cross section for the process due to photons, it is then sufficient to multiply it by the number of equivalent photons in the frequency interval $d\omega$ and to integrate over the frequencies. This method is referred to as the method of equivalent photons, or the Weizsäcker-Williams method (206), (212). The underlying idea was originally proposed and applied by Fermi (59).

The equivalent photon spectrum can also be obtained from a classical investigation of the electromagnetic field of a rapidly moving particle.

The four-potential of the field produced by a uniformly moving particle is determined by

$$\Box A = -\frac{4\pi}{c} e\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{v}t), \qquad \Box A_0 = -4\pi e\delta(\boldsymbol{r}-\boldsymbol{v}t),$$

where $\boldsymbol{v}$ is the velocity of the particle (we employ both here and in subsequent formulas of this subsection the usual CGS system of units). On expanding the potentials into Fourier integrals we obtain the following expressions for the fields (118):

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\frac{ie}{2\pi^2} \int \frac{\boldsymbol{k} - \dfrac{(\boldsymbol{k}\boldsymbol{v})\boldsymbol{v}}{c^2}}{k_y^2 + k_z^2 + k_x^2\left(1 - \dfrac{v^2}{c^2}\right)} e^{i[k_y y + k_z z + k_x(x - vt)]} \, d\boldsymbol{k}, \tag{34·6}$$

$$H(r, t) = \left[\frac{v}{c}, E\right],$$

where the $x$-axis is directed parallel to $v$.

On making the substitution $k_x \to \xi k_x$, $k_y \to \xi k_y$, $k_z \to \xi k_z$, where $\xi = 1\sqrt{1-(v^2/c^2)}$, we rewrite (34.6) as

$$E_\perp(r, t) = -\frac{ie\xi}{2\pi^2}\int \frac{k_\perp}{k^2} e^{i[k_\perp r + k_1 \xi(x-vt)]}\, dk,$$

$$E_\parallel(r, t) = -\frac{ie}{2\pi^2}\int \frac{k_\parallel}{k^2} e^{i[k_\perp r + k_1 \xi(x-vt)]}\, dk, \tag{34.6'}$$

$$H(r, t) = \left[\frac{v}{c}, E\right],$$

where

$$k_\perp = k - \frac{v(kv)}{v^2}, \qquad k_\parallel = k - k_\perp$$

and $E_\perp$ and $E_\parallel$ denote the perpendicular and the parallel components of $E$ (with respect to the direction of the velocity $v$) $k_1 = |k_\parallel|$.

We assume that $\xi \gg 1$. In this case, obviously, $|E_\perp| \gg |E_\parallel|$, i.e., the field remains practically transverse, just as in the case of a light wave. The formula (34.6′) determine the expansion of this field into plane monochromatic waves of frequencies $\omega = kv_1\xi = k_1 c\xi$ which are propagated in the direction of $v$.

We now determine the number of equivalent photons corresponding to these waves. In order to do this we obtain the total flux of electromagnetic energy in the direction of $v$:

$$S = \int df \int_{-\infty}^{\infty} \frac{c}{4\pi}[E, H]_x\, dt,$$

where $df = dy\,dz = 2\pi b\, db$ is an element of area in the $yz$ plane, $b = \sqrt{y^2+z^2}$ (this quantity is known as the impact parameter). On equating the total flux $S$ to the energy of equivalent photons

$$S = \int_0^{\infty} \hbar\omega n(\omega)\, d\omega,$$

we obtain the number of photons $n(\omega)\, d\omega$ in the frequency interval $\omega$ to $\omega + d\omega$.

On substituting (34.6′) into the expression for $S$ we obtain

$$S = \frac{c}{4\pi}\int df \int\limits_{-\infty}^{\infty} dt(E_y^2+E_z^2)$$

$$= \frac{c}{4\pi}\,\frac{Z^2e^2\xi^2}{4\pi^4}\int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} dz \int\limits_{-\infty}^{\infty} dt \int d\mathbf{k}\, d\mathbf{k}'\,\frac{(-k_yk_y'-k_zk_z')}{\mathbf{k}^2\mathbf{k}'^2}$$

$$\times e^{iy(k_y+k_y')+iz(k_z+k_z')+i\xi(x-vt)(k_x+k_x')}$$

$$= \frac{c}{2\pi^2}\,\frac{Z^2e^2\xi}{v}\int \frac{k_y^2+k_z^2}{k^4}\,d\mathbf{k} \approx \frac{Z^2e^2\xi}{2\pi^2}\int \frac{k_\perp^2}{(k_\perp^2+k_1^2)^2}\,d\mathbf{k}.$$

On noting that $d\mathbf{k} = 2\pi k_\perp\, dk_\perp\, dk_1 = 2\pi k_\perp\, dk_\perp (d\omega/c\xi)$ and on comparing this with the expression for $S$ we obtain formula (34.5).

The upper limit of the integral in (34.5) can be determined from the condition $k_\perp < m$. Since $n(\omega)$ depends on $k_{\max}$, logarithmically, we can set $k_{\max} = \eta m$, where $\eta$ is a quantity of the order of magnitude unity. On carrying out the integration we obtain

$$n(\omega)\,d\omega = \frac{2}{\pi}\,\alpha Z^2\frac{d\omega}{\omega}\ln\frac{\eta m}{\omega}. \tag{34.7}$$

Formulas (34.4), (34.7) are evidently valid for

$$\omega \ll \frac{m}{\sqrt{1-v^2}}, \quad \frac{1}{\sqrt{1-v^2}} \gg 1. \tag{34.8}$$

The fact that the integral for $n(\omega)$ diverges at large $\omega$ has a simple physical meaning: it is related to the divergence of the expression for the energy flux $S$ for small values of the impact parameter $b$.

The lower limit for $b$, and consequently, the upper limit for $k_\perp$, can be evaluated in the following manner. In quantum mechanics in order to be able to speak of a definite value of the impact parameter $b$ it is obviously necessary that $b$ should be considerably greater than the transverse dimensions $\Delta y$ of the wave packet associated with the particle: $b \gg \Delta y$.

On the other hand, according to the uncertainty principle $\Delta y > \hbar/\Delta p_y$ where $\Delta p_y$ is the uncertainty in the transverse component of the momentum of the particle. The transverse component of the velocity of the particle $v_y$ must be sufficiently small so that the transverse displacement of the particle during the collision time, which in order of magnitude is equal to $\tau \sim b/v \sim b/c$, will be small in comparison with $b$, $v_y(b/c) \ll b$, i.e., the transverse component of the velocity must be small compared to the velocity of light and, consequently, $\Delta p_y \ll mc$, where $m$ is the particle mass.

Thus, from the uncertainty relation it follows that $\Delta y \gg \hbar/mc$. Therefore, the impact parameter must be considerably larger than $\hbar/mc$.

In discussing the effect of the nuclear field on the electron we must obviously interpret $m$ as the mass of the lighter particle, i.e., of the electron. We assume that in this case the minimum value of the impact parameter is given by $b_{\min} = (1/\eta)(\hbar/mc)$, where in order of magnitude $\eta$ is equal to unity. The quantity $b_{\min}$ corresponds to the maximum value of $k_\perp$, which is given by $k_{\max} = \eta(mc/\hbar)$.

### 34.2. *Bremsstrahlung from a Fast Electron in the Field of a Nucleus*

By utilizing the method of equivalent photons we shall now obtain the bremsstrahlung cross section for the collision of a fast electron with a nucleus. In the coordinate system $K^*$, in which the electron is at rest before the collision, this process can be treated as the Compton scattering of the electron by the spectrum of the equivalent photons of the nuclear field.

The cross section for the scattering of the photon by the electron is given in the coordinate system $K^*$ by formula (28.18)

$$d\sigma^*_{\text{scatt}}(\omega_1^*, \omega_2^*) = \frac{r_0^2}{2}\left(\frac{\omega_2^*}{\omega_1^*}\right)^2 \left(\frac{\omega_1^*}{\omega_2^*} + \frac{\omega_2^*}{\omega_1^*} - \sin^2\theta^*\right) do^*,$$

where $\omega_1^*$ and $\omega_2^*$ are the frequencies of the primary and the scattered photons in the coordinate system $K^*$, which are related by expression (28.6)

$$\omega_2^* = \frac{\omega_1^*}{1 + (\omega_1^*/m)(1-\cos\theta^*)},$$

and $\theta^*$ is the scattering angle. The bremsstrahlung cross section in the coordinate system $K^*$ is related to the scattering cross section by

$$d\sigma^*_{\text{rad}}(\omega_2^*) = \int_{\omega^*_{1\,\min}}^{\omega^*_{1\,\max}} n(\omega_1^*) d\omega_1^* \, d\sigma^*_{\text{scatt}}(\omega_1^*, \omega_2^*), \tag{34.9}$$

where $\omega^*_{1\,\min}$ and $\omega^*_{1\,\max}$ are the minimum and the maximum frequencies of the primary photon corresponding to a given frequency $\omega_2^*$ of the scattered photon.

Since the scattering cross section is relativistically invariant, in order to determine $d\sigma_{\text{rad}}$ in the coordinate system $K$ we only need to express the cross section, determined by formula (34.9), in terms of the frequency $\omega_2$ in the coordinate system $K$, which corresponds to the frequency $\omega_2^*$ in the coordinate system $K^*$. The relation between the two frequencies is given by the Doppler formula $\omega_2 = \omega_2^*(1-v\cos\theta^*)\xi$.

On introducing $\xi_1 = \omega_1^*/m$, $\xi_2 = \omega_2/m\xi$ and on noting that

$$n(\omega_1^*)\,d\omega_1^* = \frac{2Z^2\alpha}{\pi}\,\frac{d\xi_1}{\xi_1}\ln\frac{\eta\xi}{\xi_1},$$

$$\cos\theta^* = \frac{1-\xi_2\big(1+(1/\xi_1)\big)}{v-\xi_2},$$

$$d\sigma_{\text{scatt}} = \pi r_0^2\frac{d\xi_1}{\xi_1}\left\{1-\xi_2+\frac{1}{1-\xi_2}+\frac{\xi_2}{\xi_1(1-\xi_2)}\left[\frac{\xi_2}{\xi_1(1-\xi_2)}-2\right]\right\},$$

we rewrite $d\sigma_{\text{rad}}$ in the form

$$d\sigma_{\text{rad}}(\omega_2) = 2r_0^2Z^2\alpha\,d\xi_2\int\limits_{\frac{\xi_2}{2(1-\xi_2)}}^{2\xi_2\xi^2}\frac{d\xi_1}{\xi_1^2}\ln\frac{\eta\xi}{\xi_1}\left\{1-\xi_2+\frac{1}{1-\xi_2}\right.$$
$$\left.+\frac{\xi_2}{\xi_1(1-\xi_2)}\left[\frac{\xi_2}{\xi_1(1-\xi_2)}-2\right]\right\}, \qquad (34.9')$$

where the upper and lower limits of integration over $\xi_1$ correspond to $\theta^* = 0$ and $\pi$.

The principal contribution to the integral (34.9′) obviously comes from the values of $\xi_1$ near the lower limit. On carrying out the calculation with logarithmic accuracy, i.e., on assuming that $\ln(\eta\xi/\xi_1)$ is a large quantity, and on taking it outside the integral sign, after setting it equal to its value at $\xi_1 = \xi_2/2(1-\xi_2)$, we obtain

$$d\sigma_{\text{rad}}(\omega_2) = 4r_0^2Z^2\alpha\frac{d\omega_2}{\omega_2}\,\frac{\varepsilon_2}{\varepsilon_1}\left(\frac{\varepsilon_1}{\varepsilon_2}+\frac{\varepsilon_2}{\varepsilon_1}-\frac{2}{3}\right)\ln\eta\frac{2\varepsilon_1\varepsilon_2}{m\omega}, \qquad (34.10)$$

where $\varepsilon_1$ and $\varepsilon_2$ are the values of the electron energy in the initial and the final states.

Formula (34.10) is valid if $\varepsilon_1 \gg m$, $\varepsilon_2 \gg m$. As should have been expected, it coincides with formula (29.25).

### 34.3. *Radiation Emitted in an Electron-Electron Collision*

By utilizing the method of equivalent photons we shall now obtain the radiation emitted in an electron-electron collision.

If a fast electron, which we shall refer to as electron $f$, collides with an electron at rest, which we shall refer to as electron $r$, then in order to determine the cross section for the resultant bremsstrahlung by the method of equivalent photons we must, in the first place, investigate the scattering of the equivalent photons of the field of the fast electron $f$ by the stationary electron $r$ and, secondly, on going over to the coordinate system associated with electron $f$ investigate the scattering by electron $f$ of the equivalent photons of the field due to electron $r$.

The second of these two processes does not differ from the process discussed earlier of the scattering of the equivalent photons of the nuclear field by an electron in its own rest system—the process to which the bremsstrahlung of an electron in the nuclear field may be reduced.

We now show that the contribution made by the first process to the cross section for bremsstrahlung accompanying an electron-electron collision is considerably smaller than the contribution made by the second process.

On assuming that the photon energies $\omega_1$ and $\omega_2$ before and after scattering in the first process are considerably greater than $m$, we can start from the following formula for the Compton scattering cross section:

$$d\sigma_{\text{scatt}} = \pi r_0^2 m \frac{d\omega_2}{\omega_1^2}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2}\right).$$

(Here we have not included the term containing sin $\theta$, which may be neglected since in the relativistic domain scattering takes place primarily into small angles $\theta \sim m/\omega$.)

By utilizing expression (34.7) for the number of equivalent photons we can represent the contribution of the first process to the cross section for the emission of radiation in the form

$$d\sigma_{\text{rad}}^{\text{I}} = 2r_0^2 a m\, d\omega_2 \int\limits_{\omega_{1\,\text{min}}}^{\omega_{1\,\text{max}}} \frac{d\omega_1}{\omega_1^3} \ln\frac{\varepsilon}{\omega_1}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2}\right), \qquad (34.11)$$

where $\varepsilon$ is the energy of electron $f$ and $\omega_{1\,\mathrm{min}}$ and $\omega_{1\,\mathrm{max}}$ are the limits between which $\omega_1$ lies for a given frequency $\omega_2$. These limits can be obtained from

$$\omega_1 = \frac{\omega_2}{1-(\omega_2/m)(1-\cos\theta)}$$

and are obviously equal to $\omega_{1\,\mathrm{min}} = \omega_2$ and $\omega_{1\,\mathrm{max}} = \infty$. Although the method is valid only for $\omega_1 \ll \varepsilon$, nevertheless since the integrand in (34.11) falls off rapidly with increasing $\omega_1$, the upper limit in (34.11) may be taken to be infinite.

The principal role in the integral (34.11) is played by the values of $\omega_1$ near the lower limit. On carrying out the integration with logarithmic accuracy, just as in (34.9′) we obtain

$$d\sigma^{\mathrm{I}}_{\mathrm{rad}}(\omega_2) = \frac{8}{3} r_0^2 \alpha m \frac{d\omega_2}{\omega_2^2} \ln \frac{\varepsilon}{\omega_2}. \tag{34.12}$$

On setting $Z=1$ in formula (34.10) for the cross section for the bremsstrahlung emitted by an electron, we obtain the contribution of the second of the two processes described earlier to the cross section for the radiation emitted in a collision of two electrons

$$d\sigma^{\mathrm{II}}_{\mathrm{rad}}(\omega_2) = 4 r_0^2 \alpha \frac{d\omega_2}{\omega_2} \frac{\varepsilon_2}{\varepsilon_1}\left(\frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1} - \frac{2}{3}\right) \ln \frac{2\varepsilon_1\varepsilon_2}{m\omega_2}. \tag{34.12′}$$

Comparison of $d\sigma^{\mathrm{I}}_{\mathrm{rad}}$ and $d\sigma^{\mathrm{II}}_{\mathrm{rad}}$ shows that $d\sigma^{\mathrm{I}}_{\mathrm{rad}} \sim (m/\omega_2)\, \mathrm{d}\sigma^{II}_{\mathrm{rad}} \ll d\sigma^{II}_{\mathrm{rad}}$. This relation is associated with the fact that in both integrals (34.9) and (34.11) the principal contributions come from values of the variables near the lower limits, while the lower limit in (34.19), which is equal to $m\omega_2/2\varepsilon$, is considerably smaller than the lower limit in (34.11), which is equal to $\omega_2$.

### 34.4. *Pair Production by a Photon in the Field of a Nucleus*

We now apply the method of equivalent photons to the evaluation of the cross section for pair production. We start by determining the cross section for pair production by a photon in the field of a nucleus.

We assume that a photon of energy $\omega = \xi m$, $\xi \gg 1$ is incident on a nucleus at rest in the coordinate system $K$. We go over to the coordinate system $K'$, in which the nucleus is moving towards the photon with a velocity $v$ equal to

$$v = \sqrt{1-\frac{4}{\xi^2}}, \qquad \frac{1}{\sqrt{1-v^2}} = \frac{\xi}{2} \gg 1.$$

In this coordinate system the photon energy is equal to

$$\omega' = \omega\frac{1-v}{\sqrt{1-v^2}} = \xi m\frac{\xi}{2}\left(1-\sqrt{1-\frac{4}{\xi^2}}\right) \approx m.$$

The field of the nucleus in the system $K'$ can be represented in the form of a superposition of pseudophotons, whose spectrum is determined in accordance with (34.7) by

$$n(\omega_0)d\omega_0 = \frac{2}{\pi}\alpha Z^2\frac{d\omega_0}{\omega_0}\ln\left(\eta\frac{\xi m}{\omega_0}\right).$$

The process of pair production by a photon in the field of a nucleus can be treated as pair production in the collision of the incident photon with the photons of this spectrum. In other words, the cross section for pair production by a photon in the field of a nucleus is equal to

$$\sigma = \int n(\omega_0)d\omega_0\sigma_{\gamma\gamma}, \tag{34.13}$$

where $\sigma_{\gamma\gamma}$ is the cross section for pair production by two photons given by formula (32.27') into which in place of $x$ we must substitute

$$x = \sqrt{1-\frac{m^2}{\omega'\omega_0}} = \sqrt{1-\frac{m}{\omega_0}}.$$

For the limits of integration in (34.13) we must take $m$ and $\xi m$. On introducing in place of $\omega_0$ the variable $x$ we rewrite (34.13) in the form

$$\sigma = 2r_0^2\alpha Z^2\int\limits_0^{\sqrt{1-\frac{1}{\xi}}} dx\cdot x\ln\left[\eta\xi(1-x^2)\right]\left\{(3-x^4)\ln\left|\frac{1+x}{1-x}\right|+2x(x^2-2)\right\}.$$

On assuming that $\ln\xi \gg 1$, and on retaining only the main terms (of order $\ln\xi$) we obtain after integration

$$\sigma = \frac{28}{9}\alpha Z^2 r_0^2\ln f\xi, \quad f\sim 1. \tag{34.14}$$

This expression coincides with formula (32.9).

### 34.5. *Pair Production in a Collision of Two Fast Particles*

We shall now also investigate the process of pair production in a collision of two fast charged particles. We assume that the relative velocity of the particles $v$ is sufficiently large, so that $\xi = (1/\sqrt{1-v^2}) \gg 1\cdot$

In the coordinate system $K_1$, in which one of the particles $A_1$ is at rest, the distribution of photons representing the field of the second particle $A_2$ is determined by

$$n_2(\omega)d\omega = \frac{2}{\pi}\alpha Z_2^2 \ln\left(\eta\frac{\xi m}{\omega}\right)\frac{d\omega}{\omega},$$

where $Z_2e$ is the charge of the particle $A_2$.

The process of pair production in the collision of particles $A_1$ and $A_2$ can be regarded as pair production in the collision of the photons of this spectrum with the stationary particle $A_1$. The cross section for pair production in the collision of a photon with a particle is, in accordance with (34.14), given by

$$\sigma_\gamma(\omega) \approx \frac{28}{9}r_0^2\alpha Z_1^2 \ln\left(\eta'\frac{\omega}{m}\right), \quad \eta' \sim 1,$$

where $\omega$ is the energy of the pair and $Z_1e$ is the charge of the particle $A_1$. Therefore, the cross section for pair production in the collision of particles $A_1$ and $A_2$ in which we are interested can be represented in the form

$$\sigma = \int \sigma_\gamma(\omega)n_2(\omega)d\omega. \tag{34.15}$$

The maximum frequency of the photons of the field due to particle $A_2$ is equal in order of magnitude to $\xi m$. Therefore, as limits in this integral we must take $2m$ and $\xi m$:

$$\sigma = \int\limits_{2m}^{\xi m} \frac{2}{\pi}\alpha Z_2^2 \ln\left(\frac{\xi m}{\omega}\right)\frac{d\omega}{\omega}\cdot\frac{28}{9}r_0^2\alpha Z_1^2 \ln\left(\eta'\frac{\omega}{m}\right)$$

$$\approx \frac{28}{27}\frac{r_0^2}{\pi}\alpha^2 Z_1^2 Z_2^2 (\ln f\xi)^3, \quad f \sim 1. \tag{34.16}$$

This formula coincides with formula (32.34).

## § 35. Scattering of a Photon by a Bound Electron. Emission of Two Photons

### 35.1. *The Dispersion Formula*

In § 28 we studied the scattering of photons by free electrons. We now go over to the study of the scattering of photons by bound electrons.

For the determination of the scattering cross section in this case we can, as we have done previously, utilize the part of the scattering matrix $S^{(2)}$ defined by the general formula (28.1), but now in place of the function $S^c(x_1-x_2)$ the expression for $S^{(2)}$ must contain the Green's function for the Dirac equation for the electron in the external field $S_c^{(e)}(x_1, x_2)$:

$$S^{(2)} = e^2 \int N\{\overline{\psi}(x_1)\hat{A}(x_1)S_c^{(e)}(x_1, x_2)\hat{A}(x_2)\psi(x_2)\}\, d^4x_1\, d^4x_2. \qquad (35.1)$$

The function $S_c^{(e)}(x_1, x_2)$ is defined in the following manner:

$$S_c^{(e)}(x_1, x_2) = \begin{cases} \sum\limits_n \psi_n^{(+)}(x_1)\overline{\psi}_n^{(+)}(x_2), & t_1 > t_2, \\ -\sum\limits_n \psi_n^{(-)}(x_1)\overline{\psi}_n^{(-)}(x_2), & t_1 < t_2, \end{cases} \qquad (35.2)$$

where

$$\psi_n^{(+)}(x) = \psi_n^{(+)}(\boldsymbol{r})e^{-iE_n^{(+)}t} \quad \text{and} \quad \psi_n^{(-)}(x) = \psi_n^{(-)}(\boldsymbol{r})e^{-iE_n^{(-)}t}$$

are the positive and negative frequency solutions of the corresponding Dirac equation.

The electron Green's function $S_c^{(e)}(x_1, x_2)$ can be written in a more compact form if we note that

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{e^{i\omega t}}{a(1-i0)+\omega}\, d\omega = \begin{cases} 0, & a < 0, \\ e^{-iat}, & a > 0, \quad t > 0, \\ -e^{-iat}, & a < 0, \quad t < 0. \end{cases}$$

By utilizing these formulas we can easily show that

$$S_c^{(e)}(x_1, x_2) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} d\omega\, e^{i\omega(t_1-t_2)} \sum_n \frac{\psi_n(\boldsymbol{r}_1)\overline{\psi}_n(\boldsymbol{r}_2)}{E_n(1-i0)+\omega}, \qquad (35.3)$$

where the summation over $n$ is taken over all states corresponding both to positive and to negative frequencies.

From formula (35.3) and from the Dirac equation it follows that $S_c^{(e)}(x_1, x_2)$ satisfies

$$\left\{\gamma_\mu\left[\frac{\partial}{\partial x_\mu} - ieA_\mu^{(e)}(x)\right] + m\right\} S_c^{(e)}(x, x') = -i\delta(x-x'), \qquad (35.4)$$

and since $S^c(x)$ satisfies

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) S^c(x) = -i\delta(x),$$

the function $S_c^{(e)}(x_1, x_2)$ satisfies the integral equation

$$S^{(e)}(x_1, x_2) = S^c(x_1, x_2) - \int S^c(x_1, x_3)\, e\hat{A}(x_3) S_c^{(e)}(x_3, x_2)\, d^4x_3. \quad (35.5)$$

We now apply these formulas to the determination of the cross section for the scattering of a photon by bound electron.

We consider the scattering of a photon of frequency $\omega_i$, propagation vector $\boldsymbol{k}_i$, and polarization $\boldsymbol{e}_i$ by an electron in the state $\psi_i(x) = \psi_i(\boldsymbol{r}) \exp(-iE_i t)$. As a result of scattering, a photon appears of frequency $\omega_f$, propagation vector $\boldsymbol{k}_f$, polarization $\boldsymbol{e}_f$, while the electron goes over into the state $\psi_f(x) = \psi_f(\boldsymbol{r}) \exp(-iE_f t)$.

The frequency of the scattered photon $\omega_f$ may either coincide with the frequency of the primary photon $\omega_i$, or differ from it (in the latter case we speak of Raman scattering). On noting that photon states under consideration correspond to the potentials

$$A_i(x) = A_i(\boldsymbol{r})\, e^{-i\omega_i t} = \frac{\boldsymbol{e}_i}{\sqrt{2\omega_i}}\, e^{i(\boldsymbol{k}_i \boldsymbol{r} - \omega_i t)},$$

$$A_f(x) = A_f(\boldsymbol{r})\, e^{i\omega_f t} = \frac{\boldsymbol{e}_f}{\sqrt{2\omega_f}}\, e^{-i(\boldsymbol{k}_f \boldsymbol{r} - \omega_f t)},$$

we obtain in accordance with (35.1) the following expression for the matrix element for this transition:

$$S_{i\to f}^{(2)} = \frac{e^2}{2\sqrt{\omega_i \omega_f}} \int d^4x_1 \int d^4x_2 \{\overline{\psi}_f(\boldsymbol{r}_1)\, e^{-i\boldsymbol{k}_f \boldsymbol{r}_1} \hat{e}_f S_c^{(e)}(x_1, x_2)$$
$$\times \hat{e}_i e^{i\boldsymbol{k}_i \boldsymbol{r}_2} \psi_i(\boldsymbol{r}_2)\, e^{it_1(E_f + \omega_f)} e^{-it_2(E_i + \omega_i)}$$
$$+ \psi_f(\boldsymbol{r}_1)\, e^{i\boldsymbol{k}_i \boldsymbol{r}_1} \hat{e}_i S_c^{(e)}(x_1, x_2)\, \hat{e}_f e^{-i\boldsymbol{k}_f \boldsymbol{r}_2} \psi_i(\boldsymbol{r}_2)\, e^{it_1(E_f - \omega_i)} e^{-it_2(E_i - \omega_f)}\}. \quad (35.6)$$

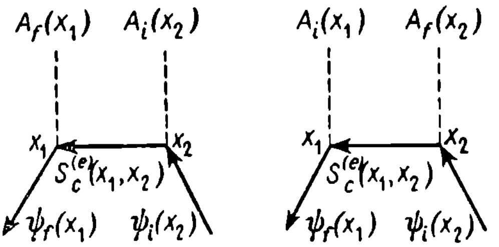


Fig. 46.

The two terms in this formula correspond to the diagrams shown in Fig. 46. The difference between these diagrams and the diagrams representing the scattering of a photon by a free electron consists of the fact that in Fig. 46 the external electron lines represent bound states, while the internal electron line corresponds to the function $S_c^{(e)}(x_1, x_2)$, and not to $S^c(x_1-x_2)$.

On substituting into (35.6) expression (35.3) for $S_c^{(e)}(x_1, x_2)$, we write $S_{i\to f}^{(2)}$ in the form

$$S_{i\to f}^{(2)} = -\frac{a}{i\sqrt{\omega_i\omega_f}} \sum_n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} d\omega \left\{ \frac{(f|\hat{e}_f e^{-i\mathbf{k}_f\mathbf{r}}|n)(n|\hat{e}_i e^{i\mathbf{k}_i\mathbf{r}}|i)}{E_n+\omega} \right.$$

$$\times\, e^{it_1(E_f+\omega_f+\omega)} e^{it_2(E_i+\omega_i+\omega)}$$

$$\left. + \frac{(f|\hat{e}_i e^{i\mathbf{k}_i\mathbf{r}}|n)(n|\hat{e}_f e^{-i\mathbf{k}_f\mathbf{r}}|i)}{E_n+\omega} e^{it_1(E_f-\omega_i+\omega)} e^{-it_2(E_i-\omega_f+\omega)} \right\}, \tag{35.7}$$

where

$$(m|\hat{e}\, e^{i\mathbf{k}\mathbf{r}}|n) = \int \bar{\psi}_m(\mathbf{r}) \hat{e}\, e^{i\mathbf{k}\mathbf{r}} \psi_n(\mathbf{r})\, d\mathbf{r}. \tag{35.7'}$$

On carrying out the integration over $t_1$, $t_2$ and $\omega$, we obtain

$$S_{i\to f}^{(2)} = -2\pi i U_{i\to f}\, \delta(E_i+\omega_i-E_f-\omega_f), \tag{35.8}$$

where

$$U_{i\to f} = \frac{2\pi a}{\sqrt{\omega_i\omega_f}} \sum_n \left\{ \frac{(f|\hat{e}_f e^{-i\mathbf{k}_f\mathbf{r}}|n)(n|\hat{e}_i e^{i\mathbf{k}_i\mathbf{r}}|i)}{E_n-E_i-\omega_i} \right.$$

$$\left. + \frac{(f|\hat{e}_i e^{i\mathbf{k}_i\mathbf{r}}|n)(n|\hat{e}_f e^{-i\mathbf{k}_f\mathbf{r}}|i)}{E_n-E_f+\omega_i} \right\} \tag{35.8'}$$

and the summation is taken over all the states $n$ corresponding both to the positive and to the negative frequencies.

The most important application of this formula is to the scattering of a photon by an atomic system.

We shall first of all consider the nonrelativistic case when the photon energies are small compared to the electron rest energy $\omega_i \ll m$, $\omega_f \ll m$. In this case we can assume that the electron energies both in the initial and the final states $E_i$ and $E_f$ differ little from $m$; $|E_i-m| \ll m$ and $|E_f-m| \ll m$. Moreover, we assume that in the sum (35.8′) an important role is played only by those states $n$ ("intermediate" states) for which the energies are also nonrelativistic, i.e., $||E_n|-m| \ll m$.

These assumptions enable us to simplify considerably the part of the sum $U_{i\to f}$ in (35.8′) which refers to negative frequencies. First of all, in each term of this part of the sum, which we denote by $U^{(-)}_{i\to f}$, we can replace the denominator by $2m$:

$$U^{(-)}_{i\to f} = \frac{-\pi\alpha}{m\sqrt{\omega_i\omega_f}} \sum_{n^{(-)}} \{(f|\hat{e}_f e^{-ik_f r}|n^{(+)})(n^{(-)}|\hat{e}_i e^{ik_i r}|i) + (f|\hat{e}_i e^{ik_i r}|n)(n|\hat{e}_f e^{-ik_f r}|i)\},$$

and after this we can transform $U^{(-)}_{i\to f}$ into a sum over all states, including not only negative, but also positive frequencies. In order to do this we must in the expression

$$(f|\hat{e}_f e^{-ik_f r}|n^{(-)}) = \int \bar{\psi}_f(r)\hat{e}_f e^{-ik_f r}\psi_n^{(-)}(r)dr$$

make the substitution

$$\psi_n^{(-)}(r) \to \frac{m-\mathrm{H}}{m-E_n^{(-)}}\psi_n^{(-)}(r) \approx \frac{m-\mathrm{H}}{2m}\psi_n^{(-)}(r),$$

where H is the Hamiltonian for the electron. In the approximation in which we are interested we have

$$\mathrm{H} \approx m\gamma_4 \quad \text{and} \quad \mathrm{H}\psi^{(\pm)}(r) = \pm m\psi^{(\pm)}(r).$$

Therefore

$$\frac{m-\mathrm{H}}{2m}\psi_n^{(-)}(r) = \psi_n^{(-)}(r), \qquad \frac{m-\mathrm{H}}{2m}\psi_n^{(+)}(r) = 0$$

and

$$U^{(-)}_{i\to f} = \frac{-\pi\alpha}{m\sqrt{\omega_i\omega_f}} \sum_n \left\{\left(f\left|\hat{e}_f\frac{1-\gamma_4}{2}e^{-ik_f r}\right|n\right)(n|\hat{e}_i e^{ik_i r}|i) + \left(f\left|\hat{e}_i\frac{1-\gamma_4}{2}e^{ik_i r}\right|n\right)(n|\hat{e}_f e^{-ik_f r}|i)\right\},$$

where the sum is taken over both signs of the frequency. On utilizing the rule for matrix multiplication and the definition (35.7′) we can rewrite $U^{(-)}_{i\to f}$ in the form

$$U^{(-)}_{i\to f}=\frac{\pi\alpha}{m\sqrt{\omega_i\omega_f}}\left\{\left(f\left|\hat{e}_f\frac{\gamma_4-1}{2}e^{-ir(k_f-k_i)}e_i\right|i\right)\right.$$
$$\left.+\left(f\left|\hat{e}_i\frac{\gamma_4-1}{2}e^{-ir(k_f-k_i)}\hat{e}_f\right|i\right)\right\}.$$

Finally, on noting that

$$\gamma_4 e=-e\gamma_4,$$
$$\hat{e}_f\hat{e}_i+\hat{e}_i\hat{e}_f=2e_ie_f$$

and that in our approximation $\gamma_4\psi_i\sim\psi_i$, we obtain

$$U^{(-)}_{i\to f}=\frac{2\pi\alpha}{m\sqrt{\omega_i\omega_f}}e_ie_f\int\psi_f^*(r)\psi_i(r)e^{i(k_i-k_f)r}\,dr. \tag{35.9}$$

We now consider the part of the sum $U_{i\to f}$ which refers to positive frequencies. In the nonrelativistic approximation in the numerators of the fractions appearing in this part of $U_{i\to f}$ we can, in accordance with subsection 15.1, replace $\boldsymbol{\alpha}A$ by $(1/m)Ap+\boldsymbol{\mu}$ curl $A$ and $\boldsymbol{\alpha}e$ by $(1/m)ep+i\boldsymbol{\mu}[k,e]$, where $\boldsymbol{\mu}=(e\hbar/2mc)\boldsymbol{\sigma}$ is the spin magnetic moment of the electron. On carrying out this replacement, and on utilizing formula (35.9) for $U^{(-)}_{i\to f}$ we obtain the following expression for $U_{i\to f}$:

$$U_{i\to f}=\frac{4\pi\alpha}{m}\left\{\frac{1}{m}\sum_n\left[\frac{(p'A_f^*)_{fn}(p'A_i)_{ni}}{E_i-E_n+\omega_i}\right.\right.$$
$$\left.\left.+\frac{(p'A_i)_{fn}(p'A_f^*)_{ni}}{E_i-E_n-\omega_f}\right]+(A_f^*A_i)_{fi}\right\}, \tag{35.10}$$

where $p'=p+\frac{1}{2}e[\boldsymbol{\sigma}\nabla]$ and the parentheses denote matrix elements evaluated using nonrelativistic wave functions:

$$(L)_{ss'}=\int\varphi_s^*L\varphi_{s'}\,dr.$$

We note that the matrix element of the form (35.10) can also be obtained in a different manner, viz., if from the outset we utilize the nonrelativistic operator for the energy of interaction of the electron with the electromagnetic field (cf. (15.3)):

$$V=-\frac{e}{m}p'A+\frac{e^2}{2m}A^2,\quad p'=p+\frac{e}{2}[\boldsymbol{\sigma}\nabla].$$

The term quadratic in $A$ in this operator determines the "direct transitions" and in the first approximation of perturbation theory leads to the last term in (35.10), while the linear term determines transitions via the "intermediate" states, and in the second approximation of perturbation theory leads to the sum in (35.10)

If the wavelength of the scattered photon is large compared to the atomic dimensions, then we can take $A_i(r)$ and $A_f(r)$ to be constant, since $\exp(ik_{i,f}r) \approx \exp(ik_{i,f}R)$ ($R$ is the radius-vector of the center of the atom). In this case $U_{i\to f}$ assumes the form

$$U_{i\to f} = \frac{2\pi a e^{i(k_i-k_f)R}}{m\sqrt{\omega_i\omega_f}}\left\{\frac{1}{m}\sum_n\left[\frac{(pe_f)_{fn}(pe_i)_{ni}}{E_i-E_n+\omega_i}+\frac{(pe_i)_{fn}(pe_f)_{ni}}{E_i-E_n-\omega_f}\right]+e_i e_f \delta_{if}\right\}. \tag{35.11}$$

Here the last term differs from zero only in the case of coherent scattering when $\omega_i = \omega_f$.

Formula (35.11) defines the dispersion law, i.e., the dependence of the scattering ability of the atom on the frequency of the light. It can be represented in a somewhat different form, viz., in such a way that instead of the matrix elements of the momentum $U_{i\to f}$ will contain matrix elements of the atomic dipole moment. In order to do this we must utilize the relation

$$(p)_{ss'} = im(E_s - E_{s'})(r)_{ss'},$$

and also the commutation relations between the operators for the momentum and the coordinate, whence it follows that

$$e_i e_f \delta_{if} = i[(pe_i)(re_f)-(re_f)(pe_i)]_{fi} = i\sum_n [(pe_i)_{fn}(re_f)_{ni}-(re_f)_{fn}(pe_i)_{ni}].$$

Further, on adding inside the figure bracket in (35.11) the zero term

$$0 = \omega_f[(re_i)(re_f)-(re_f)(re_i)]_{fi} = \omega_f \sum_n [(re_i)_{fn}(re_f)_{ni}-(re_f)_{fn}(re_i)_{ni}],$$

we finally obtain

$$U_{i\to f} = 2\pi a e^{i(k_i-k_f)R}\sqrt{\omega_i\omega_f}\sum_n\left[\frac{(re_f)_{fn}(re_i)_{ni}}{E_i-E_n+\omega_i}+\frac{(re_i)_{fn}(re_f)_{ni}}{E_i-E_n-\omega_f}\right]. \tag{35.12}$$

The differential scattering cross section is related to $U_{i\to f}$ by

$$d\sigma = 2\pi|U_{i\to f}|^2\,\delta(E_i+\omega_i-E_f-\omega_f)\frac{dk_f}{(2\pi)^3},$$

On eliminating the $\delta$-function by integration over $\omega_f$ we obtain

$$d\sigma = \left| \sum_n \left[ \frac{(Qe_f)_{fn}(Qe_i)_{ni}}{E_i - E_n + \omega_i} + \frac{(Qe_i)_{fn}(Qe_f)_{ni}}{E_i - E_n - \omega_f} \right] \right|^2 \omega_i \omega_f^3 \, do_f \,, \quad (35.13)$$

where $Q = er$ is the atomic dipole moment.

If the scattering system contains not one, but several, electrons, then we must interpret $Q$ as the total dipole moment for the system.

Formula (35.13) holds if the photon wavelength $\lambda$ is large compared to the atomic dimensions $a$, $\lambda \gg a$.

### 35.2. *Resonance Scattering*

The dispersion formula (35.13) contains a sum over all the excited states of the atom. If the photon frequency $\omega_i$ is equal to the difference between the energies of one of the excited states and the ground state of the atom, i.e., if $\omega_i = E_n - E_i$, the scattering cross section becomes infinite, which shows that the foregoing formula is not valid for $\omega_i = E_n - E_i$. This case is known as resonance.

We shall not develop here a rigorous theory of resonance, but shall confine ourselves merely to explaining the physical reason for the inapplicability of formula (35.13) near resonance, and we shall indicate an approximate method for discussing the phenomenon of resonance. A consistent discussion of resonance scattering utilizing the general theory of radiation corrections will be given in subsection 53.4.

The reason for the inapplicability of formula (35.13) near resonance lies in the fact that we have treated $\psi_n(x)$ as stationary state wave functions, whose time dependence is given by exp $(-iE_n t)$.

But the excited states are only approximately stationary, since, because of the interaction with the electromagnetic field, there exists a finite probability for the transition of the atom to the ground state (this probability was found in § 27).

Stationary states can be described approximately (cf., for example, (119)) as states of complex energy whose wave functions have the following time dependence:

$$\psi_n \sim e^{-i\left(E_n - \frac{i}{2}\Gamma_n\right)t},$$

where $\Gamma_n$ is a positive real quantity. The probability of finding the atom in an excited state is proportional to $|\psi_n|^2$, and thus decays according to exp $(-\Gamma_n t)$. This shows that $\Gamma_n = w_n$, where $w_n$ is the emission probability determined by formulas (27.14), (27.22).

At frequencies close to the resonance frequency we may neglect in formula (35.13) all the terms except for the resonance term and replace in it $E_n$ by $E_n - i/2\Gamma_n$. Thus, we obtain in place of (35.12) the following expression for the scattering matrix element:

$$U_{i\to f} = -2\pi\alpha\sqrt{\omega_i\omega_f}\,\frac{\sum\limits_s (\mathbf{r}\mathbf{e}_f)_{fs}(\mathbf{r}\mathbf{e}_i)_{si}}{E_n - E_i - \omega_i - (i/2)\Gamma_n},$$

where the sum is taken over all the states of energy $E_n$. The corresponding differential scattering section is given by

$$d\sigma = \frac{\left|\sum\limits_s (Q\mathbf{e}_f)_{fs}(Q\mathbf{e}_i)_{si}\right|^2}{(E_n - E_i - \omega_i)^2 + (1/4)\Gamma_n^2}\,\omega_i\omega_f^3\,do_f. \tag{35.14}$$

In order to obtain the total scattering cross section we must integrate expression (35.14) over the angles, average it over the polarizations of the incident photon, and over the angular momentum components of the initial state, and sum it over the polarizations of the scattered photon, and over the angular momentum components of the final states. Such a cross section is determined, in addition to $E_i$, $E_n$ and $\omega$, only by the level width $\Gamma_n$ and by the values of the angular momentum. In the case of coherent scattering the total cross section is equal to

$$\sigma = \frac{2j_n + 1}{2(2j_i + 1)}\,\frac{4\pi}{\omega^2}\,\frac{\Gamma_n^2}{(E_n - E_i - \omega)^2 + (\Gamma_n^2/4)}, \tag{35.15}$$

where $j_n$ and $j_i$ are the angular momenta of the system in the states $n$ and $i$ and $\omega \equiv \omega_i \equiv \omega_f$. This formula preserves its form also in the case when the state $n$ differs in an arbitrary way from the state $i$ with respect to its angular momentum and parity. In this case $\Gamma_n = \omega_n$ must be interpreted as the emission probability for the corresponding multipole radiation.

### 35.3. *Compton Scattering by Bound Electrons*

We now investigate the scattering of an energetic photon by a bound electron on the assumption that the photon energy is considerably greater than the electron binding energy.

We denote by $\Phi$ the state vector describing one bound electron in the state $\psi_i = \psi_i(\mathbf{r})\exp(-iE_i t)$ of energy $E_i$ and polarization $\sigma_i$, and one photon of energy $\omega_i$, momentum $\mathbf{k}_i$ and polarization $\mathbf{e}_i$. Since it has been assumed that $\omega_i \gg m$ we may neglect the electron binding energy and approximately write $\Phi$ as a superposition of states $\chi_{k_i;\mathbf{p},\sigma}$, describing the same photon $\mathbf{k}_i$ and a "free" electron of momentum $\mathbf{p}$ and polarization $\sigma$, whose energy, however, is equal not to $\sqrt{m^2+\mathbf{p}^2}$, but to the energy of the bound electron $E_i$: $\Phi = \sum\limits_{p\sigma} C_{p,\sigma}\chi_{k_i;p,\sigma}$.

The expansion coefficients $C_{p,\sigma}$ are obviously equal to

$$C_{p,\sigma} = (\chi_{k_i;p,\sigma,}\Phi) = \int \psi_i(\mathbf{r})u_\sigma(\mathbf{p})^* e^{-i\mathbf{pr}}d\mathbf{r} = \psi_{\mathbf{p}}^{\sigma_i}u^\sigma(\mathbf{p})^*,$$

where $\psi_{\mathbf{p}}^{\sigma_i}$ is the Fourier component of the bound state wave function $\psi_i(\mathbf{r})$

$$\psi_{\mathbf{p}}^{\sigma_i} = \int \psi_i(\mathbf{r})e^{-i\mathbf{pr}}d\mathbf{r}$$

and $u^\sigma(\mathbf{p})\exp(i\mathbf{pr})$ is the orthonormal system of solutions of the Dirac equation for free electron satisfying the normalization conditions $u^\sigma(\mathbf{p})u^{\sigma'}(\mathbf{p})^* = \delta_{\sigma\sigma'}$:

We are interested in the transition to the state in which there exists one photon of energy $\omega_f$, momentum $\mathbf{k}_f$ and polarization $\mathbf{e}_f$, and a free electron of momentum $\mathbf{p}_f$, energy $\varepsilon_f$ and polarization $\sigma_f$. The matrix element for this transition is evidently equal to

$$(\chi_{k_f;p_f,\sigma_f}, S\Phi) = \sum_{p,\sigma} C_{p,\sigma}(\chi_{k_f;p_f\sigma_f}, S\chi_{k_i;p,\sigma}),$$

where $\chi_{k_f;p_f,\sigma_f}$ is the state vector describing the state containing one photon of momentum $\mathbf{k}_f$, polarization $\mathbf{e}_f$, and one free electron of energy $\varepsilon_f = \sqrt{m^2+\mathbf{p}_f^2}$, momentum $\mathbf{p}_f$ and polarization $\sigma_f$.

The element of the scattering matrix $(\chi_{k_f;p_f,\sigma_f}, S\chi_{k_i;p,\sigma})$ between the free states $\mathbf{k}_i;\mathbf{p},\sigma$ and $\mathbf{k}_f;\mathbf{p}_f,\sigma_f$ (but without the usual relation between the electron energy in the initial state, and the momentum $\mathbf{p}$) can be obtained with the aid of formula (35.7), in which we interpret the states $\psi_n(x)$ as the free electron states $\psi_n^{(\pm)}(x) = u^{\sigma'}(\mathbf{p}')e^{ip'x}$, where the values $\sigma' = 1$, 2 correspond to the two polarization states of different spin orientation and of positive frequency, and the values $\sigma' = 3$, 4 correspond to the two polarization states of negative frequency.

In the case under discussion the quantities $(f|\hat{e}_i e^{-i\boldsymbol{k}_i\boldsymbol{r}}|n)$ have a simple form, for example:

$$(f|\hat{e}_i e^{-i\boldsymbol{k}_i\boldsymbol{r}}|n) = -i\boldsymbol{e}_i \int u^{\sigma_f}(\boldsymbol{p}_f)^* \boldsymbol{\alpha} u^{\sigma'}(\boldsymbol{p}')\mathrm{e}^{i\boldsymbol{r}(\boldsymbol{k}_i+\boldsymbol{p}'-\boldsymbol{p}_f)}\,d\boldsymbol{r}$$

$$= -i(2\pi)^3\delta(\boldsymbol{k}_i+\boldsymbol{p}'-\boldsymbol{p}_f)u^{\sigma_f}(\boldsymbol{p}_f)^*\boldsymbol{\alpha}\boldsymbol{e}_i u^{\sigma'}(\boldsymbol{p}').$$

Summation over $n$ in (35.8′) reduces to integration over $\boldsymbol{p}'$, and to summation over $\sigma' = 1, 2, 3, 4$. Integration over $\boldsymbol{p}'$ can be carried out immediately due to the presence of a $\delta$-function, and leads to the formula

$$(\chi_{\boldsymbol{k}_f;\boldsymbol{p}_f,\sigma_f}, S\chi_{\boldsymbol{k}_i;\boldsymbol{p},\sigma}) = \frac{ie^2(2\pi)^6}{2\sqrt{\omega_i\omega_f}}\delta(\boldsymbol{k}_f+\boldsymbol{p}_f-\boldsymbol{p}-\boldsymbol{k}_i)\,\delta(E_i+\omega_i-\varepsilon_f-\omega_f)$$

$$\times\sum_{\sigma'=1}^{4}\left\{\frac{u^{\sigma_f}(\boldsymbol{p}_f)^*\boldsymbol{\alpha}\boldsymbol{e}_f u^{\sigma'}(\boldsymbol{k}_i+\boldsymbol{p})u^{\sigma'}(\boldsymbol{k}_i+\boldsymbol{p})\boldsymbol{\alpha}\boldsymbol{e}_i u^{\sigma}(\boldsymbol{p})}{E^{\sigma'}(\boldsymbol{k}_i+\boldsymbol{p})-\omega_i-E_i}\right.$$

$$\left.+\frac{u^{\sigma_f}(\boldsymbol{p}_f)^*\boldsymbol{\alpha}\boldsymbol{e}_i u^{\sigma'}(\boldsymbol{p}-\boldsymbol{k}_f)u^{\sigma'}(\boldsymbol{p}-\boldsymbol{k}_f)\boldsymbol{\alpha}\boldsymbol{e}_f u^{\sigma}(\boldsymbol{p})}{E^{\sigma'}(\boldsymbol{p}-\boldsymbol{k}_f)+\omega_f-E_i}\right\}.$$

On noting that

$$\sum_{\sigma=1}^{4} u^{\sigma}_{\beta}(\boldsymbol{p})\psi^{\sigma_i}_{\alpha}(\boldsymbol{p})u^{\sigma}_{\alpha}(\boldsymbol{p})^* = \psi^{\sigma_i}_{\beta}(\boldsymbol{p}),$$

we obtain

$$S^{(2)}_{i\to f} = \frac{ie^2(2\pi)^6}{2\sqrt{\omega_i\omega_f}}\delta(E_i+\omega_i-\varepsilon_f-\omega_f)\,B^{\sigma_i\sigma_f}, \tag{35.16}$$

where

$$B^{\sigma_i\sigma_f} = \sum_{\sigma'=1}^{4}\left\{\frac{u^{\sigma_f}(\boldsymbol{p}_f)^*\boldsymbol{\alpha}\boldsymbol{e}_f u^{\sigma'}(\boldsymbol{k}_f+\boldsymbol{p}_f)u^{\sigma'}(\boldsymbol{k}_f+\boldsymbol{p}_f)\boldsymbol{\alpha}\boldsymbol{e}_i\psi^{\sigma_i}(\boldsymbol{k}_f+\boldsymbol{p}_f-\boldsymbol{k}_i)}{E^{\sigma'}(\boldsymbol{k}_f+\boldsymbol{p}_f)-\omega_i-E_i}\right.$$

$$\left.+\frac{u^{\sigma_f}(\boldsymbol{p}_f)^*\boldsymbol{\alpha}\boldsymbol{e}_f u^{\sigma'}(\boldsymbol{p}_f-\boldsymbol{k}_i)u^{\sigma'}(\boldsymbol{p}_f-\boldsymbol{k}_i)\boldsymbol{\alpha}\boldsymbol{e}_f\psi^{\sigma'}(\boldsymbol{k}_f+\boldsymbol{p}_f-\boldsymbol{k}_i)}{E^{\sigma}(\boldsymbol{p}_f-\boldsymbol{k}_i)+\omega_f-E_i}\right\}. \tag{35.16′}$$

The transition probability per unit time is equal to

$$dw = \frac{e^4(2\pi)^5}{4\omega_i\omega_f}\delta(E_i+\omega_i-\varepsilon_f-\omega_f)\,|B^{\sigma_i\sigma_f}|^2\,d\boldsymbol{p}_f\,d\boldsymbol{k}_f$$

$$= \frac{e^4(2\pi)^5}{4}\frac{\omega_f}{\omega_i}\delta(E_i+\omega_i-\varepsilon_f-\omega_f)\,|B^{\sigma_i\sigma_f}|^2\,d\boldsymbol{p}_f\,do\,d\omega_f,$$

On dividing $dw$ by the flux density of the incident photons, which is equal to their velocity, and on summing over final, and on averaging over the initial polarizations of the particles, we obtain the averaged scattering cross section

$$d\sigma = \frac{e^4(2\pi)^5}{4} \frac{\omega_f}{\omega_i} \delta(E_i+\omega_i-\varepsilon_f-\omega_f) \sum_{\sigma_i,\sigma_f=1}^{2} \overline{|B^{\sigma_i\sigma_f}|^2}\, d\boldsymbol{p}_f\, do\, d\omega_f, \qquad (35.17)$$

where the bar denotes averaging over all polarizations.

On eliminating the $\delta$-function by integration over the angle between $\boldsymbol{k}_f$ and $\boldsymbol{k}_i$, we obtain the following expression for the averaged cross section expressed per unit energy of the scattered photon:

$$d\sigma = \frac{2\pi e^4}{\omega_1^2} \left(\int \varepsilon_f \sum_{\sigma_i,\sigma_f=1}^{2} \overline{|B^{\sigma_i\sigma_f}|^2}\, d\boldsymbol{p}_i\right) d\omega_f. \qquad (35.17')$$

In order to obtain this formula it is necessary to transform from the variables $\boldsymbol{p}_f$, $\omega_f$ to the variables $\boldsymbol{p}_i = \boldsymbol{k}_1 - \boldsymbol{k}_f - \boldsymbol{p}_f$ and $\delta w = -\varepsilon_f + \omega_f - E_i - \omega_i$ and to evaluate the Jacobian for this transformation for $\delta w = 0$.

Since the photon energy is assumed to be large in comparison with $m$, the principal role will be played by small scattering angles $\vartheta$. Therefore in the evaluation of $\sum_{\sigma_i,\sigma_f=1}^{2} \overline{|B^{\sigma_i\sigma_f}|^2}$ we can set $\vartheta = 0$.

On selecting for $\psi_i$ the wave function of the ground state of the electron in the Coulomb field of the nucleus (cf. § 13) and on expanding $d\sigma$ into a power series in $Z\alpha$, we obtain (38)

$$d\sigma = \pi r_0^2 \frac{m}{\omega_1^2}\left(\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f}\right)\left\{1 + \frac{1}{2}(Z\alpha)^2 + \frac{25}{24}(Z\alpha)^4 + O(1)(Z\alpha)^5 + \frac{(Z\alpha)^5}{\pi} \ln\frac{\omega_i}{m}\left(1 + O(Z^2\alpha^2)\right)\right\} d\omega_f. \qquad (35.18)$$

### 35.4. *Emission of Two Photons. The Metastable $2s_{\frac{1}{2}}$ State of the Hydrogen Atom*

The matrix $S^{(2)}$ defined by formula (35.1) can also be applied to the problem of the simultaneous emission of two photons. The matrix element corresponding to this process can be easily obtained from the scattering matrix element $S^{(2)}_{i\to f}$ defined by formulas (35.11) and (35.12), if we replace in (35.12) the potentials $A_i$ and $A_f$ by $A_1^*$ and $A_2$, and the photon momenta $k_i$ and $k_f$ by $k_1$ and $k_2$. We thus obtain in the dipole approximation the following expression for the matrix element for the emission of two photons of momenta $k_1$ and $k_2$:

$$U_{i\to f} = 2\pi\alpha\sqrt{\omega_1\omega_2}\sum_s\left\{\frac{(\boldsymbol{re}_1)_{fs}(\boldsymbol{re}_2)_{si}}{E_i-E_s-\omega_2}+\frac{(\boldsymbol{re}_2)_{fs}(\boldsymbol{re}_1)_{si}}{E_f-E_s-\omega_2}\right\}. \tag{35.19}$$

The probability of emission of two photons of momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ lying within the ranges $d\boldsymbol{k}_1$ and $d\boldsymbol{k}_2$ is related to $U_{i\to f}$ by the expression

$$dw'_{k_1,k_2} = 2\pi|U_{i\to f}|^2\,\delta(E_i-E_f-\omega_1-\omega_2)\frac{d\boldsymbol{k}_1\,d\boldsymbol{k}_2}{(2\pi)^6}.$$

On eliminating the $\delta$-function by integration over $\omega_2$, we obtain

$$dw'_{k_1,k_2} = \frac{1}{(2\pi)^3}\left|\sum_s\left\{\frac{(\boldsymbol{Q}\boldsymbol{e}_2)_{fs}(\boldsymbol{Q}\boldsymbol{e}_1)_{si}}{E_i-E_s-\omega_1}+\frac{(\boldsymbol{Q}\boldsymbol{e}_1)_{fs}(\boldsymbol{Q}\boldsymbol{e}_2)_{si}}{E_i-E_s-\omega_2}\right\}\right|^2\omega_1^3\omega_2^3\,d\omega_1\,do_1\,do_2, \tag{35.20}$$

where $\boldsymbol{Q}$ is the atomic dipole moment.

On integrating (35.20) over the angles, and on summing over the photon polarizations, we obtain the probability for the emission of two photons of energies $\omega_1$ and $\omega_2 = E_i-E_f-\omega_1$

$$dw_{\omega_1\omega_2} = \frac{8}{9\pi}\sum_{M_1 M_2}\left\{\left|\sum_s\frac{(Q^{(1)*}_{1M_2})_{fs}(Q^{(1)}_{1M_1})_{si}}{E_i-E_s-\omega_1}+\frac{(Q^{(1)*}_{1M_1})_{fs}(Q^{(1)}_{1M_2})_{si}}{E_i-E_s-\omega_2}\right|^2\right\}(\omega_1\omega_2)^3\,d\omega_1, \tag{35.21}$$

where $Q^{(1)}_{1M}$ are the components of the dipole moment defined in § 27.

The probability for the emission of two photons of frequencies $\omega_1$ and $\omega_2$ is usually very small compared to the probability for the emission of a single photon of total frequency $\omega = \omega_1+\omega_2$. An exception exists in the case when the emission of a single photon is not permitted by the selection rules. This occurs, for example, if the angular momentum of the emitting system (an atom or a nucleus) in the initial and in the final states is equal to zero. In this case the probability of emission for a single photon is exactly equal to zero, since the photon has no states of angular momentum zero.

We note that states of angular momentum zero are possible in atomic systems only in the case when they consist of an even number of electrons, since the electron spin is equal to one-half. This conclusion also applies to nuclei, whose angular momentum can be equal to zero only for an even number of nucleons.

For a single electron transitions between states of orbital angular momentum equal to zero are also strongly forbidden.

A typical example in which the emission of two photons is considerably more probable than the emission of a single photon is the $2s_{1/2}$ excited state of hydrogen ($n = 2$, $l = 0$). A transition from this state to the $1s_{1/2}$ ground state ($n = 1$, $l = 0$) is possible with the emission of one magnetic-dipole photon ($L = 1$, $\lambda = 1$). However, in the nonrelativistic approximation the probability of such a transition given by formula (27.22) vanishes in virtue of the orthogonality of the radial wave functions with different principal quantum numbers. A nonvanishing expression for the probability of this transition is obtained only when exact relativistic wave functions are used, and turns out to be of the order of $10^{-7}$ sec$^{-1}$.

The transition $2s_{1/2} \to 1s_{1/2}$ with the emission of two electric dipole photons is considerably more probable. The probability of such a transition may be obtained with the aid of formula (35.21). In this case we should interpret $s$ both as the principal quantum number $n$, and as the magnetic quantum number $m$.

The matrix elements of the dipole moment differ from zero if the difference in the orbital angular momenta of the states, between which it is evaluated, is equal to unity. Since the initial and the final states are both $s$-states, it follows that the "intermediate" states must be $p$-states with three values of the magnetic quantum number $m = -1, 0, 1$, while the principal quantum number in the intermediate state may take on any arbitrary value starting with 2, i.e., $n = 2, 3 \ldots$.

The wave functions for the initial, final, and intermediate states are of the form

$$\psi_i(r) = \frac{1}{\sqrt{4\pi}} R_{10}(r),$$

$$\psi_f(r) = \frac{1}{\sqrt{4\pi}} R_{20}(r),$$

$$\psi_s(r) = R_{n1}(r) Y_{1m}, \qquad m = -1, 0, 1, \; n = 2, 3, \ldots,$$

where $R_{nl}(r)$ is the radial and $Y_{lm}$ is the angular part of the electron wave function. Therefore, the matrix elements of the electric dipole moment

$$Q_{1M}^{(1)} = \frac{e}{\sqrt{4\pi}} \sqrt{\frac{4\pi}{3}}\, r Y_{1M}$$

can be written in the form

$$(Q_{1M}^{(1)})_{fs} = \frac{e}{\sqrt{12\pi}} Q_{2n}\,\delta_{Mm} \quad \text{and} \quad (Q_{1M}^{(1)})_{si} = \frac{e}{\sqrt{12\pi}} Q_{n1}\,\delta_{Mm},$$

where

$$Q_{2n} = \int_0^\infty R_{20}(r) R_{n1}(r) r^3 dr,$$

$$Q_{n1} = \int_0^\infty R_{10}(r) R_{n1}(r) r^3 dr.$$

On substituting these quantities into (35.21) we obtain

$$dw_{\omega_1,\,\omega_2} = \frac{8a^2}{27\pi} \left| \sum_{n=2}^{\infty} Q_{2n} Q_{n1} \left( \frac{1}{E_i - E_n - \omega_2} + \frac{1}{E_i - E_n - \omega_1} \right) \right|^2 (\omega_1 \omega_2)^3 \, d\omega_1 . \tag{35.22}$$

The integrals $Q_{2n}$ and $Q_{n1}$ can be evaluated analytically, while summation over $n$ can be carried out only numerically. The total probability of the transition $2s_{1/2}$—$1s_{1/2}$ turns out to be approximately given by (35)

$$w = \int_{\omega_1 = 0}^{\omega_1 = E_i - E_f} dw_{\omega_1,\,\omega_2} \approx 7 \text{ sec}^{-1}$$

(more accurately it lies between 6.5 and 8.7 sec$^{-1}$).

We note that the long lifetime of the hydrogen atom in the metastable $2s_{1/2}$ state was utilized for the experimental observation of the radiation shift of the hydrogen levels (111).

## CHAPTER VI

# Retarded Interaction between Two Charges

## § 36. Electron-Electron and Positron-Electron Scattering

### 36.1. *Electron-Electron Scattering*

In this chapter we shall investigate processes of interaction between two charges which do not lead to emission or absorption of photons.

We restrict ourselves here to the investigation of such processes in the first nonvanishing approximation. Since the first order scattering matrix $S^{(1)}$ contains only elements corresponding to emission or absorption of a photon, we must investigate the second order matrix $S^{(2)}$.

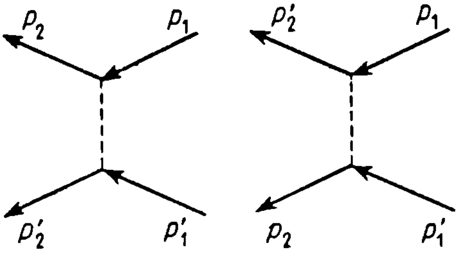


Fig. 47.

The simplest process of this type is a collision of two electrons. It is described by the two diagrams shown in Fig. 47. In accordance with the general rules formulated in § 25 we obtain the following expression for the matrix element:

$$S_{i\to f}^{(2)} = ie^2(2\pi)^4 M\,\delta(p_1+p_1'-p_2-p_2'),$$

where

$$M = \frac{(\bar{u}_2\gamma_\mu u_1)\,(\bar{u}_2'\gamma_\mu u_1')}{(p_2-p_1)^2} - \frac{(\bar{u}_2'\gamma_\mu u_1)\,(\bar{u}_2\gamma_\mu u_1')}{(p_2'-p_1)^2}. \tag{36.1}$$

Here $p_1$ and $p_1'$ are the four-momenta of the initial states of the two electrons, $p_2$ and $p_2'$ are the momenta of their final states; $u_1, u_1', u_2$

and $u_2'$ are the normalized bispinor amplitudes corresponding to these states. The normalization volume is taken equal to unity.

In accordance with the rules obtained in § 26 the expression for the differential cross section has the form

$$d\sigma = \frac{e^4}{(2\pi)^2}|M|^2 d\mathbf{p}_2 d\mathbf{p}_2' \frac{1}{J}\,\delta(\mathbf{p}_1+\mathbf{p}_1'-\mathbf{p}_2-\mathbf{p}_2')\,\delta(\varepsilon_1+\varepsilon_1'-\varepsilon_2-\varepsilon_2'), \qquad (36.2)$$

where $J$ is the particle flux density. In the center of mass system of the two electrons $(\mathbf{p}_1+\mathbf{p}_1' = \mathbf{p}_2+\mathbf{p}_2' = 0)$,

$$J = 2v$$

where $v$ is the velocity of the electrons.

On eliminating the $\delta$-functions from (36.2) we obtain

$$d\sigma = \frac{e^4}{(2\pi)^2}\frac{\mathbf{p}^2}{(2v)^2}|M|^2 do. \qquad (36.3)$$

In the case of unpolarized particles we must average expression (36.3) over the polarizations of the initial state and sum it over the final polarizations, i.e., we must replace $|M|^2$ by $\frac{1}{4}\sum|M|^2$, where $\sum$ denotes summation over all the polarization states. This summation can be carried out with the aid of the method described in § 10 which is not significantly complicated by the fact that the matrix element (36.1) contains four, rather than two, amplitudes.

In analogy with formula (10.40) we obtain

$$\sum |M|^2 = \frac{1}{16\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}\left\{\frac{1}{(p_2-p_1)^4}\,\mathrm{Sp}\,\gamma_\mu(m-i\hat{p}_1)\bar{\gamma}_\nu(m-i\hat{p}_2)\right.$$

$$\times\,\mathrm{Sp}\,\gamma_\mu(m-i\hat{p}_1')\bar{\gamma}_\nu(m-i\hat{p}_2')$$

$$+\frac{1}{(p_2'-p_1)^4}\,\mathrm{Sp}\,\gamma_\mu(m-i\hat{p}_1)\bar{\gamma}_\nu(m-i\hat{p}_2')\,\mathrm{Sp}\,\gamma_\mu(m-i\hat{p}_1')\bar{\gamma}_\nu(m-i\hat{p}_2)$$

$$\left.-\frac{2}{(p_2-p_1)^2(p_2'-p_1)^2}\,\mathrm{Sp}\,\gamma_\mu(m-i\hat{p}_1)\bar{\gamma}_\nu(m-i\hat{p}_2')\gamma_\mu(m-i\hat{p}_1')\bar{\gamma}_\nu(m-i\hat{p}_2)\right\}.$$

In this expression we can replace everywhere $\bar{\gamma}_\mu = \gamma_4\gamma_\mu\gamma_4$ by $\gamma_\mu$. Indeed, we are summing over $\mu$ $(\mu = 1, 2, 3, 4)$, i.e., we are dealing with terms of the type

$$\bar{\gamma}_\mu(\ldots)\bar{\gamma}_\mu = \sum_{\mu=1}^{3}\bar{\gamma}_\mu(\ldots)\bar{\gamma}_\mu+\bar{\gamma}_4(\ldots)\bar{\gamma}_4.$$

But $\bar{\gamma}_\mu = -\gamma_\mu$ $(\mu = 1, 2, 3)$; $\bar{\gamma}_4 = \gamma_4$ and, consequently,

$$\bar{\gamma}_\mu(\ldots)\bar{\gamma}_\mu = \gamma_\mu(\ldots)\gamma_\mu.$$

Moreover in order to simplify this expression we can utilize relations (10.43), as a result of which we obtain a sum of terms each of which contains traces of products of not more than four matrices $\gamma_\mu$. Their values were given in (10.42).

On carrying out the summation we obtain

$$d\sigma = \frac{\alpha^2}{2\varepsilon^2}\,do\left\{\frac{(p_1p_1')^2+(p_1p_2')^2+2m^2(p_1p_1')-2m^2(p_1p_2')}{(p_2-p_1)^4}\right.$$

$$\left.+\frac{(p_1p_1')^2+(p_1p_2)^2+2m^2(p_1p_1')-2m^2(p_1p_2)}{(p_2'-p_1)^4}+2\frac{(p_1p_1')^2+2m^2(p_1p_1')}{(p_2-p_1)^2(p_2'-p_1)^2}\right\}$$

or

$$d\sigma = r_0^2\frac{[2(\varepsilon^2/m^2)-1]^2}{4v^4(\varepsilon/m)^6}\left\{\frac{4}{\sin^4\vartheta}-\frac{3}{\sin^2\vartheta}\right.$$

$$\left.+\frac{([\varepsilon^2/m^2]-1)^2}{(2[\varepsilon^2/m^2]-1)^2}\left(1+\frac{4}{\sin^2\vartheta}\right)\right\}do, \qquad (36.4)$$

where $v$, $\varepsilon$ and $\vartheta$ are the velocity, the energy and the scattering angle of the electron in the center of mass system, $r_0 = \alpha/m$ is the classical electron radius.

In the nonrelativistic limit $v \ll 1$, $\varepsilon/m \approx 1$ and, as can be easily seen, (36.4) goes over into the Rutherford formula with exchange taken into account (139)

$$d\sigma = \frac{r_0^2}{16v^4}\left[\frac{1}{\sin^4(\vartheta/2)}+\frac{1}{\cos^4(\vartheta/2)}-\frac{1}{\sin^2(\vartheta/2)\cos^2(\vartheta/2)}\right]do. \qquad (36.5)$$

We express $v$ and $\varepsilon$ in (36.4) in terms of the velocity $v_1$ and the energy $\varepsilon_1$ of the incident electron in the coordinate system in which one of the electrons is at rest before the collision $(v_1' = 0)$. From the invariance of the square of the total momentum $p^2$ of the system it follows directly that

$$\frac{\varepsilon}{m} = \sqrt{\frac{1}{2}\left(\frac{\varepsilon_1}{m}+1\right)}.$$

Thus we have (137)

$$d\sigma = 2r_0^2\frac{\frac{\varepsilon_1}{m}+1}{v_1^4(\varepsilon_1/m)^2}\left\{\frac{4}{\sin^4\vartheta}-\frac{3}{\sin^2\vartheta}+\frac{\left(\frac{\varepsilon_1}{m}-1\right)^2}{4(\varepsilon_1/m)^2}\left(1+\frac{4}{\sin^2\vartheta}\right)\right\}do, \qquad (36.6)$$

where $\vartheta$ is the scattering angle given, as before, in the center of mass system. It is related to the scattering angle $\theta$ in the laboratory system of coordinates by

$$\cos\vartheta = \left(2-\left(\frac{\varepsilon_1}{m}+3\right)\sin^2\theta\right)\left(2+\left(\frac{\varepsilon_1}{m}-1\right)\sin^2\theta\right)^{-1}$$

In the nonrelativistic limit $\vartheta = 2\theta$ and

$$d\sigma = 4\left(\frac{r_0}{v_1^2}\right)^2\left[\frac{1}{\sin^4\theta}+\frac{1}{\cos^4\theta}-\frac{1}{\cos^2\theta\sin^2\theta}\right]\cos\theta\, do. \qquad (36.7)$$

The energy lost by the electron as a result of its collision with an electron at rest is uniquely determined by the scattering angle in the center of mass system. We denote by $\Delta$ the ratio of the energy transferred from one electron to the other one, which was initially at rest, to the kinetic energy of the first electron: $\Delta = (\varepsilon_1-\varepsilon_2)/(\varepsilon_1-m) = (\varepsilon_2'-m)/(\varepsilon_1-m)$. From the laws of conservation of energy and momentum it follows that $\Delta = \frac{1}{2}(1-\cos\vartheta)$. On substituting this expression into (36.6), we obtain

$$d\sigma = \frac{2\pi r_0^2}{v_1^2(x-1)}\frac{d\Delta}{\Delta^2(1-\Delta)^2}\left\{1-\left[3-\left(\frac{x-1}{x}\right)^2\right]\Delta(1-\Delta) + \left(\frac{x-1}{x}\right)^2\Delta^2(1-\Delta)^2\right\}, \qquad (36.8)$$

where $x = \varepsilon_1/m$.

Formula (36.8) gives the energy distribution of the secondary electrons arising as a result of the passage of a fast electron through matter. For small values of $\Delta$

$$d\sigma = 4\pi r_0^2\frac{1}{v_1^4}\frac{d\Delta}{\Delta^2}. \qquad (36.9)$$

### 36.2. *Positron-Electron Scattering*

Formula (36.1) for the matrix element can also be applied to positron-electron collisions. In this case we must make the following substitution:

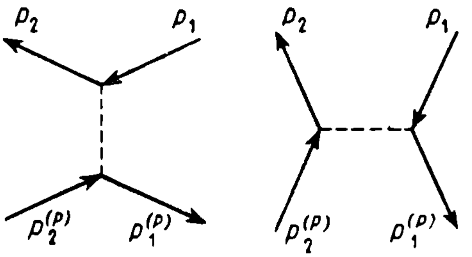
$$u_1' \to v_2, \quad p_1' \to -p_2^{(p)},$$

$$u_2' \to v_1, \quad p_2' \to -p_1^{(p)},$$

where $p_1^{(p)}$ and $p_2^{(p)}$ are the initial and the final four-momenta of the positron and $v_1$ and $v_2$ are the amplitudes of the corresponding wave functions with negative frequencies. Thus we have

$$-M = \frac{(\bar{u}_2\gamma_\mu u_1)(\bar{v}_1\gamma_\mu v_2)}{(p_2-p_1)^2} - \frac{(\bar{v}_1\gamma_\mu u_1)(u_2\gamma_\mu v_2)}{(p_1^{(p)}+p_1)^2}.$$

The diagrams which describe the scattering of a positron by an electron are shown in Fig. 48. The diagram of Fig. 48*b* describes the virtual annihilation of an electron-positron pair with momenta $p_1$ and $p_1^{(p)}$. This pair is converted into a virtual photon of momentum $k = p_1+p_1^{(p)}$ which again creates a pair of momenta $p_2$ and $p_2^{(p)}$.



(*a*) (*b*)

Fig. 48.

The evaluation of $\sum|M|^2$ can be carried out in the same way as in the case of two electrons. We give here the result of this calculation (24):

$$d\sigma = \frac{r_0^2}{16(\varepsilon/m)^2}\left\{\frac{1}{[(\varepsilon/m)^2-1]^2\sin^4\frac{\vartheta}{2}}\left[1+4\left(\frac{\varepsilon^2}{m^2}-1\right)\cos^2\frac{\vartheta}{2}\right.\right.$$
$$\left.+2\left(\frac{\varepsilon^2}{m^2}-1\right)^2\left(1+\cos^4\frac{\vartheta}{2}\right)\right]+\left(\frac{m}{\varepsilon}\right)^4\left[3+4\left(\frac{\varepsilon^2}{m^2}-1\right)\right.$$
$$\left.+\left(\frac{\varepsilon^2}{m^2}-1\right)^2(1+\cos^2\vartheta)\right]-\frac{m^4}{\varepsilon^2(\varepsilon^2-m^2)\sin^2\frac{\vartheta}{2}}\left[3+4\left(\frac{\varepsilon^2}{m^2}-1\right)(1+\cos\vartheta)\right.$$
$$\left.\left.+\left(\frac{\varepsilon^2}{m^2}-1\right)^2(1+\cos\vartheta)^2\right]\right\}do, \qquad (36.10)$$

where $\varepsilon$ and $\vartheta$ are the energy and the scattering angle in the center of mass system.

In the nonrelativistic limit (36.10) reduces to the first term of formula (36.5), i.e., to the Rutherford formula without exchange. The exchange (i.e., the annihilation) diagram (Fig. 48*b*) makes a small contribution in the nonrelativistic limit.

For the energy distribution of the secondary electrons we obtain

$$\frac{d\sigma}{d\Delta} = 2\pi r_0^2 \frac{x}{(x-1)^2 \Delta^2} F(x; \Delta), \tag{36.11}$$

$$\begin{aligned} F(x; \Delta) &= \frac{1}{x(x+1)} \left\{ \left[ 1+2(x-1)(1-\Delta)+(x-1)^2\left(1-\Delta+\frac{\Delta^2}{2}\right) \right] \right. \\ &\quad + \frac{(x-1)^2 \Delta^2}{(x+1)^2} \left[ 3+2(x-1)+(x-1)^2\left(\frac{1}{2}-\Delta+\Delta^2\right) \right] \\ &\quad \left. - \frac{(x-1)\Delta}{(x+1)} [3+4(x-1)(1-\Delta)+(x-1)^2(1-\Delta)^2] \right\}, \end{aligned}$$

where

$$x = \frac{\varepsilon_1^{(p)}}{m} \quad \text{and} \quad \Delta = \frac{\varepsilon_2 - m}{\varepsilon_1^{(p)} - m}$$

(the electron is at rest before the collision). For $\Delta \ll 1$ we obtain formula (36.9).

### 36.3. *Scattering of Polarized Electrons and Positrons*

We now investigate the scattering of polarized particles. The difference between these scattering cross sections and the formulas given earlier for the case of unpolarized beams and unpolarized targets can be utilized for measuring the degree of polarization of the particles.

We have seen (cf. §14) that in the case of scattering of electrons by an external field there is no polarization effect in the Born approximation. Similarly, in the cases of electron-electron or positron-electron scattering the unpolarized beam can become polarized only in the case when the target is polarized, while the cross section will differ from the formulas obtained previously only when both the beam and the target are polarized.

We assume that the colliding electrons are characterized by the polarizations $\zeta^{(1)}$ and $\zeta^{(2)}$. By employing the general methods for the evaluation of cross sections in the case of polarized particles described in § 26 and already utilized in Chapter V, we can obtain the expression for the differential cross section

$$d\sigma = d\sigma_0\left(1+\frac{T_{ik}}{\tau}\zeta_i^{(1)}\zeta_k^{(2)}\right), \tag{36.12}$$

where $d\sigma_0$ is the scattering cross section for unpolarized electrons determined by formula (36.4), while $T_{ik}$ is the tensor whose components in the coordinate system with axes parallel to the vectors $\boldsymbol{p}_1$, $[\boldsymbol{p}_1\boldsymbol{p}_1']$ $[\boldsymbol{p}_1[\boldsymbol{p}_1\boldsymbol{p}_1']]$, have the values (108)

$$\begin{aligned}
T_{11} &= 4x(2x-1)-(x-1)(x+3)\sin^2\vartheta,\\
T_{22} &= 4x+(x-1)(x+3)\sin^2\vartheta,\\
T_{33} &= 4(2x-1)-(x-1)^2\sin^2\vartheta,\\
T_{12} &= T_{21} = (x-1)\sqrt{2(x+1)}\sin 2\vartheta,\\
T_{31} &= T_{13} = T_{23} = T_{32} = 0,\\
\tau &= \frac{4x^2(1+3\cos^2\vartheta)+(x-1)^2(4+\sin^2\vartheta)\sin^2\vartheta}{\sin^2\vartheta},
\end{aligned}$$

where, as in the preceding cases, $\vartheta$ is the scattering angle in the center of mass system, $x = \varepsilon_1/m$, $\varepsilon_1$ is the electron energy in the system in which the target is at rest.

In the nonrelativistic limit

$$d\sigma = d\sigma_0\left(1-\frac{\sin^2\vartheta}{1+3\cos^2\vartheta}\boldsymbol{\zeta}^{(1)}\boldsymbol{\zeta}^{(2)}\right). \tag{36.13}$$

For $\vartheta = \pi/2$ the cross section vanishes in the case of complete polarization and when the two electrons have the same spin orientation. In the extreme relativistic case

$$d\sigma = d\sigma_0\left\{1-\frac{\sin^2\vartheta}{(3+\cos^2\vartheta)^2}\left[(8-\sin^2\vartheta)\zeta_1^{(1)}\zeta_1^{(2)} + \sin^2\vartheta\left(\zeta_2^{(1)}\zeta_2^{(2)}-\zeta_3^{(1)}\zeta_3^{(2)}\right)\right]\right\}. \tag{36.14}$$

We see from the last formula that if, for example, the colliding electrons are completely polarized either parallel or antiparallel to the direction of the original beam, the cross section depends very strongly on the relative orientation of their spins. For example, for $\vartheta = \pi/2$

$$\frac{d\sigma(\zeta_1^{(1)} = 1,\ \zeta_1^{(2)} = -1)}{d\sigma(\zeta_1^{(1)} = 1,\ \zeta_1^{(2)} = 1)} = 8.$$

Figure 49 shows the dependence of the coefficients $T_{ik}/\tau$ on $x$ for $\vartheta = \pi/2$ in the case of electron-electron scattering.

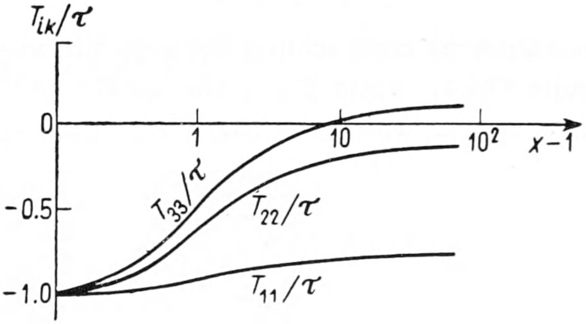


Fig. 49.

For the case of positron-electron scattering we write the cross section also in the form (36.12), where $\boldsymbol{\zeta}^{(1)}$ and $\boldsymbol{\zeta}^{(2)}$ are respectively the polarization vectors for the positron and the electron, while $d\sigma_0$ is the cross section for unpolarized particles determined by formula (36.10). The components of $T_{ik}$ in the same coordinate system have the following values:

$$\begin{aligned}
T_{11} &= (x+1)^2(7x+1)+(x+1)(7x^2-4x+5)\cos\vartheta \\
&\qquad +(x-1)(x+3)^2\cos^2\vartheta+(x-1)^2(x+3)\cos^3\vartheta, \\
T_{22} &= (x+1)^2(x+7)+(x+1)(x^2+4x-13)\cos\vartheta \\
&\qquad -(x-1)(x+3)^2\cos^2\vartheta-(x-1)^2(x+3)\cos^3\vartheta, \\
T_{33} &= (x+1)(17+8x-x^2)-(x^2-1)(x-11)\cos\vartheta \\
&\qquad +(x-1)^2(x+3)\cos^2\vartheta+(x-1)^3\cos^3\vartheta, \\
T_{12} &= T_{21} = 2\sqrt{2(x+1)}\sin\vartheta[2(x+1) \\
&\qquad +(x-1)(x+3)\cos\vartheta+(x-1)^2\cos^2\vartheta], \\
T_{31} &= T_{13} = T_{23} = T_{32} = 0, \\
\tau &= \frac{1}{(x-1)(1-\cos\vartheta)}[(x+1)(9x^3-x^2-x+25) \\
&\qquad +4(x^2-1)(3x-11)\cos\vartheta+2(x-1)^2(3x^2+12x+11)\cos^2\vartheta \\
&\qquad +4(x-1)^3\cos^3\vartheta+(x-1)^4\cos^4\vartheta].
\end{aligned}$$

In the nonrelativistic limit $d\sigma$ coincides with $d\sigma_0$, i.e., the cross section does not depend on the polarization of the particles (this dependence in the case of electron-electron scattering is associated only with the exchange effect). In the extreme relativistic limit the dependence of the cross section on the polarization differs only by its sign from the corresponding dependence in the case of electron-electron scattering (36.14)

$$d\sigma = d\sigma_0 \left\{ 1 + \frac{\sin^2\theta}{(3+\cos^2\theta)^2} \left[ (8-\sin^2\vartheta)\, \zeta_1^{(1)}\zeta_1^{(2)} + \sin^2\vartheta\, (\zeta_2^{(1)}\zeta_2^{(2)} - \zeta_3^{(1)}\zeta_3^{(2)}) \right] \right\}. \qquad (36.14')$$

Figure 50 shows the dependence of the coefficients $T_{ik}/\tau$ on $x$ for $\vartheta = \pi/2$ in the case of positron-electron scattering.

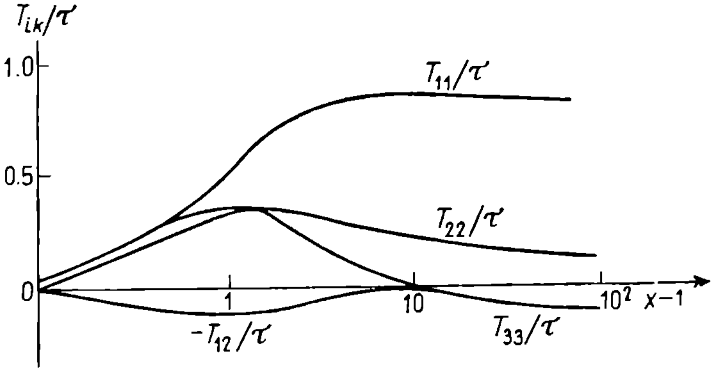


Fig. 50.

If the incident electron beam is unpolarized ($\boldsymbol{\zeta}^{(1)} = 0$) then in the case of scattering by polarized electrons the incident electrons acquire a polarization $\boldsymbol{\zeta}^{(1)\prime}$ related linearly to the polarization of the target electrons $\boldsymbol{\zeta}^{(2)}$

$$\zeta_i^{(1)\prime} = M_{ik}\zeta_k^{(2)}. \qquad (36.15)$$

We confine ourselves to stating the values of $M_{ik}$ in limiting cases. In the nonrelativistic case we have

$$M_{ik} = -\frac{(1-\cos\vartheta)\cos\vartheta}{2(1+3\cos^2\vartheta)}\delta_{ik}.$$

In the case of positron-electron scattering in the nonrelativistic limit $M_{ik} = 0$.

In the extreme relativistic case for electron-electron scattering the coefficients $M_{ik}$ are equal to

$$M_{11} = -\frac{(1-\cos\vartheta)}{(3+\cos^2\vartheta)^2}(2+\cos\vartheta+\cos^2\vartheta),$$

$$M_{22} = -M_{33} = -\frac{(1-\cos\vartheta)^2}{(3+\cos^2\vartheta)^2},$$

$$M_{ik} = 0 \quad (i \neq k).$$

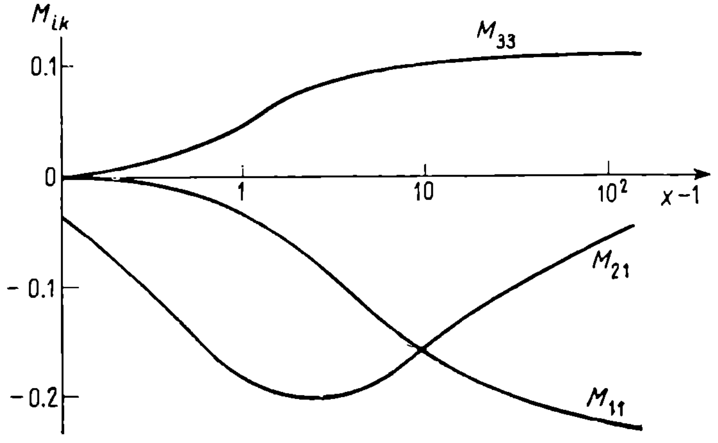


Fig. 51.

In the case of positrons the coefficients $M_{ik}$ are of the opposite sign. Figure 51 shows the dependence of $M_{ik}$ on $x$ for $\vartheta = \pi/2$ in the case of electron-electron scattering.

### 36.4. *Annihilation of an Electron-Positron Pair into a μ-Meson Pair*

As is well known, the $\mu$-meson which has an electronic charge and spin $\frac{1}{2}$ does not participate in interactions stronger than the electromagnetic interactions, i.e., in nuclear interactions. Therefore, all the electrodynamic formulas obtained for the case of electrons are also valid for $\mu$-mesons.

We can also investigate the interaction between electrons and $\mu$-mesons. It will differ from the interaction between electrons by the absence of the exchange terms, i.e., of the second diagrams in Figs. 47 or 48.

The diagram of Fig. 48*b* can be used to describe the peculiar process of the annihilation of an electron-positron pair resulting in the creation of a $\mu$-meson pair.

The minimum energy which the positron must have in order to create such a pair in a collision with an electron at rest is equal to

$$\varepsilon_0 = \left(\frac{2\mu}{m}\right)\mu = 4\cdot 10^{10}\,\mathrm{eV},$$

where $\mu$ is the $\mu$-meson mass.

We give the expression for the total effective cross section for this process (14)

$$\sigma = \frac{2\pi}{3} r_0^2 \frac{m}{\varepsilon}\left(1+\frac{\varepsilon_0}{2\varepsilon}\right)\left(1-\frac{\varepsilon_0}{\varepsilon}\right)^{1/2} \tag{36.16}$$

The maximum cross section (at $\varepsilon \approx 1.7\,\varepsilon_0$) is less by a factor of approximately 20 than the cross section for two-photon annihilation at the same energy.

## § 37. Retarded Potentials

### 37.1. *Interaction Function for Two Charges*

In the preceding section we have utilized the expression for the matrix element in the momentum representation for the scattering of one charge by another. We shall now investigate this matrix element in the coordinate representation. This will enable us to investigate the interaction between two charges in the presence of external fields, and also to investigate the bound states of the electron-positron system (positronium).

In accordance with the results of § 23 the general expression for the second order scattering matrix $\mathrm{S}^{(2)}$ has the form:

$$\mathrm{S}^{(2)} = -\frac{1}{2}\int T[\mathrm{j}_\mu(x)\mathrm{j}_\nu(y)]\,T[\mathrm{A}_\mu(x)\mathrm{A}_\nu(y)]\,d^4x\,d^4y, \tag{37.1}$$

where $T$ is the chronological operator, j is the current operator, $\mathrm{j}_\mu = ieN(\bar{\psi}\gamma_\mu\psi)$ ($N$ is the symbol for the normal product), A is the operator for the potential.

We denote the set of elements of this matrix between states containing no photons by ${}^e\mathrm{S}^{(2)}$. We shall obtain the expression for ${}^e\mathrm{S}^{(2)}$ from (37.1) if we replace in (37.1) the operator $T[\mathrm{A}_\mu(x)\mathrm{A}_\nu(y)]$ by its expectation value for the vacuum state of the electromagnetic field:

$\langle T[\mathrm{A}_\mu(x)\mathrm{A}_\nu(y)]\rangle_0$. In accordance with (17.10) this expectation value is equal to

$$\langle T[\mathrm{A}_\mu(x)\mathrm{A}_\nu(y)]\rangle_0 = \delta_{\mu\nu} D^c(x-y),$$

where the function $D^c$ is defined by formulas (17.16), (17.18).

The operator $\mathrm{j}_\mu(x)\mathrm{j}_\mu(y)$ consists of products of four electron (or positron) creation or annihilation operators. In accordance with the foregoing we shall be interested in those matrix elements of $\mathrm{S}^{(2)}$ which correspond to the presence of two particles both in the initial and in the final states. In this case results different from zero can come only from those terms of the operator $\mathrm{j}_\mu(x)\mathrm{j}_\mu(y)$ which contain two creation operators and two annihilation operators, with all the operators referring to different individual electron states. Therefore, all these four factors anticommute with each other, while their products in pairs commute, i.e., $\mathrm{j}_\mu(x)\mathrm{j}_\nu(y) = \mathrm{j}_\nu(y)\mathrm{j}_\mu(x)$, and the operator $T$ may be omitted from (37.1).

Thus, the scattering matrix for two charges is of the form

$$^e\mathrm{S}^{(2)} = -\frac{1}{2}\int\int \mathrm{j}_\mu(y) D^c(y-x)\mathrm{j}_\mu(x)\, d^4y\, d^4x. \tag{37.2}$$

We note the following fundamental fact in connection with the structure of expression (37.2). In nonrelativistic quantum mechanics interaction between particles is described by the potential energy of interaction which is a function of the distance between the particles. The operator for the potential energy U (in the occupation number representation) has the form

$$\mathrm{U} = \frac{1}{2}\int \varrho(\boldsymbol{r}_1) U(\boldsymbol{r}_1, \boldsymbol{r}_2)\varrho(\boldsymbol{r}_2)\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2,$$

where $\varrho = \psi^*\psi$ is the operator for the particle density.

In electrodynamics there exists no potential energy of interaction which depends on the coordinates of the interacting particles taken at same instant of time, since the electrons interact not directly, but via the electromagnetic field. The operator for the interaction energy is replaced in the first approximation by the scattering matrix $^e\mathrm{S}^{(2)}$ defined by formula (37.2) which plays an analogous role. The integrand in (37.2) contains the coordinates (however, including also the time coordinate) of only the electrons, while the electromagnetic field has been eliminated from this formula. Therefore, the function $e^2D^c(y-x)$ may be referred to as the interaction function for two charges.

We shall see later (cf. § 38) that in the nonrelativistic approximation it does, in fact, lead to the expression for the energy of interaction between two charges, i.e., to Coulomb's law and to corrections to this law.

### 37.2. *General Form of the Matrix Element*

We construct the general expression for the matrix element of the operator ${}^{e}\mathrm{S}^{(2)}$. We first consider the case when both particles are electrons.

Let $A$ and $B$ denote the sets of quantum numbers of the initial electron states, and $C$ and $D$ the sets of quantum numbers of the final states. Then in the operator

$$\mathrm{j}_\mu(x)\mathrm{j}_\mu(y) = -e^2 N[\overline{\psi}(x)\gamma_\mu\psi(x)]N[\overline{\psi}(y)\gamma_\mu\psi(y)]$$

nonzero values will come only from

$$\begin{aligned} -\frac{1}{e^2}\mathrm{j}_\mu(x)\mathrm{j}_\mu(y) \to\ & \mathrm{a}_C^+\mathrm{a}_A\mathrm{a}_D^+\mathrm{a}_B\big(\overline{\psi}_C(x)\gamma_\mu\psi_A(x)\big)\big(\overline{\psi}_D(y)\gamma_\mu\psi_B(y)\big) \\ & +\mathrm{a}_D^+\mathrm{a}_B\mathrm{a}_C^+\mathrm{a}_A\big(\overline{\psi}_C(y)\gamma_\mu\psi_A(y)\big)\big(\overline{\psi}_D(x)\gamma_\mu\psi_B(x)\big) \\ & +\mathrm{a}_C^+\mathrm{a}_B\mathrm{a}_D^+\mathrm{a}_A\big(\overline{\psi}_C(x)\gamma_\mu\psi_B(x)\big)\big(\overline{\psi}_D(y)\gamma_\mu\psi_A(y)\big) \\ & +\mathrm{a}_D^+\mathrm{a}_A\mathrm{a}_C^+\mathrm{a}_B\big(\overline{\psi}_C(y)\gamma_\mu\psi_B(y)\big)\big(\overline{\psi}_D(x)\gamma_\mu\psi_A(x)\big), \end{aligned} \qquad (37.3)$$

where $\mathrm{a}^+$ and a are the creation and annihilation operators for electrons in corresponding states, while $\psi_A$ are the corresponding solutions of the Dirac equation.

By utilizing the fact that all the operators appearing in this expression anticommute we can arrange them in the form of the normal product $\mathrm{a}_C^+\mathrm{a}_D^+\mathrm{a}_B\mathrm{a}_A$, whose matrix element is equal to unity.

We can then pick out the operator product in (37.3) as a common factor, with the first two terms appearing with a plus sign, while the second two appear with a minus sign. We note further that the first term in (37.3) differs from the second only by the interchange of the arguments $(x)$ and $(y)$. On substitution into the integral (37.2) they will yield identical results since the function $D^c(x-y)$ does not change when the sign of its argument is reversed. The third and the fourth terms in (37.3) also differ from each other only by an interchange of arguments.

Thus, we obtain the following expression for the matrix element ${}^{e}S^{(2)}_{i\to f}$:

$$
\begin{aligned}
{}^{e}S^{(2)}_{i\to f} = e^2 \int \{&\bar{\psi}_D(y)\gamma_\mu\psi_B(y)D^c(y-x)\bar{\psi}_C(x)\gamma_\mu\psi_A(x) \\
&-\bar{\psi}_C(y)\gamma_\mu\psi_B(y)D^c(y-x)\bar{\psi}_D(x)\gamma_\mu\psi_A(x)\}\, d^4y\, d^4x. \qquad (37.4)
\end{aligned}
$$

In accordance with the general rules for graphical representation of matrix elements the two terms of (37.4) correspond to the two diagrams shown in Fig. 52. They differ only by the interchange of the indices of the final states. Therefore, the second term in (37.4) is said to be the exchange term with respect to the first one, and the second diagram of Fig. 52 is said to be the exchange diagram with respect to the first one. The minus sign of the exchange term in (37.4) corresponds to the antisymmetry of the two-electron wave function in configuration space.

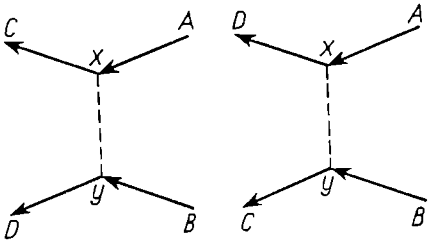


Fig. 52.

We write the matrix element (37.4) in the form

$$
{}^{e}S^{(2)}_{i\to f} = S^{(2)}_{AB;\,CD} - S^{(2)}_{AB;\,DC}, \qquad (37.5)
$$

where

$$
S^{(2)}_{AB;\,CD} = e^2 \int \bar{\psi}_D(y)\gamma_\mu\psi_B(y)D^c(y-x)\bar{\psi}_C(x)\gamma_\mu\psi_A(x)\,d^4y\,d^4x, \qquad (37.6)
$$

while $S^{(2)}_{AB;\,DC}$ differs from $S^{(2)}_{AB;\,CD}$ by the interchange of the indices $C$ and $D$.

We now proceed to investigate the interaction between two positrons. In this case $\psi$ contains creation operators, while $\bar{\psi}$ contains annihilation operators. Therefore, if we denote (in contrast to the electron case) the initial states by $C$ and $D$, and the final states $A$ and $B$, we obtain a formula which differs from (37.3) by the replacement of the electron operators $\mathrm{a}_A$, $\mathrm{a}_B$, $\mathrm{a}_C^+$, $\mathrm{a}_D^+$ by the corresponding positron operators $\mathrm{b}_A^+$, $\mathrm{b}_B^+$, $\mathrm{b}_C$, $\mathrm{b}_D$, and also by the fact that the wave functions $\psi$ now denote the corresponding solutions of the Dirac equation with negative frequencies. On going over from wave functions with negative frequencies to positron wave functions, and on taking into account that (cf. (10.8)) $\overline{\psi}_1^{(-)}\gamma_\mu\psi_2^{(-)} = \psi_2^{(\mathrm{p})}\gamma_\mu\psi_1^{(\mathrm{p})}$ we obtain the formula for the matrix element which defines the interaction between two positrons

$$
{}^eS^{(2)}_{i\to f} = e^2\int\{\psi_B^{(\mathrm{p})}(y)\gamma_\mu\psi_D^{(\mathrm{p})}(y)D^c(y-x)\overline{\psi}_A^{(\mathrm{p})}(x)\gamma_\mu\psi_C^{(\mathrm{p})}(x)
$$

$$
-\overline{\psi}_A^{(\mathrm{p})}(y)\gamma\psi_D^{(\mathrm{p})}(y)D^c(y-x)\overline{\psi}_B^{(\mathrm{p})}(x)\gamma_\mu\psi_C^{(\mathrm{p})}(x)\}\,d^4y\,d^4x.
$$

Naturally, this formula differs from formulas (37.4) only by the relabelling of the wave functions of the initial and the final states of the particles.

In the case when one of the particles is an electron, and the other one is a positron, we shall use the following notation: $A$ is the initial electron state, $C$ is the final electron state, $B$ is the final positron state, $D$ is the initial positron state. In expression (37.3) we must at the same time replace $\mathrm{a}_D^+$ by $\mathrm{b}_D$ and $\mathrm{a}_B$ by $\mathrm{b}_B^+$.

On bringing once again the product of the operators a and b to the form of the normal product $\mathrm{a}_C^+\mathrm{a}_A\mathrm{b}_D\mathrm{b}_B^+ = \mathrm{a}_C^+\mathrm{b}_B^+\mathrm{a}_A\mathrm{b}_D$, we obtain

$$
{}^eS^{(2)}_{i\to f} = -(S^{(2)}_{AB;CD} - S^{(2)}_{AB;DC}). \tag{37.7}
$$

The matrix element (37.7) differs from (37.5) by its sign, and this corresponds to the electron and the positron having charges of opposite sign. The exchange term in (37.7) (the second diagram of Fig. 52) now corresponds to virtual annihilation and subsequent creation of an electron-positron pair.

We shall also apply the fundamental formula (37.2) to the interaction between an electron and a proton. In this case $\mathrm{j}(x)$ can denote the electron current, while $\mathrm{j}(y)$ denotes the proton current, or vice versa. Let the states $A$ and $C$ refer to the electron, and $B$ and $D$ refer to the proton. In the matrix elements of $\mathrm{j}_\mu(x)\,\mathrm{j}_\mu(y)$ the exchange terms will be absent, i.e., in place of (37.3) we shall obtain

$$
\mathrm{j}_\mu(x)\,\mathrm{j}_\mu(y) = 2e^2\mathrm{a}_C^+\mathrm{a}_D^+\mathrm{a}_B\mathrm{a}_A\big(\overline{\psi}_C(x)\,\gamma_\mu\psi_A(x)\big)\big(\overline{\psi}_D(y)\,\gamma_\mu\psi_B(y)\big)
$$

and, consequently,

$$
S^{(2)}_{i\to f} = -S^{(2)}_{AB:CD}, \tag{37.8}
$$

where $S^{(2)}_{AB;CD}$ is determined by formula (37.6), as before. The minus sign corresponds to the opposite signs of the electron and proton charge. The matrix element (37.8) is represented by the first diagram of Fig. 52.

### 37.3. *Retarded Potentials and Transition Currents*

The matrix element $S^{(2)}_{AB:CD}$ can be written in a form which is very convenient for applications, and which permits a simple physical interpretation. In order to do this we shall explicitly write the time dependent terms in the wave functions appearing in (37.6) $\psi_A(x) = \psi_A(\boldsymbol{r}_1)e^{-i\omega_A t_1}$ and similarly for $\psi_B$, $\psi_C$ and $\psi_D$. Furthermore, we shall substitute for $D^c(y-x)$ expressions (17.16), (17.18):

$$D^c(y-x) = \frac{-i}{(2\pi)^4}\int \frac{e^{i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)-i\omega(t_1-t_2)}}{k^2-\omega^2}\,d\boldsymbol{k}\,d\omega .$$

As a result of this we obtain

$$S^{(2)}_{AB:CD} = \frac{-ie^2}{(2\pi)^4}\int \bar{\psi}_C(\boldsymbol{r}_1)\,\gamma_\mu\psi_A(\boldsymbol{r}_1)\,d\boldsymbol{r}_1\int \bar{\psi}_D(\boldsymbol{r}_2)\,\gamma_\mu\psi_B(\boldsymbol{r}_2)\,d\boldsymbol{r}_2$$
$$\times\int e^{i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)}\,d\boldsymbol{k}\int\frac{d\omega}{k^2-\omega^2}\int e^{i(\omega_C-\omega_A-\omega)t_1}\,dt_1\int e^{i(\omega_D-\omega_B+\omega)t_2}\,dt_2 . \tag{37.9}$$

We first of all carry out the integration over $t_2$:

$$\int e^{i(\omega_D-\omega_B+\omega)t_2}\,dt_2 = 2\pi\delta(\omega_D-\omega_B+\omega);$$

the integration over $\omega$ now reduces to the replacement of $\omega$ by $\omega_{BD}\equiv\omega_B-\omega_D$. We next carry out the integration over $k$. Integration over the angles yields:

$$\int\frac{e^{i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)}}{k^2-\omega}\,d\boldsymbol{k} = \frac{4\pi}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\int_0^\infty\frac{\sin\varkappa|\boldsymbol{r}_1-\boldsymbol{r}_2|}{\varkappa^2-\omega^2}\,\varkappa\,d\varkappa .$$

Integration over $\varkappa$ must be carried out taking into account the rule for going around the poles of the integrand formulated in § 17, i.e., replacing $k^2$ by $k^2-i0$; this yields

$$\int_0^\infty\frac{\sin\varkappa|\boldsymbol{r}_1-\boldsymbol{r}_2|}{\varkappa^2-\omega^2}\,\varkappa\,d\varkappa = \frac{\pi}{2}\,e^{i|\omega|\,|\boldsymbol{r}_1-\boldsymbol{r}_2|} .$$

Thus,

$$D^c(y-x) = \frac{-i}{8\pi^2}\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)+i|\omega|\,|\mathbf{r}_1-\mathbf{r}_2|}\,d\omega. \tag{37.10}$$

Substituting this expression into (37.9), and introducing for the sake of brevity the notation for the "transition currents" (here and subsequently the vector indices on $y$, $j$, and $\gamma$ have been omitted,)

$$\begin{aligned} j_{CA}(x) &= ie\bar{\psi}_C(x)\gamma\psi_A(x) = j_{CA}(\mathbf{r}_1)e^{i\omega_{CA}t}, \\ j_{DB}(y) &= ie\bar{\psi}_D(y)\gamma\psi_B(y) = j_{DB}(\mathbf{r}_2)e^{i\omega_{DB}t}, \end{aligned} \tag{37.11}$$

where

$$\begin{aligned} j_{CA}(\mathbf{r}_1) &= ie\bar{\psi}_C(\mathbf{r}_1)\gamma\psi_A(\mathbf{r}_1), \\ j_{DB}(\mathbf{r}_2) &= ie\bar{\psi}_D(\mathbf{r}_2)\gamma\psi_B(\mathbf{r}_2), \end{aligned}$$

we obtain the expression for the matrix element $S^{(2)}_{AB;CD}$

$$S^{(2)}_{AB;CD} = \frac{i}{4\pi}\int j_{CA}(\mathbf{r}_1,t)j_{DB}(\mathbf{r}_2,t)\frac{e^{i|\omega_{BD}|\,|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}\,d\mathbf{r}_1\,d\mathbf{r}_2\,dt \tag{37.12}$$

or, on carrying out the integration over $t$,

$$S^{(2)}_{AB;CD} = \frac{i}{2}\int j_{CA}(\mathbf{r}_1)\frac{e^{i|\omega_{BD}|\,|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}j_{DB}(\mathbf{r}_2)\,d\mathbf{r}_1\,d\mathbf{r}_2, \tag{37.12'}$$

with $\omega_{CA} = -\omega_{DB}$, i.e., $\omega_C-\omega_A = \omega_B-\omega_D$, which expresses the law of conservation of energy. For the sake of definiteness we take $\omega_A > \omega_C$, then $\omega_B < \omega_D$ and $\omega_{AC} = -\omega_{BD} > 0$. On taking this condition into account we can rewrite (37.12) in the form

$$S^{(2)}_{AB;CD} = i\int dt\int j_{DB}(\mathbf{r}_2,t)A_{CA}(\mathbf{r}_2,t)\,d\mathbf{r}_2 = -e\int\bar{\psi}_D(y)\hat{A}_{CA}(y)\psi_B(y)\,d^4y, \tag{37.13}$$

where

$$A_{CA}(\mathbf{r}_2,t) = \frac{1}{4\pi}\int j_{CA}(\mathbf{r}_1)\frac{e^{-i\omega_{AC}t+i\omega_{AC}|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}\,d\mathbf{r}_1$$

or

$$A_{CA}(\mathbf{r}_2,t) = \frac{1}{4\pi}\int\frac{j_{CA}(\mathbf{r}_1,t-|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|}\,d\mathbf{r}_1. \tag{37.14}$$

It is remarkable that formula (37.13) coincides with the expression for the elements of the first order scattering matrix $S^{(1)}$ in the case of the transition of an electron from state $B$ to state $D$ induced by an external field whose potential is given by $A_{CA}$. In these formulas $A_{CA}$ is determined by the transition current $j_{CA}$ corresponding to the transition of the particle from state $A$ to state $C$ in exactly the same way as in classical electrodynamics, i.e., by the formula for the retarded potentials (37.14). Naturally, the differential equations for the potentials of classical electrodynamics also hold:

$$\Box A_{CA} = -j_{CA},$$

$$\frac{\partial (A_{CA})_\mu}{\partial x_\mu} = 0.$$

We note that expression (37.12) for $S^{(2)}_{AB:CD}$ is symmetric in the two currents. In going over from this expression to (37.13) we have treated the current $j_{DB}$ as "experiencing" the external field $A_{CA}$, and the current $j_{CA}$ as giving rise to this field. Here we have taken the current with the positive frequency to be the one which gives rise to the field (we have assumed that $\omega_{AC} > 0$; the transition takes place from the higher energy level to the lower one). If we regard the current with negative frequency as generating the field, i.e., if we write (37.12) in the form

$$S^{(2)}_{AB:CD} = i \int j^{(x)}_{CA} A_{DB}(x)\, d^4x,$$

then in place of (37.14) we obtain $A_{DB}$ in the form of "advanced" potentials

$$A_{DB}(\boldsymbol{r}_1, t) = \frac{1}{4\pi} \int \frac{j_{DB}(\boldsymbol{r}_2, t+|\boldsymbol{r}_1-\boldsymbol{r}_2|)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, d\boldsymbol{r}_2.$$

We now go over from the scattering matrix to the matrix for the effective perturbation energy for the system of two charges defined in the following manner:

$$S_{i\to f} = -2\pi i U_{i\to f}\, \delta(\omega_A + \omega_B - \omega_C - \omega_D)$$

or

$$S_{AB:CD} = -2\pi i U_{AB:CD}\, \delta(\omega_{AC} + \omega_{BD}).$$

Then we obtain from (37.12′):

$$U_{AB:CD} = -\frac{1}{4\pi} \int j_{CA}(\boldsymbol{r}_1) j_{DB}(\boldsymbol{r}_2) \frac{e^{i|\omega_{AC}|\,|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2. \qquad (37.15)$$

Expression (37.15) can also be written in the form

$$U_{AB;\,CD} = -\int j_{DB}(\boldsymbol{r}_2) A_{CA}(\boldsymbol{r}_2) d\boldsymbol{r}_2, \tag{37.16}$$

where

$$A_{CA}(\boldsymbol{r}_2) = \frac{1}{4\pi}\int \frac{j_{CA}(\boldsymbol{r}_1)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, e^{i\omega_{AC}|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, d\boldsymbol{r}_1,$$

The potentials $A_{CA}$ satisfy the differential equations of classical electrodynamics for monochromatic fields

$$\Delta A_{CA}+\omega_{CA}^2 A_{CA} = -j_{CA}, \quad \operatorname{div} A_{CA}-\omega_{CA}(A_4)_{CA} = 0. \tag{37.17}$$

The results of this section provide a rigorous foundation for the semiclassical method of treating the interaction between electrons and the electromagnetic field (the so-called "correspondence principle" (27)). In this method the quantum properties of the electromagnetic field are not taken into account, and the equations of classical electrodynamics are used in which the currents are replaced by the transition currents. It can also be shown that the energy flux in the classical electromagnetic field corresponding to the potentials (37.14) is equal to the probability of emission of a photon per unit time, with the electron making the transition from state $A$ to state $C$, multiplied by the photon energy $\omega_{AC}$.

## § 38. Interaction Energy of Two Electrons to Terms of Order $v^2/c^2$

### 38.1. *The Breit Formula*

In the preceding section we have obtained an expression for the matrix element for the effective interaction energy of two electrons. It has the form

$$U_{i\to f} = U_{AB;\,CD}-U_{AB;\,DC}, \tag{38.1}$$

where $A$ and $B$ denote the initial, and $C$ and $D$ the final states of the electrons. In accordance with (37.15) the matrix element $U_{AB;\,CD}$ can be written in the form

$$U_{AB;\,CD} = \alpha\int \psi_C^*(\boldsymbol{r}_1)\,\psi_D^*(\boldsymbol{r}_2)\,\frac{1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, e^{i|\omega_{AC}|\,|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\psi_A(\boldsymbol{r}_1)\,\psi_B(\boldsymbol{r}_2) d\boldsymbol{r}_1\, d\boldsymbol{r}_2, \tag{38.2}$$

where $\alpha = e^2/4\pi$. Here the operator $\boldsymbol{\alpha}_1$ operates on the function $\psi_A(\boldsymbol{r}_1)$, while the operator $\boldsymbol{\alpha}_2$ operates on the function $\psi_B(\boldsymbol{r}_2)$. We note the structure of expression (38.2). It contains an operator which depends on the coordinates and the spin variables of both particles, $(1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)/|\boldsymbol{r}_1-\boldsymbol{r}_2|$, and the "retardation factor" $\exp(i|\omega_{AC}|\,|\boldsymbol{r}_1-\boldsymbol{r}_2|)$. The presence of this last factor, which depends explicitly on the initial and the final energies of the system, does not allow us to introduce in a general form the concept of the operator U for the interaction between two electrons whose matrix element would be given by $U_{AB;\,CD}$:

$$U_{AB;\,CD} = \int \psi_C^*(\boldsymbol{r}_1)\psi_D^*(\boldsymbol{r}_2)\,U\psi_A(\boldsymbol{r}_1)\psi_B(\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{38.3}$$

However, for small velocities ($v/c \leqslant 1$, $c$ is the velocity of light) such an operator can be constructed. In order to obtain it we expand the matrix element in powers of $v/c$ up to terms of order $(v/c)^2$. As in the case of the analogous expansion in § 15 we shall not use in this section the system of units in which $c = 1$. This is convenient from the point of view that the transition to the nonrelativistic approximation can be formally accomplished by expanding in powers of $1/c$.

The expansion of the retardation factor up to terms of order $1/c^2$ inclusive is of the form

$$\frac{e^{\frac{i}{c}|\omega_A-\omega_C|\,|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} + i\frac{|\omega_A-\omega_C|}{c} - \frac{(\omega_A-\omega_C)^2}{2c^2}\,|\boldsymbol{r}_1-\boldsymbol{r}_2|. \tag{38.4}$$

In order of magnitude the matrix elements of the operators $\boldsymbol{\alpha}$ are equal to $v/c$. Indeed, $\psi^*\boldsymbol{\alpha}\psi = \varphi^*\boldsymbol{\sigma}\chi+\chi^*\boldsymbol{\sigma}\varphi$, while $\chi \sim \varphi/c$. Therefore, in terms containing $\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2$ in (38.2) it is sufficient to retain only the first term in the expansion (38.4). The remaining terms will have as factors not $1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2$ but simply unity. The second term in (38.4) will vanish after substitution into the integral in virtue of the orthogonality of the functions $\psi_A$ and $\psi_C$. The third term can be put into symmetric form by utilizing the fact that $\omega_A-\omega_C = \omega_D-\omega_B$, and therefore

$$-(\omega_A-\omega_C)^2|\boldsymbol{r}_1-\boldsymbol{r}_2| = (\omega_A-\omega_C)(\omega_B-\omega_D)|\boldsymbol{r}_1-\boldsymbol{r}_2|.$$

We can then eliminate the frequencies from the expression for the matrix element by utilizing the Dirac equation $\mathrm{H}_1\psi_A(\boldsymbol{r}_1) = \omega_A\psi_A(\boldsymbol{r}_1)$ and similar relations for the other wave functions. Keeping in mind that expression (38.4) is multiplied on the right by $\psi_A(\boldsymbol{r}_1)\psi_B(\boldsymbol{r}_2)$, and on the left by $\psi_C^*(\boldsymbol{r}_1)\psi_D^*(\boldsymbol{r}_2)$ and integrated over $d\boldsymbol{r}_1$ and $d\boldsymbol{r}_2$, we can replace in it $\omega_A$ and $\omega_B$ by the operators $\mathrm{H}_1$ and $\mathrm{H}_2$ to the right of the factor $|\boldsymbol{r}_1-\boldsymbol{r}_2|$, and $\omega_C$ and $\omega_D$ by the operators $\mathrm{H}_1$ and $\mathrm{H}_2$ to the left of $|\boldsymbol{r}_1-\boldsymbol{r}_2|$, i.e.,

$$(\omega_A-\omega_C)(\omega_B-\omega_D)|\boldsymbol{r}_1-\boldsymbol{r}_2| \rightarrow |\boldsymbol{r}_1-\boldsymbol{r}_2|\mathrm{H}_1\mathrm{H}_2+\mathrm{H}_1\mathrm{H}_2|\boldsymbol{r}_1-\boldsymbol{r}_2|$$
$$-\mathrm{H}_1|\boldsymbol{r}_1-\boldsymbol{r}_2|\mathrm{H}_2-\mathrm{H}_2|\boldsymbol{r}_1-\boldsymbol{r}_2|\mathrm{H}_1 = \left[\mathrm{H}_1,[\mathrm{H}_2,|\boldsymbol{r}_1-\boldsymbol{r}_2|]\right].$$

(The square brackets denote commutators of the corresponding quantities.)

Since the Hamiltonian operator has the form $\mathrm{H} = c\boldsymbol{\alpha}\boldsymbol{p}+\beta mc^2+e\hat{A}^{(e)}$, where $A^{(e)}$ is the external field, it contains only a single term, viz., $\boldsymbol{\alpha}\boldsymbol{p}$ which does not commute with $|\boldsymbol{r}_1-\boldsymbol{r}_2|$. On evaluating the commutator $c^2\left[[\boldsymbol{\alpha}_1\boldsymbol{p}_1, \boldsymbol{\alpha}_2\boldsymbol{p}_2, |\boldsymbol{r}_1-\boldsymbol{r}_2|]\right]$, we find that in substituting (38.4) into the integral (38.2) we can use the substitution:

$$-\frac{(\omega_A-\omega_B)^2}{c^2}|\boldsymbol{r}_1-\boldsymbol{r}_2| \rightarrow -\frac{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2-(\boldsymbol{\alpha}_1\boldsymbol{n})(\boldsymbol{\alpha}_2\boldsymbol{n})}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}, \tag{38.5}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}.$$

On substituting (38.5) and (38.4) into (38.2) we can show that $U_{AB;CD}$ does indeed turn out to be the matrix element of the operator (33), (112), (20)

$$U = \frac{a}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}-\frac{a}{2}\,\frac{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2+(\boldsymbol{\alpha}_1\boldsymbol{n})(\boldsymbol{\alpha}_2\boldsymbol{n})}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}, \tag{38.6}$$

for which relation (38.3) holds.

The first term in (38.6) is the Coulomb interaction between the charges. The second term takes into account corrections arising because of the motion of the charges.

Since this expression holds only up to terms of order $(v/c)^2$, the electron wave functions $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$ should also be taken to terms of the same order, i.e., we must use expression (15.2)

$$\chi = \frac{\boldsymbol{\sigma}\mathbf{p}}{2mc}\varphi.$$

On substituting this approximate expression for $\chi$ and the operator (38.6) into (38.3) we obtain

$$
\begin{aligned}
U_{AB;CD} = {} & a\int \varphi_C^*(\boldsymbol{r}_1)\varphi_D^*(\boldsymbol{r}_2)\varphi_A(\boldsymbol{r}_1)\varphi_B(\boldsymbol{r}_2)\frac{d\boldsymbol{r}_1\,d\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\\
& +\frac{a}{4c^2}\int\{[\boldsymbol{\sigma}_1\mathbf{p}_1\varphi_C(\boldsymbol{r}_1)]^*[\sigma_1\mathbf{p}_1\varphi_A(\boldsymbol{r}_1)]\varphi_D^*(\boldsymbol{r}_2)\varphi_B(\boldsymbol{r}_2)\\
& +\varphi_C^*(\boldsymbol{r}_1)\varphi_A(\boldsymbol{r}_1)[\boldsymbol{\sigma}_2\mathbf{p}_2\varphi_D(\boldsymbol{r}_2)]^*\,[\boldsymbol{\sigma}_2\mathbf{p}_2\varphi_B(\boldsymbol{r}_2)]\}\frac{d\boldsymbol{r}_1\,d\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\\
& +\frac{a}{8c^2}\int\{\varphi_C^*(\boldsymbol{r}_1)\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_1\mathbf{p}_1)\varphi_A(\boldsymbol{r}_1)+[\boldsymbol{\sigma}_1\mathbf{p}\varphi_C(\boldsymbol{r}_1)]^*\boldsymbol{\sigma}_1\varphi_A(\boldsymbol{r}_1)\}\\
& \times\{\varphi_D^*(\boldsymbol{r}_2)\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_2\mathbf{p}_2)\varphi_B(\boldsymbol{r}_2)+[\boldsymbol{\sigma}_2\mathbf{p}_2\varphi_D(\boldsymbol{r}_2)]^*\,\boldsymbol{\sigma}_2\varphi_B(\boldsymbol{r}_2)\}\frac{d\boldsymbol{r}_1\,d\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\\
& -\frac{a}{8c^2}\int\{\varphi_C^*(\boldsymbol{r}_1)\,(\boldsymbol{\sigma}_1\boldsymbol{n})\,(\boldsymbol{\sigma}_1\mathbf{p}_1)\varphi_A(\boldsymbol{r}_1)+[\boldsymbol{\sigma}_1\mathbf{p}_1\varphi_C(\boldsymbol{r}_1)]^*(\boldsymbol{\sigma}_1\boldsymbol{n})\varphi_A(\boldsymbol{r}_1)\}\\
& \times\{\varphi_D^*(\boldsymbol{r}_2)\,(\boldsymbol{\sigma}_2\boldsymbol{n})\,(\boldsymbol{\sigma}_2\mathbf{p}_2)\varphi_B(\boldsymbol{r}_2)+[\boldsymbol{\sigma}_2\mathbf{p}_2\varphi_D(\boldsymbol{r}_2)]^*(\boldsymbol{\sigma}_2\boldsymbol{n})\varphi_B(\boldsymbol{r}_2)\}\frac{d\boldsymbol{r}_1\,d\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}.
\end{aligned}
\tag{38.7}
$$

We now introduce the approximate two-component electron wave function $\Phi$ related to $\varphi$ by expressions (15.7), (15.9):

$$\varphi=\left(1-\frac{\mathrm{p}^2}{8m^2c^2}\right)\Phi. \tag{38.8}$$

In doing this it is sufficient to replace $\varphi$ by $\Phi$ in all the terms of expression (38.7) with the exception of the first one, since they already contain the factor $1/c^2$.

We now rewrite expression (38.7) in such a way that it assumes the form (38.3). In other words, we find an operator $\mathrm{U}^{(e)}$ which leads to

$$U_{AB;CD}=\int\Phi_C^*(\boldsymbol{r}_1)\Phi_D^*(\boldsymbol{r}_2)\,\mathrm{U}^{(e)}\Phi_A(\boldsymbol{r}_1)\Phi_B(\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{38.9}$$

(As we have done previously, for the sake of brevity we again omit here the spin indices of the wave functions.)

In order to bring (38.7) into the form (38.9) it is necessary to integrate by parts. Since the integrand contains high powers of the quantity $|\boldsymbol{r}_1-\boldsymbol{r}_2|^{-1}$ we must exclude the origin (the point $\boldsymbol{r}_1=\boldsymbol{r}_2$). The integral over the surface surrounding the origin yields a finite quantity as

$|\boldsymbol{r}_1-\boldsymbol{r}_2| \to 0$. Since the latter depends only on the values of the integrand at $\boldsymbol{r}_1 = \boldsymbol{r}_2$ it can also be written in the form of a volume integral of an expression containing $\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)$. We finally obtain the following expression for the operator for the interaction energy of two electrons:

$$\begin{aligned}\mathrm{U}^{(e)} = \frac{\alpha}{r} - \pi\frac{\alpha}{m^2c^2}\delta(\boldsymbol{r}) - \frac{\alpha}{4m^2c^2}\frac{1}{r^3}([\boldsymbol{r}\mathbf{p}_1]\boldsymbol{\sigma}_1 - [\boldsymbol{r}\mathbf{p}_2]\boldsymbol{\sigma}_2 + 2[\boldsymbol{r}\mathbf{p}_1]\boldsymbol{\sigma}_2 \\ -2[\boldsymbol{r}\mathbf{p}_2]\boldsymbol{\sigma}_1) - \frac{\alpha}{2m^2c^2}\left(\frac{1}{r}\mathbf{p}_1\mathbf{p}_2 + \frac{1}{r^3}\boldsymbol{r}(\boldsymbol{r}\mathbf{p}_1)\mathbf{p}_2\right) \\ + \frac{\alpha}{4m^2c^2}\left[\frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{r^3} - \frac{3(\boldsymbol{\sigma}_1\boldsymbol{r})(\boldsymbol{\sigma}_2\boldsymbol{r})}{r^5} - \frac{8\pi}{3}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\,\delta(\boldsymbol{r})\right],\end{aligned} \qquad (38.10)$$

where $\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2$.

The $\delta$-functions contained in this expression have arisen in the manner just explained. In this expression (since the region in the neighborhood of $\boldsymbol{r} = 0$ has already been taken into account) we need not pay any attention to the seeming divergence of certain integrals arising, for example, as a result of substituting the last square bracket of (38.10) into (38.9). In order to avoid divergence we can first carry out the integration over the angles.

Naturally, the presence of $\delta$-functions in (38.10) does not indicate a strong interaction. Terms involving $\delta$-functions contain $1/c^2$ in their coefficients and, therefore, from the point of view of the foregoing expansion must be treated as small compared to the first term (the Coulomb interaction).

We shall return to the interpretation of the individual terms of the operator (38.10) in the next section when we apply it to a specific problem. We note here a certain similarity between this operator and the energy of the electron in an external field up to terms of order $(v/c)^2$ (cf. (15.10), (15.11)). Thus, the second term in (38.10) is similar to the last term in (15.10) if $A_0$ appearing therein is the Coulomb potential $A_0 = e/4\pi r$. This term is followed in (38.10) by terms of the same type as the penultimate term in formula (15.10), if we interpret $\boldsymbol{E}$ appearing therein as the Coulomb field $\boldsymbol{E} = e\boldsymbol{r}/4\pi r^3$. The last terms in (38.10) correspond to the last term of formula (15.11) in which $\boldsymbol{H}$ denotes the field of a magnetic dipole of magnetic moment $e\boldsymbol{\sigma}/2mc$.

We note that the operator for the interaction energy of two electrons (38.10) can be obtained from expression (36.1) for the matrix element for electron-electron scattering. In the nonrelativistic approximation (including terms of order $(p/m)^2$) we can write, on the basis of (10.9′) and (10.10), the plane wave amplitude $u$ in the form

$$u = \left(1 - \frac{p^2}{8m^2}\right)\begin{pmatrix} w \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\,\omega \end{pmatrix}.$$

On substituting this expression, and also the approximate equation

$$\frac{1}{q^2} = \frac{1}{\mathbf{q}^2} + \frac{1}{4m^2} + \frac{(\mathbf{q}\mathbf{p}_1)(\mathbf{q}\mathbf{p}_2)}{m^2(\mathbf{q}^2)^2} \qquad (\mathbf{q} \equiv \mathbf{p}_1' - \mathbf{p}_1 = \mathbf{p}_2 - \mathbf{p}_2')$$

into the first term of (36.1) we obtain

$$(\bar{u}_1'\gamma_\mu u_1)(\bar{u}_2'\gamma_\mu u_2)\frac{e^2}{q^2} = w_1'^{*}\, w_2'^{*}\, \mathrm{U}^e(\mathbf{q}) w_1 w_2,$$

$$\mathrm{U}^{(e)}(\mathbf{q}) = \alpha\left\{\frac{1}{\mathbf{q}^2} - \frac{1}{4m^2} + \frac{i}{4\mathbf{q}^2 m^2}([\mathbf{q}\mathbf{p}_1]\boldsymbol{\sigma}_1 - [\mathbf{q}\mathbf{p}_2]\boldsymbol{\sigma}_2 + 2\,[\mathbf{q}\mathbf{p}_1]\boldsymbol{\sigma}_2 - 2\,[\mathbf{q}\mathbf{p}_2]\boldsymbol{\sigma}_1)\right.$$
$$\left. + \frac{(\mathbf{q}\mathbf{p}_1)(\mathbf{q}\mathbf{p}_2)}{m^2(\mathbf{q}^2)^2} - \frac{\mathbf{p}_1\mathbf{p}_2}{m^2\mathbf{q}^2} + \frac{(\mathbf{q}\boldsymbol{\sigma}_1)(\mathbf{q}\boldsymbol{\sigma}_2)}{4m^4} - \frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{4m^2}\right\}.$$

This quantity is the momentum representation of the interaction energy (38.10) $\mathrm{U}^e(\mathbf{r}) = \int e^{i\mathbf{q}\mathbf{r}}\mathrm{U}^e(\mathbf{q})\,d\mathbf{q}$.

### 38.2. *Schrödinger Equation for a Two-Electron System*

The exchange matrix element in (38.1) $U_{AB;DC}$ can be brought into the form (38.9) in exactly the same way as $U_{AB;CD}$

$$U_{AB;DC} = \int \Phi_D^*(\mathbf{r}_1)\Phi_C^*(\mathbf{r}_2)\,\mathrm{U}^{(e)}\Phi_A(\mathbf{r}_1)\,\Phi_B(\mathbf{r}_2)\,d\mathbf{r}_1\,d\mathbf{r}_2, \qquad (38.11)$$

where $\mathrm{U}^{(e)}$ is the same operator (38.10).

If we introduce the antisymmetrized two-electron wave functions

$$\Phi_{AB}(\mathbf{r}_1, \lambda_1, \mathbf{r}_2, \lambda_2)$$
$$= \frac{1}{\sqrt{2}}[\Phi_A(\mathbf{r}_1, \lambda_1)\,\Phi_B(\mathbf{r}_2, \lambda_2) - \Phi_B(\mathbf{r}_1, \lambda_1)\,\Phi_A(\mathbf{r}_2, \lambda_2)], \qquad (38.12)$$

where $\lambda_1$ and $\lambda_2$ are the spin variables of the particles, and analogous functions for other pairs of states, then the matrix element (38.1) can be written in the form

$$U_{i\to f} = \int \Phi^*_{CD} \mathrm{U}^e \Phi_{AB}\, dr_1\, dr_2. \tag{38.13}$$

If we interpred $A$ and $B$ as arbitrary stationary states of the electron situated in a given external field, then the wave functions $\Phi_{AB}$ form a complete system of antisymmetrized eigenfunctions of the operator $\mathrm{H}_1+\mathrm{H}_2$, where $\mathrm{H}_1$ and $\mathrm{H}_2$ are, as before, the Hamiltonian operators for a single electron.

Due to the existence of an interaction between electrons the state described by the wave function $\Phi_{AB}$ is not stationary. If at $t=-\infty$ the electrons were in the state $\Phi_{AB}$ then at $t=+\infty$ they can turn out to be in some other state $\Phi_{CD}$; the probability of such a transition is, as we have seen, proportional to

$$\left|\int \Phi^*_{CD} \mathrm{U}^e \Phi_{AB}\, dr_1\, dr_2\right|^2.$$

We can formulate the problem of determining the stationary states of a two-electron system taking their interaction into account (up to terms of order $v^2/c^2$). Such states are superpositions of states (38.12), and the wave functions $\Phi_n$ corresponding to them are eigenfunctions of the operator

$$\mathrm{H} = \mathrm{H}_1+\mathrm{H}_2+\mathrm{U}^e. \tag{38.14}$$

Thus we have obtained the Schrödinger equation for two electrons in configuration space

$$\mathrm{H}\Phi_n = E_n\Phi_n, \tag{38.15}$$

where $E_n$ is the energy of a stationary state. In addition to satisfying equation (38.15) the wave functions $\Phi_n$ must satisfy the requirement of antisymmetry.

### 38.3. *Interaction between an Electron and a Positron*

We now proceed to the determination of the interaction energy between an electron and a positron to terms of order $v^2/c^2$.

In accordance with (37.7) the matrix element is in this case of the form $U_{i\to f} = -(U_{AB;CD}-U_{AB;DC})$, where $A$ and $C$, as before, denote the initial and final states of the electron, while $D$ and $B$ denote the initial and final states of the positron described by the wave functions

$\psi_B$ and $\psi_D$ with negative frequencies. In place of these functions we introduce the positron wave functions in accordance with subsection 10.3, $\psi^{(\mathrm{p})} = \mathrm{C}\overline{\psi}^{(-)}$, and utilize the fact that $\overline{\psi}_D^{(-)}\gamma_\mu\psi_B^{(-)} = \overline{\psi}_B^{(\mathrm{p})}\gamma_\mu\psi_D^{(\mathrm{p})}$. Then the matrix element $U_{AB;CD}$ assumes the form

$$U_{AB;CD} = a\int \psi_C^*(\boldsymbol{r}_1)\,\psi_B^{(p)*}(\boldsymbol{r}_2)\,\frac{1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,e^{i\frac{|\omega_{AC}|}{c}|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\psi_A(\boldsymbol{r}_1)\psi_D^{(\mathrm{P})}(\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{38.16}$$

In (38.16), as in (38.2), we have arranged on the left the wave functions of the final states of the particles, and on the right the wave functions of the initial states, with all these functions corresponding to positive frequencies. Therefore, by proceeding in exactly the same way in which (38.2) was transformed into (38.11), we can obtain from (38.16) the formula

$$U_{AB;CD} = \int \Phi_C^*(\boldsymbol{r}_1)\Phi_B^{(p)*}(\boldsymbol{r}_2)\,\mathrm{U}^{(e)}\Phi_A(\boldsymbol{r}_1)\Phi_D^{(\mathrm{P})}(\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2, \tag{38.17}$$

where $\mathrm{U}^{(e)}$ is the operator (38.10).

### 38.4. *Exchange Interaction between an Electron and a Positron*

After expansion in powers of $1/c^2$ the exchange matrix element

$$U_{AB;DC} = a\int \psi_D^{(-)*}(\boldsymbol{r}_1)\,\psi_C^*(\boldsymbol{r}_2)\,\frac{1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,e^{\frac{i}{c}|\omega_A-\omega_D|\,|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\psi_A(\boldsymbol{r}_1)\psi_B^{(-)}(\boldsymbol{r}_2)\,d\boldsymbol{r}_1 d\boldsymbol{r}_2, \tag{38.18}$$

assumes a form quite pifferent from the two-electron case. Since $\omega_D$ and $\omega_B$ are negative, the "retardation factor" now contains not the difference, but the sum of the particle energies $\omega_A-\omega_D = \varepsilon_A+\varepsilon_D^{(p)}$, including the rest energy $2mc^2$. Thus, $(\omega_A-\omega_D)/c$ is not a small quantity of order $1/c$, but a large one—of order $c$—and expansion (38.4) ceases to be applicable.

In this case the transition to the low velocity approximation can be easily carried out by replacing the energy in the exponential by the rest energy $\varepsilon_A+\varepsilon_D^{(p)} \approx 2mc^2$. We then obtain

$$U_{AB;DC} = a\int \psi_D^{(-)*}(\boldsymbol{r}_1)\psi_C^*(\boldsymbol{r}_2)\frac{1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,e^{2imc|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\psi_A(\boldsymbol{r}_1)\psi_B^{(-)}(\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{38.19}$$

We note that in (38.19) the initial and the final states of the particles enter only through the wave functions, i.e., this expression already has a structure of the form of (38.3).

The retardation factor in (38.19) contains in the exponent the large coefficient $2mc$ and, therefore, oscillates with a wavelength $(1/2mc)$ ($1/mc$ is the "Compton wavelength" for the electron). For this integration the important contributions come from the region $|\boldsymbol{r}_1-\boldsymbol{r}_2| \sim 1/mc$. Since the electron and the positron wave functions (in accordance with the assumptions that the velocities are small) do not vary significantly over such a distance, we can segregate in (38.19) the integration over $\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2$ by setting $\boldsymbol{r}_1 = \boldsymbol{r}_2$ in the arguments of the wave functions:

$$U_{AB;\,DC} = a \int \Big[\psi_D^{(-)*}(\boldsymbol{r}_1)\psi_A(\boldsymbol{r}_1)\psi_C^*(\boldsymbol{r}_1)\psi_B^{(-)}(\boldsymbol{r}_1) \\ -\big(\psi_D^{(-)*}(\boldsymbol{r}_1)\boldsymbol{\alpha}\psi_A(\boldsymbol{r}_1)\big)\big(\psi_C^*(\boldsymbol{r}_1)\boldsymbol{\alpha}\psi_B^{(-)}(\boldsymbol{r}_1)\big)\Big]\, d\boldsymbol{r}_1 \int \frac{e^{i2mcr}}{r}\, d\boldsymbol{r}.$$

The last improper integral can be evaluated in the manner in which expressions containing oscillating integrands are usually evaluated:

$$\int \frac{e^{i2mcr}}{r}\, d\boldsymbol{r} = \lim_{\lambda=0} \int e^{i2mcr-\lambda r}\, \frac{d\boldsymbol{r}}{r} = -\frac{\pi}{(mc)^2},$$

as a result of which the exchange matrix element assumes the form

$$U_{AB:\,DC} = -\pi \frac{a}{m^2c^2} \int \big(\psi_D^{(-)*}(\boldsymbol{r}_1)\,\psi_A(\boldsymbol{r}_1)\big)\big(\psi_C^*(\boldsymbol{r}_1)\,\psi_B^{(-)}(\boldsymbol{r}_1)\big) \\ -\big(\psi_D^{(-)*}(\boldsymbol{r}_1)\,\boldsymbol{\alpha}\psi_A(\boldsymbol{r}_1)\big)\big(\psi_C^*(\boldsymbol{r}_1)\boldsymbol{\alpha}\psi_B^{(-)}(\boldsymbol{r}_1)\big)\, d\boldsymbol{r}_1$$

or

$$U_{AB:\,DC} = -\pi \frac{a}{m^2c^2} \int \psi_D^{(-)}(\boldsymbol{r}_1)\,\psi_C^*(\boldsymbol{r}_2)(1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2) \\ \times \delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\psi_A(\boldsymbol{r}_1)\,\psi_B^{(-)}(\boldsymbol{r}_2)\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2. \qquad (38.20)$$

In (38.20) we express the wave functions with negative frequencies in terms of the positron wave functions $\psi^{(-)} = -\beta C\psi^{(p)*}$; $\psi^{(-)*} = -\psi^{(p)}C\beta$; we then obtain

$$U_{AB;\,DC} = -\frac{\pi a}{m^2c^2} \int \{(\psi_D^{p}\, C\beta\psi_A)(\psi_C^*\beta\, C\psi_B^{(p)*}) \\ -(\psi_D^{p} C\beta\boldsymbol{\alpha}\psi_A)(\psi_C^*\boldsymbol{\alpha}\beta\, C\,\psi_B^{(p)*})\}\, d\boldsymbol{r}_1. \qquad (38.21)$$

The last expression already contains the coefficient $1/c^2$. Therefore, in the wave functions

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

we can omit $\chi$, and to the desired approximation replace $\varphi$ by the two-component wave function $\Phi$. Each four-rowed matrix

$$\Gamma = \begin{pmatrix} \Gamma_1 \Gamma_2 \\ \Gamma_3 \Gamma_4 \end{pmatrix},$$

where $\Gamma_1$, $\Gamma_2, \ldots$ are two-rowed matrices, can be replaced here by $\Gamma_1$. Utilizing the explicit expression for the matrices $\mathrm{C} = \alpha_y$ (cf. § 10) and $\boldsymbol{\alpha}$, we obtain

$$\begin{aligned} \mathrm{C}\beta &\rightarrow 0, \\ \mathrm{C}\beta\alpha_y &= \alpha_y\beta\mathrm{C} \rightarrow -1, \\ \mathrm{C}\beta\alpha_x &= -\alpha\beta_x\mathrm{C} \rightarrow i\sigma_z, \\ \mathrm{C}\beta\alpha_z &= -\alpha_z\beta\mathrm{C} \rightarrow -i\sigma_x. \end{aligned}$$

Using these expressions in (38.21), we obtain

$$U_{AB,DC} = -\pi\frac{a}{m^2c^2}\int \{\Phi_D^{(p)}(\mathbf{r}_1, \lambda_1)\,\Phi_C^*(\mathbf{r}_1, \lambda_2)\,[1+\sigma_{1z}\sigma_{2z} + \sigma_{1x}\sigma_{2x}]\,\Phi_A(\mathbf{r}_1, \lambda_1)\,\Phi_B^{(p)}(\mathbf{r}_1, \lambda_2)\}\,d\mathbf{r}_1,$$

where $\lambda_1$ and $\lambda_2$ are spin variables ($\boldsymbol{\sigma}_1$ operates on $\lambda_1$, while $\boldsymbol{\sigma}_2$ operates on $\lambda_2$).

In order to put the exchange matrix elements into a form analogous to (38.17) we bring the integrand in $U_{AB;DC}$ into the form

$$\Phi_C^*(\lambda_1)\,\Phi_B^{(p)*}(\lambda_2)\mathrm{T}\Phi_A(\lambda_1)\,\Phi_D^{(p)}(\lambda_2).$$

It can be easily shown that the operator T must have the structure $\mathrm{T} = c_1 + c_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, where $c_1$ and $c_2$ are constants. Indeed, the matrix element $U_{AB;DC}$ must be a scalar (in three-dimensional space). The operator T, which is a general bilinear combination of four two-rowed matrices $(1, \sigma_x, \sigma_y, \sigma_z)$, guarantees that this condition is satisfied.

By utilizing directly the explicit form of the matrices $\sigma_i$ we obtain the following values for the coefficients $c_1$ and $c_2$: $c_1 = \frac{3}{2}$ and $c_2 = \frac{1}{2}$.

Thus we have

$$U_{AB;DC} = \int \Phi_C^*(\mathbf{r}_1)\Phi_B^{(p)*}(\mathbf{r}_2)\mathrm{U}^{(\mathrm{exch})}\Phi_A(\mathbf{r}_1)\Phi_D^{(p)}(\mathbf{r}_2)\,d\mathbf{r}_1\,d\mathbf{r}_2,$$

where (152), (12)

$$\mathrm{U}^{(\mathrm{exch})}=\frac{\pi}{2}\frac{a}{m^2c^2}(3+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2). \qquad (38.22)$$

The matrix element for the transition $U_{i\to f}$ can now be written in the form

$$U_{i\to f}=\int \Phi^*_{CB}\mathrm{U}^{(\mathrm{p})}\Phi_{AD}d\boldsymbol{r}_1 d\boldsymbol{r}_2, \qquad (38.23)$$

where the operator

$$\mathrm{U}^{(\mathrm{p})}=-\mathrm{U}^{(e)}+\mathrm{U}^{(\mathrm{exch})} \qquad (38.24)$$

is the complete operator for the interaction between an electron and a positron and

$$\Phi_{AD}(\boldsymbol{r}_1,\lambda_1;\boldsymbol{r}_2,\lambda_2)=\Phi_A(\boldsymbol{r}_1,\lambda_1)\Phi_D^{(\mathrm{p})}(\boldsymbol{r}_2,\lambda_2);$$
$$\Phi_{CB}(\boldsymbol{r}_1,\lambda_1;\boldsymbol{r}_2,\lambda_2)=\Phi_C(\boldsymbol{r}_1,\lambda_1)\Phi_B^{(\mathrm{p})}(\boldsymbol{r}_2,\lambda_2).$$

The functions $\Phi_{AD}$ ($A$ and $D$ are the quantum numbers of arbitrary states of the electron and the positron in a given external field) form a complete system of eigenfunctions of the operator $\mathrm{H}^{(\mathrm{e})}+\mathrm{H}^{(\mathrm{p})}$, where $\mathrm{H}^{(\mathrm{e})}$ is the Hamiltonian for the electron, and $\mathrm{H}^{(\mathrm{p})}$ is the Hamiltonian for the positron (in the presence of an external field they differ by the sign of their charges).

The wave function $\Phi_n(\boldsymbol{r}_1\boldsymbol{r}_2)$ of the stationary state of the electron-positron system satisfies the Schrödinger equation

$$[\mathrm{H}^{(\mathrm{e})}(\boldsymbol{r}_1)+\mathrm{H}^{(\mathrm{p})}(\boldsymbol{r}_2)+\mathrm{U}^{(\mathrm{p})}]\Phi_n=E_n\Phi_n, \qquad (38.25)$$

and is not restricted by any symmetry requirements. The electron and the positron appear in equation (38.25) as nonidentical particles.

The symmetry of the system with respect to the reversal of the sign of the charge of both particles leads, as was shown in subsection 23.3, to a definite charge parity. In particular, the symmetric states $\Phi_n(x,y)=\Phi_n(y,x)$ ($x$ and $y$ denote the set of the coordinates and the spin variable) are charge-odd, while the antisymmetric states $\Phi_n(x,y)=-\Phi_n(y,x)$ are charge-even.

## § 39. Positronium

### 39.1. *Hamiltonian Operator and the Unperturbed Equation*

The Schrödinger equation (38.25) can be applied to the problem of positronium, i.e., to the problem of the bound states of the electron-positron system.

The operator for the interaction energy between the particles is determined by formulas (38.24), (38.22), (38.10), while the Hamiltonians for the electron and the positron are determined by formula (15.10) with $A_0 = 0$ (there is no external field).

The Hamiltonian operator in equation (38.25) consists of terms of different orders of magnitude. We write it in the form $\mathrm{H} = \mathrm{H}^{(0)} + \mathrm{H}^{(1)}$, where $\mathrm{H}^{(0)}$ includes terms which do not contain $1/c^2$:

$$\mathrm{H}^{(0)} = \frac{1}{2m}(\mathrm{p}_1^2 + \mathrm{p}_2^2) - \frac{a}{|r_1 - r_2|}, \tag{39.1}$$

while $\mathrm{H}^{(1)}$ is proportional to $1/c^2$.

The problem of the positronium energy levels can be solved by perturbation theory. The operator $\mathrm{H}^{(0)}$ can be treated as the unperturbed Hamiltonian, while $\mathrm{H}^{(1)}$ can be regarded as the perturbation.

The unperturbed problem is exceedingly simple; it reduces to the problem of the hydrogen atom in nonrelativistic quantum mechanics. Indeed, we use the system of coordinates in which the center of mass of positronium is at rest. Then we have $p_1 = -p_2 = p$, where $p$ is the operator for the momentum corresponding to the relative radius-vector $r = r_1 - r_2$. The unperturbed Schrödinger equation has the form

$$\mathrm{p}^2 \Phi = m\left(E + \frac{a}{r}\right)\Phi. \tag{39.2}$$

It coincides with the equation of motion for the electron in the hydrogen atom, if we replace the electron mass by the reduced mass for two electrons $m/2$. Therefore, the values of the positronium energy are smaller by a factor two in absolute value compared to the corresponding energies of the hydrogen atom, while the orbit radii are twice as large.

As is well known, the unperturbed levels depend only on the principal quantum number $n$ and do not depend on the quantum numbers $j$ and $l$, which define the total and the orbital angular momenta.

## 39.2. *Perturbation Operator*

We now proceed to investigate the fine structure of the positronium levels, i.e., their displacement and splitting due to the perturbing energy $\mathrm{H}^{(1)}$ (10).

We write $\mathrm{H}^{(1)}$ in the form of a sum of five terms

$$\mathrm{H}^{(1)} = \mathrm{V}_1+\mathrm{V}_2+\mathrm{V}_3+\mathrm{V}_4+\mathrm{V}_5, \tag{39.3}$$

where $\mathrm{V}_1, \ldots, \mathrm{V}_5$ are defined below by formulas (39.4)–(39.8). The term $\mathrm{V}_1$ is a correction of order $1/c^2$ to the kinetic energy of the particles:

$$\mathrm{V}_1 = -\frac{1}{8m^3c^2}(\mathbf{p}_1^4+\mathbf{p}_2^4) = -\frac{\mathbf{p}^4}{4m^3c^2}. \tag{39.4}$$

The remaining terms are contained in the operator for the interaction energy $\mathrm{U}^{(n)}$. In $\mathrm{V}_2$ we have combined all those terms from the operator $\mathrm{U}^{(n)}$ in (38.24), which contain no spin operators. The operator $\mathrm{V}_2$ describes the interaction associated with the orbital motion of the particles[1]:

$$\mathrm{V}_2 = \pi\left(\frac{e\hbar}{mc}\right)^2 \delta(r)+\frac{e^2}{2m^2c^2}\left\{\frac{1}{r}\mathbf{p}_1\mathbf{p}_2+\frac{1}{r^3}r(r\mathbf{p}_1)\mathbf{p}_2\right\}.$$

This operator can also be written in the following form from which it can be clearly seen that it is selfconjugate:

$$\mathrm{V}_2 = \pi\left(\frac{e\hbar}{mc}\right)^2 \delta(r)-\frac{e^2}{2m^2c^2}\left\{\frac{1}{r}\mathbf{p}^2+\mathbf{p}^2\frac{1}{r}-4\pi\hbar^2\,\delta(r)-\frac{1}{r^3}[r\mathbf{p}]^2\right\}.$$

We write $\mathrm{V}_2$ in still another form which is more convenient for further calculations:

$$\mathrm{V}_2 = \pi\left(\frac{e\hbar}{mc}\right)^2 \delta(r)-\frac{i\hbar e^2}{m^2c^2}\frac{1}{r^3}r\mathbf{p}-\frac{e^2}{m^2c^2}\frac{1}{r}\mathbf{p}^2+\frac{1}{2}\left(\frac{e\hbar}{mc}\right)^2\frac{1}{r^3}\mathbf{L}^2, \tag{39.5}$$

where $\hbar\mathbf{L} = [r\mathbf{p}]$ is the operator for orbital angular momentum.

In $\mathrm{V}_3$ we have collected those terms which contain both the momentum operator and the spin operator:

$$\mathrm{V}_3 = \left(\frac{e\hbar}{2mc}\right)^2\frac{1}{\hbar r^3}\{[r\mathbf{p}_1]\sigma_1-[r\mathbf{p}_2]\sigma_2+2[r\mathbf{p}_1]\sigma_2-2\,[r\mathbf{p}_2]\sigma_1\}.$$

This operator describes the spin-orbit interaction. It can be written in the form

$$\mathrm{V}_3 = \frac{3}{2}\left(\frac{e\hbar}{mc}\right)^2\frac{1}{r^3}\mathbf{L}\mathbf{s}, \quad \text{where} \quad \mathbf{s} = \frac{1}{2}(\sigma_1+\sigma_2) \tag{39.6}$$

is the operator for the spin of the system.

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[1] In the following discussion we use the ordinary system of units and introduce the constant $\hbar$ into the formulas; $e$ denotes the charge in ordinary Gaussian units.

The operator $V_4$ describes the interaction between the spin magnetic moments of electron and the positron:

$$V_4 = \left(\frac{e\hbar}{2mc}\right)^2 \left[\frac{3(\sigma_1 r)(\sigma_2 r)}{r^5} - \frac{\sigma_1\sigma_2}{r^3}\right] + \frac{8\pi}{3}\left(\frac{e\hbar}{2mc}\right)^2 \sigma_1\sigma_2\,\delta(r).$$

If we introduce the spin operator $s$, then $V_4$ can be written in the form

$$V_4 = 6\left(\frac{e\hbar}{2mc}\right)^2 \frac{1}{r^3} s_i s_k \left(\frac{x_i x_k}{r^2} - \frac{1}{3}\delta_{ik}\right) + \frac{8\pi}{3}\left(\frac{e\hbar}{2mc}\right)^2 (2s^2-3)\,\delta(r). \quad (39.7)$$

Finally, we denote by $V_5$ the exchange interaction operator (38.22): $V_5 = U^{(\mathrm{exch})}$. It can also be written as

$$V_5 = \pi\left(\frac{e\hbar}{mc}\right)^2 s^2\,\delta(r). \quad (39.8)$$

It is very essential that, as can be seen from (39.3)–(39.8), the perturbation energy $H^{(1)}$ contains operators acting on the spin variables of the particles only in the form of the total spin of the system $s$. This means that the matrix of the perturbing energy is diagonal with respect to the eigenvalues of the operator $s^2$, which are given by $s^2 = s(s+1)$. From this is follows that positronium levels can be divided into singlet or parapositronium levels ($s = 0$), and triplet or orthopositronium levels ($s = 1$).

### 39.3. *Fine Structure*

The operators $V_1$ and $V_2$ are diagonal with respect to the orbital quantum numbers $l$ and do not depend on the spin variables. The interaction described by these operators removes the degeneracy with respect to $l$. In order to determine the correction to the energy level we must evaluate the expectation values of these operators ($\overline{V}_1$ and $\overline{V}_2$) in the unperturbed states. By utilizing the unperturbed wave equation (39.2) we can easily obtain $\overline{V}_1$:

$$\overline{V}_1 = -\frac{1}{4mc^2}\left(E^2 + 2Ee^2\overline{r^{-1}} + e^4\overline{r^{-2}}\right) - \frac{e^2}{4m^2c^2}\overline{\left(\mathbf{p}^2\frac{1}{r} - \frac{1}{r}\mathbf{p}^2\right)}. \quad (39.9)$$

The last term vanishes. Indeed,

$$\mathbf{p}^2\frac{1}{r} - \frac{1}{r}\mathbf{p}^2 = 4\pi\hbar^2\,\delta(r) + \frac{2i\hbar}{r^3}\mathbf{r}\mathbf{p},$$

Furthermore, we have

$$\overline{\delta(\mathbf{r})} = \Phi^2(0).$$

and

$$\frac{i}{\hbar}\overline{\frac{1}{r^3}(\mathbf{rp})} = \int \Phi^* \frac{1}{r^2}\frac{\partial \Phi}{\partial r}\,dr = \int_0^\infty R\frac{dR}{dr}\,dr = -\frac{1}{2}R^2(0),$$

where $R$ is the radial wave function; since $R(0)$ differs from zero only when $l = 0$, then $R^2(0) = 4\pi\Phi^2(0)$. Consequently,

$$\overline{\mathbf{p}^2\frac{1}{r} - \frac{1}{r}\mathbf{p}^2} = 0.$$

For the evaluation of the different expectation values we can use the well-known formulas which hold for the hydrogen atom (taking the reduced mass into account);

$$E = -\frac{\varepsilon_0}{4n^2}, \qquad \Phi^2(0) = \delta_{l0}\frac{1}{8\pi a_0^3 n^3},$$

$$\overline{r^{-1}} = \frac{1}{2a_0 n^2}, \qquad \overline{r^{-2}} = \frac{1}{2a_0^2 n^3(2l+1)}, \tag{39.10}$$

$$\overline{r^{-3}} = \frac{1}{4a_0^3 n^3 l(l+1)(2l+1)},$$

where $\varepsilon_0 = e^2/a_0$, $a_0 = \hbar^2/me^2$ (the Bohr radius) and $n$ is the principal quantum number.

On utilizing these values we obtain

$$\overline{\mathrm{V}}_1 = -\alpha^2\varepsilon_0\frac{1}{8n^3}\left(\frac{1}{2l+1} - \frac{3}{8n}\right), \qquad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}. \tag{39.11}$$

Similarly, in accordance with (39.5) and on utilizing (39.2) we obtain

$$\overline{\mathrm{V}}_2 = 3\pi\left(\frac{e\hbar}{mc}\right)^2\Phi^2(0) - \frac{e^2}{mc^2}\overline{\left(\frac{E+(e^2/r)}{r}\right)} + \frac{1}{2}\left(\frac{e\hbar}{mc}\right)^2 l(l+1)\overline{r^{-3}}.$$

We note that $\overline{lr^{-3}}$ must be taken equal to zero for $l = 0$, in spite of the fact that $\overline{r^{-3}}$ diverges for $l = 0$ as $r \to 0$. Indeed, in the derivation of the expression for $\mathrm{U}^{(e)}$ the terms for which the origin played an important role in the matrix elements were treated separately. In the product $\overline{lr^{-3}}$ the factor $l$ arises as a result of integration over the angles which, as we have already noted in § 38, can be carried out first in the case of singular terms.

On utilizing (39.10) we obtain

$$\overline{\mathrm{V}}_2 = \alpha^2\varepsilon_0\left[\frac{1}{4n^3}\,\delta_{l0} - \frac{1}{8n^3}\left(\frac{3}{2l+1} - \frac{1}{n}\right)\right]. \tag{39.12}$$

The operator $\mathrm{V}_3$ is also diagonal with respect to $l$ and $s$. Its eigenvalues are equal to zero in singlet states, and depend on the value of the total angular momentum $j$ in triplet states. On utilizing the fact that $2\mathbf{ls} = j(j+1)-l(l+1)-s(s+1)$, we may directly write·

$$\overline{\mathrm{V}}_3 = \begin{cases} \dfrac{3}{16}\alpha^2\varepsilon_0\dfrac{j(j+1)-l(l+1)-2}{n^3 l(l+1)(2l+1)} & \text{for } s = 1,\ l \neq 0, \\ 0 & \text{for } s = 0, \text{ or } l = 0. \end{cases} \tag{39.13}$$

In the operator $\mathrm{V}_4$ of (39.7) we treat the two terms separately. The second of these is diagonal with respect to $s$ and differs from zero only for $l = 0$. It determines the expectation value of $\mathrm{V}_4$ for $l = 0$:

$$\overline{\mathrm{V}}_4 = \frac{8\pi}{3}\left(\frac{e\hbar}{mc}\right)^2\frac{1}{4}\,[2s(s+1)-3]\Phi^2(0),$$

or

$$\overline{\mathrm{V}}_4 = \frac{1}{4n^3}\,\alpha^2\varepsilon_0\begin{cases} -1 & \text{for} \quad s = 0 \\ \dfrac{1}{3} & \text{for} \quad s = 1 \end{cases} \quad (l = 0). \tag{39.14}$$

The first term in (39.7) has matrix elements different from zero only for $l \neq 0$, $\mathrm{s} = 1$, both in the case of terms diagonal and nondiagonal with respect to $l$. From the laws of conservation of total angular momentum, spin angular momentum and parity it can be easily seen that the only nonvanishing nondiagonal elements of $\mathrm{V}_4$ are those which connect states with the same values of $j$ and $M$ for $s = 1$ and $l = j \pm 1$.

For the evaluation of the matrix elements we must write the positronium wave functions $\Phi_{jlsM}$ in the form

$$\Phi_{jlsM} = \sum_{m+\mu=M} C^{jM}_{lm\,s\mu}\varphi_{lm}\chi_{s\mu},$$

where $\varphi_{lm}$ and $\chi_{s\mu}$ are the orbital and the spin functions, as a result of which we obtain

$$(l'|\mathrm{V}_4|l) = 6\left(\frac{e\hbar}{2mc}\right)^3 \sum C^{jM}_{lm1\mu} C^{jM}_{l'm'1\mu'} (l'|r^{-3}|l)$$

$$\times \left(l'm'\left|\frac{x_i x_k}{r^2} - \frac{1}{3}\delta_{ik}\right|lm\right)(\mu'|\mathrm{s}_i\mathrm{s}_k|\mu).$$

The last two factors in this expression can be easily evaluated, since the angular and the spin functions are of a simple form. After this we can easily carry out the summation over $m$, $m'$ and $\mu$ and express $(l'|\mathrm{V}_4|l)$ in terms of only the radial matrix elements. We shall state the result.

After the application of (39.10) the diagonal elements assume the form

$$(l|\mathrm{V}_4|l) = \frac{\alpha^2\varepsilon_0}{8l(l+1)(2l+1)n^3}\begin{cases} 1 & \text{for} \quad j=l, \\ -\dfrac{l}{2l+3} & \text{for} \quad j=l+1, \\ -\dfrac{l+1}{2l-1} & \text{for} \quad j=l-1 \end{cases} \tag{39.15}$$

$$(l\neq 0, s=1),$$

while the nondiagonal elements become

$$(j+1|\mathrm{V}_4|j-1) = \alpha^2\varepsilon_0\frac{\sqrt{j(j+1)}}{2j+1}(j+1|r^{-3}|j-1). \tag{39.16}$$

We shall not carry out the evaluation of $(j+1|r^{-3}|j-1)$ here since in the general form this does not lead to simple expressions, while in all special cases referring to the discrete spectrum the calculations are elementary. In particular, for $j=1$ and $n=3$ (39.16) vanishes, so that the orbital quantum number remains a good quantum number in the case of the first three principal quantum numbers.

Finally, the expectation values of $\mathrm{V}_5$ differ from zero only for $l = 0$, $s=1$:

$$\overline{\mathrm{V}}_5 = \begin{cases} 0 & \text{for} \quad s=0, \\ \left(\dfrac{\alpha^2}{4n^3}\right)\varepsilon_0\delta_{l0} & \text{for} \quad s=1. \end{cases} \tag{39.17}$$

Thus, of all the terms making up the perturbation energy $H^{(1)}$ only $V_4$ contains nondiagonal elements belonging to the case $l=j+1$, $l'=j-1$ and $s=1$. Therefore both singlet and triplet states for which $l=j$ (parity $(-1)^j$) can be classified by their orbital angular momentum. We denote the shift of each such level with respect to its unperturbed value by $w({}^{2s+1}l_j)$ (the spectroscopic symbol of the term is indicated in brackets). Then we have

$$w({}^1j_j) = -\alpha^2\varepsilon\frac{1}{2n^3}\left(\frac{1}{2j+1}-\frac{11}{32n}\right), \tag{39.18}$$

$$w({}^3j_j) = w({}^1j_j)-\frac{\alpha^2\varepsilon}{8n^3j(j+1)\,(2j+1)}. \tag{39.19}$$

Terms of parity $(-1)^{j+1}$ are superpositions of unperturbed terms for which $l=j\pm1$. For these terms we have

$$w=\frac{\bar{V}_+-\bar{V}_-}{2}\pm\sqrt{\left(\frac{\bar{V}_++\bar{V}_-}{2}\right)^2+(j+1|V_4|j-1)^2}, \tag{39.20}$$

where

$$\begin{aligned}V_-&\equiv\bar{V}\left({}^3(j-1)_j\right)\\&\equiv w\left({}^1(j-1)_{j-1}\right)+\alpha^2\varepsilon_0\left[\frac{2}{3n^2}\delta_{j-1,0}+\frac{1}{4n^3j\,(2j-1)}\left(\frac{3}{2}-\frac{1}{2j+1}\right)\right],\end{aligned}$$

$$V_+\equiv\bar{V}\left({}^3(j+1)_j\right)\equiv w\left({}^1(j+1)_{j+1}\right)-\frac{\alpha^2\varepsilon_0}{4n^3(j+1)\,(2j+3)}\left(\frac{3}{2}+\frac{1}{2j+1}\right)$$

and $(j+1|V_4|j-1)$ is defined by formula (39.16).

These formulas enable us, in particular, to determine the energy difference between the ground states of ortho- and parapositronium. The dependence of the energy on $s$ for $l=0$ is contained only in formulas (39.14) and (39.17) which define $V_4$ and $V_5$. From these formulas it follows that the ground level of orthopositronium lies above the ground level of parapositronium, with the energy difference equal to

$$\Delta=\frac{1}{4}\varepsilon_0\alpha^2\left(\frac{4}{3}+1\right)=8.2\cdot10^{-4}\text{eV}. \tag{39.21}$$

The first term in brackets corresponds here to the magnetic interaction $V_4$, and the second term to the exchange interaction $V_5$.

### 39.4. *Zeeman Effect*

We investigate the Zeeman effect in positronium (10). The special features of this effect are associated with the fact that the orbital magnetic moment of positronium is equal to zero, while its spin magnetic moment is not proportional to the spin angular momentum.

Indeed, since in positronium $[\boldsymbol{r}_1\mathbf{p}_1] = [\boldsymbol{r}_2\mathbf{p}_2]$, the orbital magnetic moment operator is given by

$$\boldsymbol{\mu}_L = \frac{e\hbar}{2mc}([\boldsymbol{r}_1\mathbf{p}_1]+[\boldsymbol{r}_2\mathbf{p}_2]) = 0.$$

The spin magnetic moment operator

$$\boldsymbol{\mu} = \frac{e\hbar}{2mc}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$$

does not commute with the total spin operator $\boldsymbol{s} = \frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$. Therefore, the eigenstates of the operators $\boldsymbol{\mu}$ and $\boldsymbol{s}$ do not coincide in the general case.

We consider the four eigenfunctions $\chi_{s\mu}$ of the operators $\boldsymbol{s}^2$ and $s_z$, corresponding to definite values of the spin $s$ and of its component $s_z$:

$$\chi_{11} = \alpha(1)\alpha(2),$$

$$\chi_{1,-1} = \beta(1)\beta(2),$$

$$\chi_{10} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2)+\beta(1)\alpha(2)],$$

$$\chi_{00} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2)-\beta(1)\alpha(2)],$$

where $\alpha$ and $\beta$ are the two spin functions for one particle corresponding to the values $\sigma_z = \pm 1$, while the numbers in brackets denote: (1) an electron, (2) a positron. With the aid of these functions we evaluate the matrix elements for the operator of the perturbing energy for a positron in an external magnetic field H; $V^{(H)} = -\mu_z H$. We can easily show that all the matrix elements $(ss_z\,|\,V^{(H)}\,|\,s's'_z)$ vanish with the exception of

$$(00\,|\mu_z|\,10) = (10|\mu_z|00) = \frac{e\hbar}{mc}H.$$

Thus, the matrix for the energy of interaction with the magnetic field contains only elements which connect singlet states with triplet states. But, as we have seen, the perturbing energy V which determines the fine structure of the positronium levels does not contain such elements. Therefore, in weak fields when $(e\hbar/mc)H \ll |W_1 - W_0|$, where $W_1$ and $W_0$ are the values of the energy for the positronium triplet and singlet levels, there is no effect proportional to the field.

In the opposite limiting case of a very strong field the interaction between the spins may be neglected and the splitting of the levels will be given by $\pm(e\hbar/mc)H$ corresponding to states described by the wave functions $\alpha(1)\beta(2)$ and $\alpha(2)\beta(1)$.

For the ground state (and also for $S$-states with $n=2$ and 3) we can easily write the exact formula for splitting in a magnetic field. In this case the wave functions for the stationary states are given by superpositions of the functions $\chi_{10}$, $\chi_{00}$:

$$\chi^{(i)} = C_1^{(i)}\chi_{10} + C_0^{(i)}\chi_{00}, \qquad i = 1,\ 2, \tag{39.22}$$

where $i = 1$ corresponds to the state which reduces to the triplet state in the absence of a magnetic field ($C_1^{(1)} = 1$, $C_0^{(1)} = 0$, $H = 0$), while $i = 2$ corresponds to the state which reduces to the singlet state for ($C_1^{(2)} = 0$, $C_2^{(2)} = 1$, $H = 0$). The coefficients $C^{(i)}$ satisfy the equations

$$\begin{aligned} (E^{(i)} - W_1)C_1^{(i)} + \frac{e\hbar}{mc} HC_0^{(i)} &= 0, \\ \frac{e\hbar}{mc} HC_1^{(i)} + (E^{(i)} - W_0)C_0^{(i)} &= 0, \end{aligned} \tag{39.23}$$

where $E^{(i)}$ is the positronium energy in the field $H$.

It follows from (39.23) that

$$E^{(i)} = \frac{W_1 - W_0}{2} \pm \sqrt{\frac{(W_1 - W_0)^2}{4} + \left(\frac{e\hbar}{mc} H\right)^2}. \tag{39.24}$$

The ratio $|C_1^{(i)}/C_0^{(i)}|^2$ defines the relative weight of the triplet and singlet states in a given stationary state of positronium in a magnetic field.

The magnetic field has an important effect on the probability of positronium decay. We denote by $w_0$ and $w_1$ the decay probabilities per unit time for para- and orthopositronium. In the presence of a magnetic field the probability of positronium decay in the state $\chi^{(i)}$ is given by

$$w^{(i)} = |C_1^{(i)}|^2 w_1 + |C_0^{(i)}|^2 w_0 \qquad (|C_1^{(i)}|^2 + |C_0^{(i)}|^2 = 1). \tag{39.25}$$

For the state $\chi^{(1)}$ (triplet state in the absence of the magnetic field) even a small value of the coefficient $C_0$ greatly increases the probability of decay, since (cf. § 33) $w_0 \gg w_1$. In a weak magnetic field we have in accordance with (39.23), (39.24)

$$\left|\frac{C_0^{(1)}}{C_1^{(1)}}\right|^2 = \frac{\left((e\hbar/mc)H\right)^2}{(E_{10}-E_{00})^2}. \tag{39.26}$$

A measurement of the effect of the magnetic field on positronium decay enables us, by utilizing formula (39.26), to determine experimentally (45) the energy difference between the ground levels of ortho- and parapositronium $\Delta = E_{10}-E_{00}$. The experimentally found value of $\Delta$ agrees with the theoretical value (39.21).

## § 40. Internal Conversion of Gamma-Rays

### 40.1. *Expansion of Retarded Potentials in Spherical Waves*

Because of its electromagnetic interaction with external electrons a nucleus in an excited state may make a transition to a state of lower energy giving up its excitation energy to the electrons in the atomic shells, or producing an electron-positron pair. Processes which are not accompanied by emission of radiation have the general name of internal conversion of $\gamma$ rays. A similar phenomenon in atomic shells, i.e., an interaction between two electrons as a result of which one electron makes a transition to a state of lower energy, while the other makes a transition into the continuum, is called the Auger effect.

The phenomenon of internal conversion in atomic shells may be investigated on the basis of the retarded interaction between two charges described in § 37, one of the two charges being an atomic electron, while the other one is a proton in the nucleus.

We denote the electron wave functions in the initial and final states by $\psi_1$ and $\psi_2$, and the proton wave functions correspondingly by $\Psi_1$ and $\Psi_2$. In accordance with (37.8) and (37.15) the matrix element of the effective perturbing energy is given by

$$\begin{aligned} U_{i\to f} &= -\alpha \int \bar{\Psi}_2(\boldsymbol{r}_1)\gamma_\mu \Psi_1(\boldsymbol{r}_1)\frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\bar{\psi}_2(\boldsymbol{r}_2)\gamma_\mu\psi_1(\boldsymbol{r}_2)\,d\boldsymbol{r}_1 d\boldsymbol{r}_2 \\ &= -\alpha \int \Psi_2^*(\boldsymbol{r}_1)\psi_2^*(\boldsymbol{r}_2)(1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)\frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\Psi_1(\boldsymbol{r}_1)\psi_1(\boldsymbol{r}_2)\,d\boldsymbol{r}_1 d\boldsymbol{r}_2, \end{aligned} \tag{40.1}$$

where $\omega = E_1 - E_2 = \varepsilon_2 - \varepsilon_1$, $\alpha = e^2/4\pi = 1/137$, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are Dirac matrices which operate respectively on the proton and the electron spin variables, $E_{1,2}$ are the proton energies, while $\varepsilon_{1,2}$ are the electron energies.

The evaluation of these integrals is facilitated by the fact that the dimensions of the regions within which the proton and the electron motions take place are different. For the proton this region is of the order of magnitude of the nuclear radius. But for the electron the analogous region is of considerably greater dimensions. Therefore, we can assume that in the integrals of (40.1) the important contributions come from a range of variation of the variables for which $r_1 < r_2$. Under this condition the following expansion holds[1]:

$$\frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \frac{i\omega}{4\pi}\sum_{lm} \varphi^*_{lm}(\boldsymbol{r}_1)\,\Phi_{lm}(\boldsymbol{r}_2), \tag{40.2}$$

where

$$\begin{aligned}
\varphi_{lm}(\boldsymbol{r}) &= g_l(\omega r)\,Y_{lm}\left(\frac{\boldsymbol{r}}{\mathrm{r}}\right),\\
\Phi_{lm}(\boldsymbol{r}) &= G_l(\omega r)\,Y_{lm}\left(\frac{\boldsymbol{r}}{\mathrm{r}}\right),\\
g_l(z) &= (2\pi)^{3/2} i^l \frac{J_{l+\frac{1}{2}}(z)}{\sqrt{z}},\\
G_l(z) &= (2\pi)^{3/2} i^l \frac{H^{(1)}_{l+\frac{1}{2}}(z)}{\sqrt{z}}
\end{aligned} \tag{40.3}$$

and $J_n(z)$ are Bessel functions, $H^{(1)}_n(z)$ are Hankel functions of the first kind, and $Y_{lm}$ are spherical harmonics. We now show that the following expansion also holds:

$$\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 \frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \frac{i\omega}{4\pi}\sum_{LlM} \left(\boldsymbol{\alpha}_1 \boldsymbol{A}^*_{LlM}(\boldsymbol{r}_1)\right)\left(\boldsymbol{\alpha}_2 \boldsymbol{B}_{LlM}(\boldsymbol{r}_2)\right), \tag{40.4}$$

---

[1] The expansion (40.2) is the addition theorem for cylindrical functions applied to the function

$$H^{(1)}_{1/2}(\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|) \equiv -i\sqrt{\frac{2}{\pi\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}\, e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}.$$

where

$$\begin{aligned} \boldsymbol{A}_{LlM} &= g_l(\omega r)\, \boldsymbol{Y}_{LlM}, \\ \boldsymbol{B}_{LlM} &= G_l(\omega r)\, \boldsymbol{Y}_{LlM}, \end{aligned} \tag{40.5}$$

$\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are arbitrary vectors which do not depend on the coordinates, and $\boldsymbol{Y}_{LlM}$ are vector spherical harmonics defined in § 4. We note in this connection that any vector function can be expanded in terms of a complete system of vector spherical harmonics $\boldsymbol{Y}_{LlM}$. Thus, the vector

$$\boldsymbol{\alpha}_1 \frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|},$$

regarded as a function of $\boldsymbol{r}_2/r_2$ can be written in the form

$$\boldsymbol{\alpha}_1 \frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \frac{i\omega}{4\pi} \sum_{LlM} C^{LlM}(\boldsymbol{r}_1, r_2)\, \boldsymbol{Y}_{LlM}\left(\frac{\boldsymbol{r}_2}{r_2}\right). \tag{40.6}$$

In order to determine the expansion coefficients $C^{LlM}$ we multiply this equation by $\boldsymbol{Y}^*_{L'l'M'}(\boldsymbol{r}_2/r_2)$ and integrate over all directions of the unit vector $\boldsymbol{r}_2/r_2$. By utilizing expansion (40.2) and also the orthogonality property of vector spherical harmonics, and their relation to spherical harmonics (cf. § 4) we obtain

$$C^{LlM}(\boldsymbol{r}_1, r_2) = (\boldsymbol{\alpha}_1 \boldsymbol{A}^*_{LlM}(\boldsymbol{r}_1))\, G_l(\omega r_2).$$

Substituting this expression into the expansion (40.6), and multiplying it by $\boldsymbol{\alpha}_2$ we obtain (40.4).

Instead of the set of vectors $\boldsymbol{A}_{LlM}$ and $\boldsymbol{B}_{LlM}$ (for a given $L$ and $M, l = L\pm 1, L$) we can use the vectors $\boldsymbol{a}^{(\lambda)}_{LM}$ defined in (27.6), (27.16), and similarly defined vectors $\boldsymbol{B}^{(\lambda)}_{LM}$ (the difference between $\boldsymbol{B}^{(\lambda)}_{LM}$ and $\boldsymbol{a}^{(\lambda)}_{LM}$ consists of the replacement of $g_l$ by $G_l$). The vectors $\boldsymbol{a}^{(\lambda)}_{LM}$ and $\boldsymbol{B}^{(\lambda)}_{LM}$ have the property that, as was shown in § 4, they arc asymptotically (for $\omega r \gg 1$) transverse (for $\lambda = 0, 1$) or longitudinal (for $\lambda = -1$). It can be easily shown that

$$\sum_{l=L-1}^{L+1} (\boldsymbol{\alpha}_1 \boldsymbol{A}^*_{LlM})(\boldsymbol{\alpha}_2 \boldsymbol{B}_{LlM}) = \sum_{\lambda=-1}^{1} (\boldsymbol{\alpha}_1 \boldsymbol{a}^{(\lambda)*}_{LM})(\boldsymbol{\alpha}_2 \boldsymbol{B}^{(\lambda)}_{LM}).$$

On substituting these expansions into (40.1) we obtain

$$U_{i\to f} = -e^2 \frac{i\omega}{4\pi} \sum_{LM} \left\{ (\varphi^*_{LM})_{21} (\Phi_{LM})_{21} - \sum_{\lambda=-1}^{1} (\boldsymbol{\alpha} \boldsymbol{a}^{(\lambda)*}_{LM})_{21} (\boldsymbol{\alpha} \boldsymbol{B}^{(\lambda)}_{LM})_{21}, \right\}, \tag{40.7}$$

where brackets denote matrix elements of corresponding quantities evaluated using electron or proton states:

$$(\varphi^*_{LM})_{21} = \int \Psi_2^* \varphi^*_{LM} \Psi_1 \, d\boldsymbol{r},$$

$$(\Phi_{LM})_{21} = \int \psi_2^* \Phi_{LM} \psi_1 \, d\boldsymbol{r},$$

$$(\boldsymbol{\alpha a}^{(\lambda)*}_{LM})_{21} = \int \Psi_2^* \boldsymbol{\alpha a}^{(\lambda)*}_{LM} \Psi_1 \, d\boldsymbol{r},$$

$$(\boldsymbol{\alpha B}^{(\lambda)}_{LM})_{21} = \int \psi_2^* \boldsymbol{\alpha B}^{(\lambda)}_{LM} \psi_1 \, d\mathbf{r}$$

(In the future we shall sometimes omit the subscripts from these brackets.)

Expression (40.7) can be simplified if we use certain relations obtained in § 27. In accordance with (27.8) and (27.10′) we have $(\boldsymbol{\alpha a}^{(-1)*}_{LM}) = (\varphi^*_{LM})$. Moreover, under the condition $\omega R \ll 1$, which we assume to hold ($R$ is the nuclear radius), according to (27.7)

$$\boldsymbol{a}^{(-1)}_{LM} = \sqrt{\frac{L}{L+1}}\, \boldsymbol{a}^{(1)}_{LM}.$$

In virtue of this, three of the four terms appearing in the figure brackets of (40.7) can be combined in the manner

$$(\varphi^*_{LM})(\Phi_{LM}) - (\boldsymbol{\alpha a}^{(-1)*}_{LM})(\boldsymbol{\alpha B}^{(-1)}_{LM}) - (\boldsymbol{\alpha a}^{(1)*}_{LM})(\boldsymbol{\alpha B}^{(1)}_{LM}) = (\boldsymbol{\alpha a}^{(1)*}_{LM})(B^{(1)}_{LM}), \quad (40.8)$$

where

$$\begin{aligned} B^{(1)}_{LM} &= -\boldsymbol{\alpha B}^{(1)}_{LM} + \sqrt{\frac{L}{L+1}}\,(\Phi_{LM} - \boldsymbol{\alpha B}^{(-1)}_{LM}) \\ &= \sqrt{\frac{L}{L+1}}\,\Phi_{LM} - \sqrt{\frac{2L+1}{L+1}}\,\boldsymbol{\alpha B}_{L,L-1,M}. \end{aligned} \quad (40.9)$$

In the future we shall also use the notation

$$B^{(0)}_{LM} = -\boldsymbol{\alpha B}^{(0)}_{LM}. \quad (40.10)$$

We note that in spite of the fact that $\boldsymbol{B}^{(-1)}_{LM}$ and $\Phi_{LM}$ are related by an expression analogous to (27.8):

$$\boldsymbol{B}^{(-1)}_{LM} = \frac{i}{\omega} \nabla \Phi_{LM},$$

the matrix elements $(\Phi_{LM})_{21}$ and $(\boldsymbol{\alpha B}^{(-1)}_{LM})_{21}$ cannot be expressed in terms of each other in analogy with (27.10′), since the transformations which we used in § 27 are inadmissible in this case because of the singularity of the function $\Phi_{LM}$. Indeed, due to the presence of a pole at the point

$r = 0$, in the course of integration by parts which is involved in the transition from (27.8) to (27.10′) there remains an integral over the surface surrounding the pole which does not vanish as $r \to 0$. In other words, the operator H is not self-conjugate with respect to the functions $\Phi_{LM}$.

Thus, the matrix element corresponding to internal conversion can be written in the form

$$U_{i\to f} = -\alpha \frac{i\omega}{4\pi} \sum_{LM} \{(\boldsymbol{\alpha a}_{LM}^{(1)*})_{21} (B_{LM}^{(1)})_{21} + (\boldsymbol{\alpha a}_{LM}^{(0)*})_{21} (B_{LM}^{(0)})_{21}\}. \qquad (40.11)$$

## 40.2. *Conversion Coefficient*

The expression for the conversion matrix element (40.11) admits a simple physical interpretation. Here $(\boldsymbol{\alpha a}_{LM}^{(1)})_{21}$ and $(\boldsymbol{\alpha a}_{LM}^{(0)})_{21}$ are, apart from numerical coefficients, the matrix elements for the emission of a photon of angular momentum $L$ and parity $(-1)^L$ or $(-1)^{L+1}$ which are proportional to the matrix element of the electric or the magnetic $2^L$-pole nuclear moment (cf. § 27). The operators $B_{LM}^{(\lambda)}$ for which the matrix elements are evaluated with respect to the electron states can be interpreted as the perturbation energy due to the field associated with the nuclear transition. The quantities $B_{LM}^{(\lambda)}$ and $\Phi_{LM}$ can be called the potentials of the electric or the magnetic multipole. Indeed, the potentials $\Phi_{LM}$ and $B_{LM}^{(\lambda)}$ are, as can be easily shown, solutions of the wave equation (37.17) having the form of outgoing spherical waves with a singularity at the origin. This singularity corresponds to the presence at the point $r = 0$ of a radiating electric or magnetic multipole of the $L$-th order ($2^L$-pole). The process of conversion can be interpreted as the absorption of these waves by the electrons near the nucleus.

We note that the expression $(B_{LM}^{(\lambda)})$ differs from the matrix element of the operator $A_{0LM\lambda} - \boldsymbol{\alpha} A_{LM\lambda}$, which corresponds to the absorption of a photon of definite angular momentum and parity only by the replacement of the regular function $g_l$ by the function $G_l$ which has a singularity at $r = 0$. It should be noted that the potentials of the electric multipole (40.9) have the structure of (6.18) which corresponds to a definite choice of the arbitrary constant in the general expression for the photon potentials (6.17). An arbitrary choice of the constant in formula (6.17) for $A_{LM\lambda}$, which is made possible by gauge invariance, is not allowed in this case because of the singular nature of the potentials.

Let the angular momentum of the nucleus in the initial state be $J_1$, and in the final state $J_2$. Then the matrix element $(\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(\lambda)})_{21}$ will differ from zero if the condition $|J_1-J_2| \leqslant L \leqslant J_1+J_2$ is satisfied. If $\omega R \ll 1$ then only the smallest possible value of $L$ is important. For a given value of $L$ the matrix elements $(\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(0)})_{21}$ and $(\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(1)})_{21}$ cannot both simultaneously differ from zero in virtue of the parity selection rule. Cases are possible when both an electric and a magnetic multipole, of orders differing by unity, take part in the process, for example $(\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(1)})_{21}$ and $(\boldsymbol{\alpha} \boldsymbol{a}_{L+1,M}^{(1)})_{21}$. However, we shall not investigate such processes here. An estimate of such matrix elements (cf. § 27) leads to the conclusion that, generally speaking, they are of different orders of magnitude. Therefore, in the majority of cases the process of conversion is determined by only a single term from the sum (40.11). The general expression for the conversion probability

$$w = 2\pi |U_{i\to f}|^2 \, \delta(E_1-E_2-\varepsilon_1+\varepsilon_2)$$

can be simplified as a result of this.

In the case when the angular momentum of the nucleus changes[1] by $L$

$$|J_1-J_2| = L$$

and parity changes by $(-1)^L$, the conversion probability is determined by the potential of the electric $2^L$-pole and is equal to

$$w_L^{(1)} = \frac{\alpha^2}{8\pi} \omega^2 |(\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(1)*})_{21}|^2 \, |(B_{LM}^{(1)})_{21}|^2 \, \delta(E_1-E_2-\varepsilon_1+\varepsilon_2).$$

In the case when the angular momentum of the nucleus changes by $L$ and parity changes by $(-1)^{L+1}$ the conversion probability is determined by the potential of the magnetic $2^L$-pole, and is equal to

$$w_L^{(0)} = \frac{\alpha^2}{8\pi} \omega^2 |(\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(0)*})_{21}|^2 \, |(B_{LM}^{(0)})_{21}|^2 \, \delta(E_1-E_2-\varepsilon_1+\varepsilon_2).$$

The *conversion coefficient* is defined as the ratio of the probability of the conversion process to the probability of photon emission in the same nuclear transition. In accordance with the results of § 27 the probability of emission is given by

---

[1] If $J_1 = J_2 \neq 0$, and the parity changes, then the dipole transition ($L = 1$) is the most probable one. For the case $J_1 = J_2$ without a change of parity cf. § 42.

$$w_{L\,\mathrm{rad}}^{(\lambda)} = \frac{\alpha\omega}{2\pi} |(\boldsymbol{\alpha}\boldsymbol{a}_{LM}^{(\lambda)})_{21}|^2 .$$

Therefore, the conversion coefficient in the case of an electric ($\lambda = 1$) or a magnetic ($\lambda = 0$) $2^L$-pole is given by

$$\beta_L^{(\lambda)} = \frac{w_{L\,\mathrm{conv}}^{(\lambda)}}{w_{L\mathrm{rad}}^{(\lambda)}} = \frac{\alpha}{4}\omega |(B_{LM}^{(\lambda)})_{21}|^2 \,\delta(E_1 - E_2 - \varepsilon_1 + \varepsilon_2) . \qquad (40.12)$$

It is assumed here that the electron wave functions are normalized so that $\int \psi_1^* \psi_1 \, d\boldsymbol{r} = \int \psi_2^* \psi_2 \, d\boldsymbol{r} = 1$.

If the wave function $\psi_2$ (belonging to the continuous spectrum) is normalized per unit energy, i.e., if we assume that $\int \psi_\varepsilon^* \psi_{\varepsilon'} \, d\boldsymbol{r} = \delta(\varepsilon - \varepsilon')$, and at the same time retain the former normalization for $\psi_1$, the expression for the conversion coefficient assumes the form

$$\beta_L^{(\lambda)} = \frac{\alpha\omega}{4} \left| \int \psi_2^* B_{LM}^{(\lambda)} \psi_1 \, d\boldsymbol{r} \right|^2 . \qquad (40.13)$$

### 40.3. *Conversion in the K-shell*

It is important that the conversion coefficient $\beta_L^{(\lambda)}$ contains no quantities which refer to nuclear properties. In order to evaluate it, a knowledge of only the electron wave functions is sufficient. If in the initial state the electron is in the $K$-shell, then its conversion coefficient is equal to

$$\beta_L^{(\lambda)} = \frac{\alpha\omega}{4} \sum_K \sum_f |(B_{LM}^{(\lambda)})_{fK}|^2 .$$

Here $\lambda$ denotes one of the two indices (1 or 0) which indicate whether the transition is of electric or magnetic type; $\Sigma_K$ denotes summation over the two electrons of the $K$-shell, and $\Sigma_f$ denotes summation over the possible final states, i.e., those allowed by the selection rules. In order to simplify the calculations we can use the fact that for a closed shell the conversion coefficient cannot depend on the magnetic quantum number of the electron $m$. Indeed, the latter depends on the choice of the $z$-axis, while a closed shell is spherically symmetrical. We can, therefore, determine $\beta_L^{(\lambda)}$ assuming $M = 0$. Moreover, since the perturbation does not depend on the azimuth the conversion probability for $M = 0$ cannot be different for two electrons which differ only by the quantum number $m = \pm\frac{1}{2}$. Therefore,

$$\beta_L^{(\lambda)} = \frac{\alpha\omega}{2} \sum_f |(B_{L0}^{(\lambda)})|^2.$$

The final state must have the energy $\varepsilon_2 = \omega - |\varepsilon_K|$, where $|\varepsilon_K| = -\varepsilon_K$ is the binding energy of the electron in the $K$-shell (conversion can take place if $\omega > |\varepsilon_K|$).

In order to determine the angular momenta of the final states we can use the laws of conservation of angular momentum and of parity. These laws show that the change in the angular momentum of the nucleus will be exactly the same as in the case of emission of a photon of angular momentum $L$ and of parity $(-1)^L$ or $(-1)^{L+1}$, and that the selection rules for the final state of the electron will be the same as in the case of absorption of such a photon.

This can also be shown directly by a consideration of the angular part of the integral in the matrix element (40.11) which coincides with the angular part of the matrix element for photon absorption.

Since for the electron in the initial state ($K$-shell) we have $j_1 = \frac{1}{2}$ and $l_1 = 0$, then in the final state in the case of an electric multipole we have $l_2 = L$ and $j_2 = l_2 \pm \frac{1}{2} = L \pm \frac{1}{2}$.

In the case of a magnetic multipole we have either $l_2 = L+1$, $j_2 = l_2 - \frac{1}{2} = L + \frac{1}{2}$ or $l_2 = L-1$ and $j_2 = l_2 + \frac{1}{2} = L - \frac{1}{2}$. Thus

$$\begin{aligned} \beta_L^{(1)} &= \frac{\alpha}{2}\omega\{|(B_{L0}^{(1)})_{L+\frac{1}{2},L}|^2 + |(B_{L0}^{(1)})_{L-\frac{1}{2},L}|^2\}, \\ \beta_L^{(0)} &= \frac{\alpha}{2}\omega\{|(B_{L0}^{(0)})_{L+\frac{1}{2},L+1}|^2 + |(B_{L0}^{(0)})_{L-\frac{1}{2},L-1}|^2\}. \end{aligned} \tag{40.14}$$

(The two indices on the matrix elements indicate the quantum numbers of the final state of the electron: the first is $j_2$, the second is $l_2$.)

For the electron wave functions we use the general expression (12.4)

$$\psi_{jlm} = \begin{pmatrix} ig_\varkappa(r)\Omega_{jlm}(\boldsymbol{n}) \\ -f_\varkappa(r)\Omega_{jl'm}(\boldsymbol{n}) \end{pmatrix}, \tag{40.15}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{r}}{r},\ l' = 2j - l,\ \Omega_{jl'm} = -(\boldsymbol{\sigma}\boldsymbol{n})\Omega_{jlm}.$$

In particular, the electron wave function for the $K$-shell ($j = \frac{1}{2}$, $l = 0$, $\varkappa = -1$) can be written in the form

$$\psi_K = \begin{pmatrix} i g_K(r) u \\ f_K(r)(\boldsymbol{\sigma}\boldsymbol{n}) u \end{pmatrix}, \tag{40.16}$$

where $u$ is a constant spinor normalized as $u^* u = \frac{1}{4\pi}$. On utilizing the definition (40.9) for the quantity $B^{(1)}_{LM}$, we obtain

$$(B^{(1)}_{L0})_{j_2 l_2} = \sqrt{\frac{L}{L+1}}(\Phi_{L0})_{j_2 l_2} - \sqrt{\frac{2L+1}{L+1}}(\boldsymbol{\alpha}\boldsymbol{B}_{L,L-1,0})_{j_2 l_2}. \tag{40.17}$$

On substituting into this equation the expressions for the wave functions (40.15) and (40.16) and for the potentials (40.3), (40.5), we obtain

$$(\Phi_{L0})_{j_2 l_2} = \int (g^*_\varkappa g_K + f^*_\varkappa f_K) G_L r^2 dr\, u^*_{j_2 l_2} u \tag{40.18}$$

where the following notation has been introduced:

$$u^*_{jl} = \int \Omega^*_{jlm} Y_{l0} do. \tag{40.19}$$

(We note that for $l = 0$, $j = \frac{1}{2}$ and $u_{jl}$ coincides with $u$.)

The second term in (40.17) contains the quantity

$$(\boldsymbol{\alpha}\boldsymbol{B}_{L,L-1,0})_{j_2 l_2} = +i \int f^*_\varkappa g_K G_{L-1} r^2 dr \left(\int \Omega^*_{j_2 l_2 m_2}(\boldsymbol{\sigma}\boldsymbol{n})\boldsymbol{\sigma}\boldsymbol{Y}_{L,L-1,0} do\right) u$$
$$-i \int g^*_\varkappa f_K G_{L-1} r^2 dr \left(\int \Omega^*_{j_2 l_2 m_2}\boldsymbol{\sigma}(\boldsymbol{\sigma}\boldsymbol{n})\boldsymbol{Y}_{L,L-1,0} do\right) u. \tag{40.20}$$

The integrals over the angles in this expression can be brought into the form (40.19). Indeed, we expand the vector spherical harmonic $\boldsymbol{Y}_{L,L-1,0}$ into a transverse and a longitudinal component

$$\boldsymbol{Y}_{L,L-1,0} = \sqrt{\frac{L}{2L+1}}\boldsymbol{Y}^{(-1)}_{L0} + \sqrt{\frac{L+1}{2L+1}}\boldsymbol{Y}^{(1)}_{L0}.$$

Since $(\boldsymbol{\sigma}\boldsymbol{n})\boldsymbol{\sigma} = \boldsymbol{n} + i[\boldsymbol{\sigma}\boldsymbol{n}]$, we have

$$(\boldsymbol{\sigma}\boldsymbol{n})\boldsymbol{\sigma}\boldsymbol{Y}_{L,L-1,0} = \sqrt{\frac{L}{2L+1}}\boldsymbol{n}\boldsymbol{Y}^{(-1)}_{L0} + \sqrt{\frac{L+1}{2L+1}} i\boldsymbol{\sigma}[\boldsymbol{n}\boldsymbol{Y}^{(1)}_{L0}].$$

On expressing the longitudinal and the transverse vector spherical harmonics in terms of the sphesical harmonic

$$\boldsymbol{Y}^{(-1)}_{L0} = \boldsymbol{n} Y_{L0}, \quad -i[\boldsymbol{n}\boldsymbol{Y}^{(1)}_{L0}] = \boldsymbol{Y}^{(0)}_{L0} = \frac{1}{\sqrt{L(L+1)}}\mathbf{L} Y_{L0}$$

($\mathbf{L} = -i[\boldsymbol{r}\nabla]$ is the operator for the orbital angular momentum), we obtain

$$(\boldsymbol{\sigma}\boldsymbol{n})\boldsymbol{\sigma}Y_{I,L-1,0} = \left(\sqrt{\frac{L}{2L+1}} - \frac{\boldsymbol{\sigma}\mathbf{L}}{\sqrt{L(2L+1)}}\right)Y_{L0}.$$

Similary

$$\boldsymbol{\sigma}(\boldsymbol{\sigma}\boldsymbol{n})Y_{L,L-1,0} = \left(\sqrt{\frac{L}{2L+1}} + \frac{\boldsymbol{\sigma}\mathbf{L}}{\sqrt{L(2L+1)}}\right)Y_{L0}.$$

Moreover, making use of the fact that the operator $\mathbf{L}\boldsymbol{\sigma}$ is self-conjugate, we transform the integral

$$\int \Omega^*_{j_2l_2m_2}(\boldsymbol{n})\mathbf{L}\boldsymbol{\sigma}Y_{L0}(\boldsymbol{n})u\, do$$

in such a way that the operator $\boldsymbol{L\sigma}$ acts on the function $\Omega_{j_2l_2m_2}$ which is its own eigenfunction corresponding to the eigenvalue $j_2(j_2+1)-l_2(l_2+1)-\frac{3}{4}$.

We then obtain the following values for the integrals over the angles appearing in (40.20):

$$\begin{aligned}
&\int \Omega^*_{j_2l_2m_2}(\boldsymbol{\sigma}\boldsymbol{n})\boldsymbol{\sigma}Y_{L,L-1,0}\, u\, do \\
&\qquad = \frac{u^*_{j_2l_2}u}{\sqrt{L(2L+1)}}\begin{cases} 0 & \text{for} \quad j_2 = L+\frac{1}{2}, \\ 2L+1 & \text{for} \quad j_2 = L-\frac{1}{2}, \end{cases} \\
&\int \Omega_{j_2l_2m_2}\boldsymbol{\sigma}(\boldsymbol{\sigma}\boldsymbol{n})Y_{L,L-1,0}\, u\, do \qquad\qquad\qquad\qquad (40.21)\\
&\qquad = \frac{u^*_{j_2l_2}u}{\sqrt{L(2L+1)}}\begin{cases} 2L & \text{for} \quad j_2 = L+\frac{1}{2}, \\ -1 & \text{for} \quad j_2 = L-\frac{1}{2}, \end{cases} \\
&\qquad\qquad\qquad\qquad\qquad (l_2 = L).
\end{aligned}$$

On substituting (40.21) into (40.20) and (40.18) into (40.17) we obtain finally the following expression for the matrix element which determines the internal conversion coefficient for an electric multipole:

$$\begin{aligned}
&(B^{(1)}_{L0})_{L+\frac{1}{2},L} = u^*_{L+\frac{1}{2},L}u\sqrt{\frac{L}{L+1}}(R_1+R_2-2iR_4), \\
&\qquad j_2 = L+\frac{1}{2}, \qquad\qquad\qquad\qquad\qquad\qquad (40.22)\\
&(B^{(1)}_{L0})_{L-\frac{1}{2},L} = u^*_{L-\frac{1}{2},L}u\sqrt{\frac{L}{L+1}}\left(R_1+R_2-i\frac{2L+1}{L}R_3-\frac{i}{L}R_4\right), \\
&\qquad j_2 = L-\frac{1}{2},
\end{aligned}$$

where $R_i$ denote the integrals

$$
\begin{aligned}
R_1 &= \int_0^\infty g_\varkappa^* g_K G_L r^2 dr, \\
R_2 &= \int_0^\infty f_\varkappa^* f_K G_L r^2 dr, \\
R_3 &= \int_0^\infty f_\varkappa^* g_K G_{L-1} r^2 dr, \\
R_4 &= \int_0^\infty g_\varkappa^* f_K G_{L-1} r^2 dr.
\end{aligned}
\tag{40.23}
$$

The quantity $u_{jl}^* u$ appearing in (40.22) can be easily evaluated if in the integral (40.19) we express the spinor spherical harmonic in terms of the spherical harmonic in accordance with the formulas of § 4. As we have already mentioned, the value of the quantum number $m_1$ (the component of the angular momentum of the electron in the initial state) can be taken equal to $\frac{1}{2}$ for the $K$-shell. Then $m_2$ will also be equal to $\frac{1}{2}$. In this case

$$u = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$u_{j_2 l_2}^* u = \frac{1}{\sqrt{4\pi(2L+1)}} \begin{cases} \sqrt{L+1} & \text{for} \quad j_2 = L+\frac{1}{2}, \\ -\sqrt{L} & \text{for} \quad j_2 = L-\frac{1}{2}. \end{cases} \tag{40.24}$$

On substituting (40.22) and (40.23) into (40.13) we obtain the following expression for the internal conversion coefficient for an electric multipole in the $K$-shell:

$$
\begin{aligned}
\beta_L^{(1)} = \frac{\alpha\omega}{8\pi} \Bigg\{ & \frac{L}{2L+1} \left| R_1 + R_2 - 2iR_4 \right|^2_{j_2 = L+\frac{1}{2},\, l_2 = L} \\
& + \frac{L^2}{(2L+1)(L+1)} \left| R_1 + R_2 - i\frac{2L+1}{L} R_3 - i\frac{1}{L} R_4 \right|^2_{j_2 = L-\frac{1}{2},\, l_2 = L} . \Bigg\}
\end{aligned}
\tag{40.25}
$$

The matrix element which determines the conversion for a magnetic multipole is of the form

$$
\begin{aligned}
(B_{L0}^{(0)})_{j_2 l_2} = {} & i \int f_\varkappa^* g_K G_L r^2 dr \int \Omega_{j_2 l_2 m_2}^* (\boldsymbol{\sigma n}) \boldsymbol{\sigma} Y_{L0}^{(0)} u \, do \\
& - i \int g_\varkappa^* f_K G_L r^2 dr \int \Omega_{j_2 l_2 m_2}^* \boldsymbol{\sigma} (\boldsymbol{\sigma n}) Y_{L0}^{(0)} u \, do .
\end{aligned}
\tag{40.26}
$$

The angular integrals appearing in this expression can also be reduced to an integral of type (40.19). In order to do this we must express $Y^{(0)}_{L0}$ in terms of the spherical harmonic $Y_{L0}$ and utilize the fact that

$$\Omega^*_{jlm}(\boldsymbol{\sigma n}) = -\Omega^*_{jl'm}(l' = 2j-l).$$

Thus, we obtain

$$\int \Omega^*_{j_2l_2m_2}(\boldsymbol{\sigma n})\boldsymbol{\sigma} \boldsymbol{Y}^{(0)}_{L0} u \, do = -\int \Omega^*_{j_2l_2'm_2} \frac{\boldsymbol{\sigma}\mathbf{L}}{\sqrt{L(L+1)}} Y_{L0} u \, do$$

$$= \frac{-j_2(j_2+1)+l_2'(l_2'+1)+\frac{3}{4}}{\sqrt{L(L+1)}} u^*_{j_2l_2'}u.$$

The second integral over the angles in (40.26) differs from the first one only by its sign.

On introducing the notation for the radial integrals

$$\begin{aligned} R_5 &= \int f^*_{\varkappa} g_K G_L r^2 \, dr, \\ R_6 &= \int g^*_{\varkappa} f_K G_L r^2 \, dr, \end{aligned} \tag{40.27}$$

we can write (40.26) in the form

$$(B^{(0)}_{L0})_{j_2l_2} = -iu^*_{j_2l_2'}u(R_5+R_6)\frac{1}{\sqrt{L(L+1)}}\begin{cases} L & \text{for } j_2 = L+\frac{1}{2}, \\ -(L+1) & \text{for } j_2 = L-\frac{1}{2}, \end{cases}$$

where $u^*_{j_2l_2'}u$ is defined by formula (40.24).

The internal conversion coefficient for a magnetic multipole in the $K$-shell is given by

$$\beta^{(0)}_L = \frac{\alpha\omega}{8\pi}\frac{1}{2L+1}\{L|R_5+R_6|^2_{j_2=L+\frac{1}{2},\, l_2=L+1}$$

$$+(L+1)|R_5+R_6|^2_{j_2=L-\frac{1}{2},\, l_2=L-1}\}. \tag{40.28}$$

The radial integrals (40.23) and (40.27) which appear in the expressions for the conversion coefficients (40.25) and (40.28) can be simply evaluated only in the limiting case of low nuclear charge $Ze$ and of high transition frequencies $\omega$. In such a case we can, on the one hand, neglect the magnitude of the electron binding energy in the $K$-shell and, on the other hand, neglect the effect of the nuclear charge on the electron wave function in the continuum (final state). In this approximation we have

$$
\begin{aligned}
g_K &= 2(Z\alpha m)^{3/2}, \quad f_K = 0, \\
g_\varkappa &= \sqrt{\frac{\varepsilon+m}{2}}\,\frac{J_{l_2+\frac{1}{2}}(pr)}{\sqrt{r}}, \\
f_\varkappa &= \frac{\varkappa}{|\varkappa|}\sqrt{\frac{\varepsilon-m}{2}}\,\frac{J_{l_2'+\frac{1}{2}}(pr)}{\sqrt{r}}, \\
l_2' &= 2j_2-l_2, \quad p = \sqrt{\varepsilon^2-m^2}, \quad \varepsilon = \omega+m.
\end{aligned}
\tag{40.29}
$$

The substitution of (40.29) into (40.23) and (40.27) leads to the following expressions for the radial integrals

$$
\begin{aligned}
R_1 &= i^L(2\pi Z\alpha m)^{3/2}\sqrt{\frac{2(\varepsilon+m)}{\omega}}\,I_L, \\
R_2 &= 0, \\
R_3 &= i^{L-1}(2\pi Z\alpha m)^{3/2}\sqrt{\frac{2(\varepsilon-m)}{\omega}}\,I_{L-1}, \\
R_4 &= 0, \\
R_5 &= i^L(2\pi Z\alpha m)^{3/2}\sqrt{\frac{2(\varepsilon-m)}{\omega}}\,I_L\frac{\varkappa}{|\varkappa|}, \\
R_6 &= 0,
\end{aligned}
$$

where

$$
I_L = \int J_{L+\frac{1}{2}}(pr)\,H^{(1)}_{L+\frac{1}{2}}(\omega r)\,r\,dr.
$$

This integral is equal to

$$
I_L = \frac{2}{\pi i}\left(\frac{p}{\omega}\right)^{L+\frac{1}{2}}\frac{1}{p^2-\omega^2}.
$$

On utilizing these formulas we obtain the following expressions for the internal conversion coefficients:

$$
\begin{aligned}
\beta_L^{(1)} &= 2\alpha(Z\alpha)^3\,\frac{(L+1)\omega^2+4Lm^2}{(L+1)\,\omega^2}\left(1+\frac{2m}{\omega}\right)^{L-\frac{1}{2}}\frac{m}{\omega}, \\
\beta_L^{(0)} &= 2\alpha(Z\alpha)^3\left(1+\frac{2m}{\omega}\right)^{L+\frac{1}{2}}\frac{m}{\omega}.
\end{aligned}
\tag{40.30}
$$

If in these formulas we go over to the nonrelativistic approximation ($\omega \ll m$), we obtain

$$\begin{aligned} \beta_L^{(1)} &= \alpha (Z\alpha)^3 \frac{L}{L+1} \left(\frac{2m}{\omega}\right)^{L+\frac{5}{2}}, \\ \beta_L^{(0)} &= \alpha (Z\alpha)^3 \left(\frac{2m}{\omega}\right)^{L+\frac{3}{2}}. \end{aligned} \tag{40.31}$$

Formulas (40.30) and (40.31) give a quantitative idea of the nature of the dependence of the conversion coefficients on the energy and the angular momentum. The coefficients $\beta_L^{(\lambda)}$ are large for small $\omega$, and for large $L$. In some cases $\beta_L^{(\lambda)}$ can considerably exceed unity. However, quantitatively the free electron approximation gives very low accuracy even for small $Z$ and large $\omega$. The magnitude of $R_i$ can be evaluated if we use the nonrelativistic approximation for the electron wave functions (78). However, the accuracy of such results is not very great, since even at low energies the electron wave functions differ appreciably from the nonrelativistic ones at small values of $r$, which turns out to play the most important role in the integrals $R_i$.

If we use exact electron wave functions $R_i$ can be evaluated only numerically (184).

### 40.4. *Effect of Finite Nuclear Size*

In a number of cases in evaluating the internal conversion coefficients we must take finite nuclear size into account. This introduces two types of corrections into the evaluation of the matrix element which determines the conversion probability. Firstly, in the region $r_2 < R$ ($R$ is the nuclear radius) expansion (40.2) is not valid. Therefore, the exact expression for the matrix element for transitions of given $L$ will have the following form instead of (40.11):

$$\begin{aligned} U_{i\to f} = &-\alpha \frac{i\omega}{4\pi} (\boldsymbol{\alpha} \boldsymbol{a}_L^{(\lambda)})_{21} \int\limits_{r_2>R} \psi_2^*(\boldsymbol{r}_2)\, B_{LM}^{(\lambda)}\, \psi_1(\boldsymbol{r}_2)\, d\boldsymbol{r}_2 \\ &-\alpha \int \int\limits_{r_2<R} \left(\bar{\Psi}'_2(\boldsymbol{r}_1)\, \gamma_\mu \Psi'_1(\boldsymbol{r}_1)\right)\left(\bar{\psi}_2(\boldsymbol{r}_2)\, \gamma_\mu\, \psi_1(\boldsymbol{r}_2)\right) \frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2. \end{aligned} \tag{40.32}$$

(In the expressions obtained previously it was assumed that $R \to 0$.)

Secondly, the electron wave functions near the nucleus may, as we have seen in § 13, differ significantly from the electron wave functions in the Coulomb field. As actual analysis shows, the main error is due to extending the integral in the first term of (40.32) into the region $0 < r_2 < R$, and using Coulomb wave functions in this region. The latter have a singularity at $r = 0$, and therefore the region $0 < r_2 < R$ in the case of large $Z$ makes a contribution to the integral comparable to that of the region $R < r_2 < \infty$. But the true wave functions have no singularities at $r = 0$ ($r\psi \to 0$ as $r \to 0$), and therefore, in fact, the integral over the region $r_2 < R$ must be small compared to the integral over the region $r_2 > R$. Results good for practical purposes can be obtained from the expression for the matrix element (183)

$$U_{i\to f} = -\alpha^2 \frac{i\omega}{4\pi} (\boldsymbol{\alpha} \boldsymbol{a}_{LM}^{(\lambda)})_{21} \int\limits_{r_2 > R} \psi_2^* B_{LM}^{(1)} \psi_1 \, d\boldsymbol{r}_2, \tag{40.33}$$

into which Coulomb wave functions can be substituted. Thus, the lower limit $r_2 = R$ should be introduced into the integrals $R_i$. Taking finite nuclear size into account leads to corrections which are inappreciable for low values of $Z$, but which attain values of 30–40% at $Z \sim 80$–90.

### 40.5. *Effect of Electron Shells on Radiation from the Nucleus*

In § 27 we have obtained expressions for the probability of emission of a photon by a nucleus. The presence of electron shells introduces corrections into these formulas. In addition to direct emission of a photon by the nucleus, represented by the diagram of Fig. 53, there also exists the possibility of the process of emission by an electron belonging to an atomic shell, due to its interaction with the nucleus. Such a process is represented by the diagram of Fig. 54.

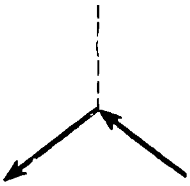

Fig. 53.

Since the latter process is a third order process we can, as a rule, neglect it in comparison with the process of direct emission which is a first order process. However, this may turn out not to be so when the conversion coefficient is very large (110). Indeed, the matrix element for the process shown by the diagram of Fig. 54 amounts, roughly speaking, to the product of the matrix element for the internal conversion process and the matrix element for the emission of radiation. If the conversion process is more probable than emission, then the effect of the electron shells on the emission by the nucleus may also turn out to be important.

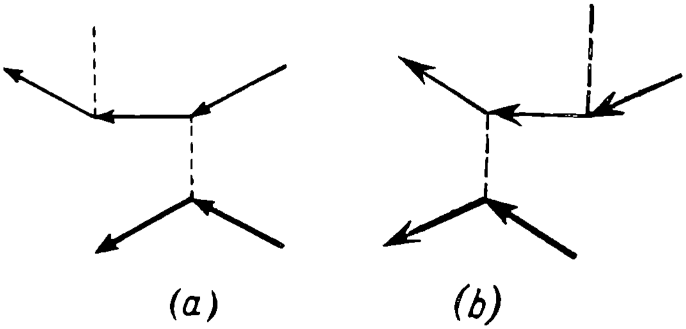

Fig. 54.

We investigate in greater detail the process represented by the diagram of Fig. 54, in which the thick lines represent the proton. As we have done before we take $\Psi_1(x) = \Psi_1(\boldsymbol{r}) \exp(iE_1 t)$ and $\Psi_2(x) = \Psi_2(\boldsymbol{r}) \exp(iE_2 t)$ to denote respectively the proton wave functions in the initial and the final states. We assume the initial and the final states of the electron to be the same, and denote the electron wave function by $\psi_0(x) = \psi_0(\boldsymbol{r}) \exp(i\varepsilon_0 t)$, and the vector potential of the emitted photon by $A(x) = A(\boldsymbol{r}) \exp(-i\omega t)$. The matrix element for this process (cf. Fig. 54) will be expressed as follows:

$$S^{(3)}_{i\to f} = e^3 \int \overline{\Psi}_2(x_1)\gamma_\mu \Psi_1(x_1) D^c(x_1 - x_2)\overline{\psi}_0(x_2)$$

$$\times \gamma_\mu S^{(e)}_c(x_2, x_3)\hat{A}^*(x_3)\psi_0(x_3)\, d^4x_1\, d^4x_2\, d^4x_3$$

$$+ e^3 \int \overline{\Psi}_2(x_1)\gamma_\mu \Psi_1(x_1) D^c(x_1 - x_3)\overline{\psi}_0(x_2)\hat{A}(x_2) S^{(e)}_c(x_2, x_3)$$

$$\times \gamma_\mu \psi_0(x_3)\, d^4x_1\, d^4x_2\, d^4x_3.$$

On substituting into this equation the expression for $D^c(x)$ in the form (37.10)

$$D^c(x_1-x_2) = \frac{-i}{8\pi^2} \frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} \int\limits_{-\infty}^{\infty} e^{-i\omega''(t_2-t_1)+i|\omega''||\boldsymbol{r}_1-\boldsymbol{r}_2|} d\omega''$$

and for $S_c^{(e)}$ in the form (35.3)

$$S_c^{(e)}(x_2, x_3) = \frac{1}{2\pi i} \sum_n \psi_n(\boldsymbol{r}_2)\overline{\psi}_n(\boldsymbol{r}_3) \int\limits_{-\infty}^{\infty} \frac{e^{i\omega'(t_2-t_3)}}{\omega'+\varepsilon_n} d\omega',$$

where $\psi_n$ are the electron wave functions in the nuclear field, and $n$ is a set of appropriate quantum numbers, we obtain the expression for $S^{(3)}_{i\to f}$

$$S^{(3)}_{i\to f} = -2\pi i\delta(E_1-E_2-\omega)\, U_{if},$$

$$U_{if} = \sum_n U_n^{(\mathrm{conv})} U_n^{(\mathrm{rad})} \left(\frac{1}{\varepsilon_n-\varepsilon_0+\omega} + \frac{1}{\varepsilon_n-\varepsilon_0-\omega}\right),$$

where

$$U_n^{(\mathrm{conv})} = a \int \overline{\Psi}_2(\boldsymbol{r}_1)\gamma_\mu \Psi_1(\boldsymbol{r}_1) \frac{e^{i(E_1-E_2)|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} \overline{\psi}_0(\boldsymbol{r}_2)\gamma_\mu \psi_n(\boldsymbol{r}_2)\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2$$

represents the matrix element for the process of internal conversion (40.3) in which the electron makes a transition from the initial state to state $n$, and

$$U_n^{(\mathrm{rad})} = e \int \overline{\psi}_0(\boldsymbol{r}) \hat{A}(\boldsymbol{r}) \psi_n(\boldsymbol{r})\, d\boldsymbol{r}$$

represents the matrix element for the emission of radiation, in which the electron makes a transition from the state $n$ to the initial state.

In the case of states belonging to the continuous spectrum we normalize $\psi_n$ per unit energy, i.e.,

$$\int \psi^*_{n_1}(\boldsymbol{r})\psi_{n_2}(\boldsymbol{r})\, d\boldsymbol{r} = \delta(\varepsilon_{n_1}-\varepsilon_{n_2})\,\delta_{n'_1 n'_2},$$

where $n'$ are all the quantum numbers, except for the energy, i.e., the quantum numbers for the orbital, spin and total angular momenta. In this case $U_{if}$ will be given in the following form:

$$U_{if}^{(3)} = \sum_{n'} \int U_n^{(\mathrm{conv})}\, U_n^{(\mathrm{rad})} \left(\frac{1}{\varepsilon-\varepsilon_0+\omega} + \frac{1}{\varepsilon-\varepsilon_0-\omega}\right) d\varepsilon$$

$$+ \sum_n U_n^{(\mathrm{conv})}\, U_n^{(\mathrm{rad})} \left(\frac{1}{\varepsilon_n-\varepsilon_0+\omega} + \frac{1}{\varepsilon_n-\varepsilon_0-\omega}\right).$$

Here the first sum refers to the continuous spectrum, and the second to the discrete spectrum.

The principal role is played by the resonance region which lies in the continuous spectrum near the energy value $\varepsilon = \varepsilon_0 + \omega$. Therefore (198) we have

$$U_{if}^{(3)} = i\pi \sum_{n'} U_n^{(\text{conv})} U_n^{(\text{rad})} \qquad (\varepsilon - \varepsilon_0 = E_1 - E_2).$$

Thus, the matrix element for the emission of radiation, taking into account the effect of the electron shells, has the form

$$U_{if}^{(1)} + U_{if}^{(2)} = U^{(1)} + i\pi \sum_{n'} U^{(\text{conv})} U^{(\text{rad})}, \qquad (40.34)$$

where $U^{(1)}$ is the matrix element for the emission of radiation by a nucleus not surrounded by electrons. If the conversion coefficient $\beta = |U^{(\text{conv})}/U^{(1)}|^2$, evaluated without taking the effect of the shells into account, is large, then the second term in (40.34) may play an important role.

## § 41. Conversion Accompanied by Pair Production. Excitation of Nuclei by Electrons

### 41.1 *Conversion of Magnetic Multipole Radiation*

If the nuclear excitation energy $\omega$ exceeds $2m$, then in addition to the phenomenon of internal conversion in the atomic shells creation of electron-positron pairs may also occur. The coefficient of conversion accompanied by pair creation can be determined from the general formula (40.13) in which for $\psi_2$ we take an electron wave function, while for $\psi_1$ we take a wave function of negative frequency corresponding to a positron. The matrix element appearing in (40.13) can be reduced to integrals over radial wave functions similarly to the way this was done in the preceding section in the case of conversion in the $K$-shell.

We confine ourselves here to the investigation of only the one case when the electron and the positron states may be regarded as free (15), (178), (165). Such an approximation is applicable for small $Z$.

We choose the electron and the positron wave functions in the form of plane waves

$$\begin{aligned} \psi_2 &= u e^{i p_- r}, \\ \psi_1 &= v e^{-i p_+ r}, \end{aligned} \qquad (41.1)$$

where $p_-$ is the electron momentum, $p_+$ is the positron momentum, and $u$ and $v$ are unit bispinor amplitudes.

On substituting (41.1) into (40.12) we obtain the following expression for the differential conversion coefficient for an electric or a magnetic multipole accompanied by the creation of an electron in the momentum interval $dp_-$ and of a positron in the momentum interval $dp_+$:

$$d\beta_L^{(\lambda)} = \frac{\alpha\omega}{4}\,\frac{dp_-dp_+}{(2\pi)^6}\,\delta(\omega-\varepsilon_--\varepsilon_+)\sum_{\mu_1,\mu_2}\left|\int e^{-i\boldsymbol{q}\boldsymbol{r}}u^*B_{LM}^{(\lambda)}v\,d\boldsymbol{r}\right|^2, \tag{41.2}$$

where $\boldsymbol{q} = \boldsymbol{p}_-+\boldsymbol{p}_+$ and $\sum\limits_{\mu_1,\mu_2}$ denotes summation over the spin states of the electron and the positron.

We begin our calculation with the case of a magnetic multipole. In this case, in accordance with (40.10), we have

$$B_{LM}^{(0)} = -\boldsymbol{\alpha}\boldsymbol{Y}_{LM}^{(0)}G_L(\omega r).$$

Further, on utilizing the expansion of $\exp(i\boldsymbol{q}\boldsymbol{r})$ in terms of spherical harmonics we obtain

$$\int e^{-i\boldsymbol{q}\boldsymbol{r}}\boldsymbol{Y}_{LM}^{(0)}G_L\,d\boldsymbol{r} = (-1)^L\boldsymbol{Y}_{LM}^{(0)}(\boldsymbol{v})I_L,$$

where $\boldsymbol{v} = \boldsymbol{q}/q$

$$I_L = \int G_L(\omega r)\,G_L(qr)\,r^2\,dr = \frac{(4\pi)^2 i}{\omega(\omega^2-q^2)}\left(\frac{q}{\omega}\right)^L.$$

Thus, we have

$$d\beta_L^{(0)} = \frac{\alpha}{\pi^2}\,\frac{1}{(\omega^2-q^2)^2\,\omega}\left(\frac{q}{\omega}\right)^{2L}dp_+dp_-\sum_{\mu_1,\mu_2}|u^*\boldsymbol{\alpha}\boldsymbol{Y}_{LM}^{(0)}(\boldsymbol{v})v|^2\,\delta(\omega-\varepsilon_--\varepsilon_+). \tag{41.3}$$

The summation over the spin states may be carried out in the usual manner in accordance with § 26. It yields

$$\sum_{\mu_1,\mu_2}|u^*\boldsymbol{\alpha}\boldsymbol{Y}_{LM}^{(0)}v|^2 = \frac{1}{\varepsilon_-\varepsilon_+}\{(\varepsilon_-\varepsilon_++m^2-\boldsymbol{p}_-\boldsymbol{p}_+)|\boldsymbol{Y}_{LM}^{(0)}|^2$$
$$+(\boldsymbol{p}_-\boldsymbol{Y}_{LM}^{(0)})(\boldsymbol{p}_+\boldsymbol{Y}_{LM}^{(0)*})+(\boldsymbol{p}_-\boldsymbol{Y}_{LM}^{(0)*})(\boldsymbol{p}_+\boldsymbol{Y}_{LM}^{(0)})\}.$$

This expression can be simplified. Since $\boldsymbol{p}_+ = \boldsymbol{q}-\boldsymbol{p}_-$ and $\boldsymbol{q}\boldsymbol{Y}_{LM}^{(0)} = 0$ we have $\boldsymbol{p}_-\boldsymbol{Y}_{LM}^{(0)} = -\boldsymbol{p}_+\boldsymbol{Y}_{LM}^{(0)}$. Further, we have $\boldsymbol{p}_-\boldsymbol{p}_+ = \boldsymbol{p}_+\boldsymbol{q}-p_+^2$. On the other hand, from the law of conservation of energy

$$\omega = \sqrt{(p_-^2+m^2)}+\sqrt{(\boldsymbol{q}-\boldsymbol{p}_-)^2+m^2}$$

we can obtain $\boldsymbol{p}_+\boldsymbol{q} = \omega\varepsilon_+ - \frac{1}{2}(\omega^2 - q^2)$. Therefore

$$\sum_{\mu_1,\mu_2} |u^* \boldsymbol{\alpha} \boldsymbol{Y}^{(0)}_{LM} v|^2 = \frac{1}{\varepsilon_- \varepsilon_+}\left[\frac{\omega^2 - q^2}{2} |\boldsymbol{Y}^{(0)}_{LM}|^2 - 2|\boldsymbol{p}_+ \boldsymbol{Y}^{(0)}_{LM}|^2\right].$$

We substitute this expression into (41.3) and go over from the variables $\boldsymbol{p}_+$, $\boldsymbol{p}_-$ to the variables $\boldsymbol{q}$, $\boldsymbol{p}_+$. We write the product of the differentials of the momenta in the form

$$d\boldsymbol{p}_+ d\boldsymbol{p}_- = d\boldsymbol{p}_+ d\boldsymbol{q} = p_+^2\, dp_+ do_+ q^2\, dq\, do_q,$$

and the argument of the $\delta$-function appearing in (41.3) in the form

$$\omega - \varepsilon_- - \varepsilon_+ = \omega - \varepsilon_+ - \sqrt{q^2 + p_+^2 + m^2 - 2qp_+ \cos\vartheta}\,,$$

where $\vartheta$ is the angle between $\boldsymbol{q}$ and $\boldsymbol{p}$. If we take the $z$-axis along $\boldsymbol{q}$ hen $do_+ = d\cos\vartheta\, d\varphi_+$ and we can eliminate the $\delta$-function:

$$\delta(\omega - \varepsilon_+ - \varepsilon_-)\, d(\cos\vartheta) \to \frac{\varepsilon_-}{p_+ q}.$$

Now the angle $\vartheta$ is determined by the law of conservation of energy:

$$(\omega - \varepsilon_+)^2 = q^2 + \varepsilon_+^2 - 2qp_+ \cos\vartheta. \tag{41.4}$$

Formula (41.3) assumes the form

$$d\beta_L^{(0)} = \frac{\alpha}{\pi^2(\omega^2 - q^2)^2}\left(\frac{q}{\omega}\right)^{2L+1}\left[\frac{\omega^2 - q^2}{2} |\boldsymbol{Y}^{(0)}_{LM}|^2 - 2p_+^2 |\boldsymbol{Y}^{(0)}_{LM}|^2 \sin\vartheta\right] d\varepsilon_+\, dq\, d\varphi_+\, do_q. \tag{41.5}$$

We now integrate (41.5) over $d\varphi_+$ and $do_q$. On utilizing for $\sin^2\vartheta$ the value given by (41.4) we obtain the following exprcssion for the coefficient of conversion accompanied by creation of a pair with given values of the positron energy $\varepsilon_+$ and of the magnitude of the total momentum $q$:

$$d\beta^{(0)}_{LM}(\varepsilon_+, q) = \frac{2\alpha}{\pi}\frac{q^{2L-1}}{\omega^{2L+1}}\left\{-\frac{1}{4} + \frac{p_+^2 + p_-^2}{2(\omega^2 - q^2)} + \frac{\omega^2}{(\omega^2 - q^2)^2}\right\} d\varepsilon_+\, dq. \tag{41.6}$$

On integrating this expression over $q$ between the limits $q_{\min}$ to $q_{\max}$, where

$$q_{\min} = |\boldsymbol{p}_+ - \boldsymbol{p}_-|, \qquad q_{\max} = |\boldsymbol{p}_+ + \boldsymbol{p}_-|,$$

we obtain the energy distribution of the converted positrons:

$$\begin{aligned}d\beta_L^{(0)}(\varepsilon_+) = \frac{\alpha}{\pi\omega^{2L+1}}\Big\{&\omega^{2(L-1)}[p_+^2+p_-^2-2(L-1)m^2]\ln\frac{m^2+p_+p_-+\varepsilon_+\varepsilon_-}{m\omega}\\ &-\frac{1}{4L}[(p_++p_-)^{2L}-(p_+-p_-)^{2L}]+\frac{1}{2}[(\varepsilon_+\varepsilon_-+p_+p_--m^2)(p_++p_-)^{2(L-1)}\\ &-(\varepsilon_+\varepsilon_--p_+p_--m^2)(p_+-p_-)^{2(L-1)}]-\frac{1}{2}[p_+^2+p_-^2-2(L-1)m^2]\\ &\times\sum_{n=1}^{L-1}\frac{\omega^{2(L-n-1)}}{n}[(p_++p_-)^{2n}-(p_+-p_-)^{2n}]\Big\}\,d\varepsilon_+. \qquad (41.7)\end{aligned}$$

We give in explicit form the corresponding expressions for the cases of a magnetic dipole ($L = 1$), quadrupole ($L = 2$) and octopole ($L = 3$):

$$\begin{aligned}d\beta_1^{(0)}(\varepsilon_+) &= \frac{\alpha}{\pi\omega^3}(p_+^2+p_-^2)\ln\frac{m^2+p_+p_-+\varepsilon_+\varepsilon_-}{m\omega}\,d\varepsilon_+;\\ d\beta_2^{(0)}(\varepsilon_+) &= \frac{\alpha}{\pi\omega^5}\Big\{\omega^2(p_+^2+p_-^2-2m^2)\ln\frac{m^2+p_+p_-+\varepsilon_+\varepsilon_-}{m\omega}\\ &\qquad +p_+p_-(\omega^2-3p_+^2-3p_-^2)\Big\}\,d\varepsilon_+; \qquad (41.8)\\ d\beta_3^{(0)}(\varepsilon_+) &= \frac{\alpha}{\pi\omega^7}\Big\{\omega^4(p_+^2+p_-^2-4m^2)\ln\frac{m^2+p_+p_-+\varepsilon_+\varepsilon_-}{m\omega}\\ &\qquad +\frac{2}{3}p_+p_-[\omega^2(\omega^2+8m^2)-2\omega^2(p_+^2+p_-^2)-5(p_+^2+p_-^2)^2]\Big\}\,d\varepsilon_+.\end{aligned}$$

The total conversion coefficient $\beta_L^{(0)}$ will be obtained by integrating (41.7) over the range from $m$ to $\omega-m$. The result of this integration in the general case cannot be expressed in terms of elementary functions. Later we shall give the results of numerical integration. However, we shall first investigate the extreme relativistic case $\omega \gg m$ when we can approximately take $p_+ \approx \varepsilon_+$, $p_- \approx \varepsilon_-$. Assuming, moreover that $(\omega/m)^2 \gg L$ we obtain the expression for the total conversion coefficient

$$\beta_L^{(0)} = \frac{2\alpha}{3\pi}\left[\ln\frac{2\omega}{m}-\frac{23}{12}-\frac{3}{L(2L+1)}-\sum_{n=1}^{L-1}\frac{4n+5}{2(2n+1)(2n+3)}\right]. \qquad (41.9)$$

In the opposite limiting case $L \gg (\omega/m)^2$ (for which also $\omega \gg m$) we obtain the asymptotic expression for $\beta_L^{(0)}$,

$$\beta_L^{(0)} = \frac{aL}{\pi\omega^2}\left[\ln\frac{Lm^2}{\omega^2}+2+\gamma\right], \tag{41.10}$$

where $\gamma$ is Euler's constant.

The distribution over the angle $\chi$ between the momenta of the components of the pair can be obtained on going over in (41.6) from the variable $q$ to $\chi$, and on integrating the resultant expression over the energy $\varepsilon_+$. The relation between $\chi$ and $q$ follows from the definition of $q$:

$$q^2 = p_+^2+p_-^2-2p_+p_-\cos\chi .$$

As an example of the application of these formulas for the conversion coefficients we now obtain the cross section for the capture of slow neutrons by protons resulting in the formation of a deuteron, an electron and a positron. On taking into account the fact that in such a process a transition occurs from a $^1S$ state to a $^3S$ state, we can easily conclude that the transition is of a magnetic dipole nature. Therefore, we have $\sigma = \sigma_c\beta_1^{(0)}$, where $\sigma$ is the cross section for the process of interest to us, while $\sigma_c$ is the photocapture cross section, which in the case of thermal neutrons is equal to $\sigma_c \approx 0.3\cdot 10^{-24}$ cm$^2$. The binding energy of the deuteron is given by $|\varepsilon| = 2.15$ MeV, i.e., $\omega = 4.2m$. Numerical integration carried out for this value of $\omega$ yields for $\beta_1^{(0)}$ the value $3\times 10^{-4}$. Consequently, we have $\sigma = 0.9\cdot 10^{-28}$ cm$^2$.

### 41.2. *Conversion of Electric Multipole Radiation*

We now proceed to investigate the conversion of electric multipole radiation. On substituting into (40.12)

$$B_{LM}^{(1)} = \sqrt{\frac{L}{L+1}}\,G_L(\omega r)Y_{LM} - \sqrt{\frac{2L+1}{L+1}}\,G_{L-1}(\omega r)\mathbf{Y}_{L,L-1,M}\boldsymbol{\alpha}$$

with the same wave functions (41.1) we obtain

$$(B_{LM}^{(1)}) = \frac{(-1)^{L+1}(4\pi)^2 i}{\omega(q^2-\omega^2)}\left(\frac{q}{\omega}\right)^L\frac{1}{\sqrt{L+1}}\Big[\sqrt{L}\,Y_{LM}u^*v - \sqrt{2L+1}\,\frac{\omega}{q}\,\mathbf{Y}_{L,L-1,M}u^*\boldsymbol{\alpha}v\Big].$$

Substitution of this expression into (40.12) and integration of (40.12) over the angle between $\boldsymbol{p}_+$ and $\boldsymbol{q}$ yields the expression for the differential conversion coefficient

$$d\beta_L^{(1)} = \frac{\alpha q^{2L+1}}{\pi^2 \omega^{2L+1}(\omega^2-q^2)^2 (L+1)} \sum_\mu \Big| \sqrt{L}\, Y_{LM} u^* v - \sqrt{2L+1}\,\frac{\omega}{q}\, \boldsymbol{Y}_{L,L-1,M} u^* \boldsymbol{\alpha} v \Big|^2 \varepsilon_+ \varepsilon_-\, d\varepsilon_+\, dq\, do_q\, d\varphi_+, \qquad (41.11)$$

where $\sum_\mu$ denotes summation over the electron polarizations. On carrying out the summation, we obtain

$$\sum_\mu |\ldots|^2 = L|Y_{LM}|^2(\varepsilon_+\varepsilon_- + p_+p_- - m^2) + (2L+1)\frac{\omega^2}{q^2}|\boldsymbol{Y}_{L,L-1,M}|^2$$
$$\times(\varepsilon_+\varepsilon_- - \boldsymbol{p}_+\boldsymbol{p}_- + m^2) - \sqrt{L(2L+1)}\,\frac{\omega}{q}\,(Y_{LM}\boldsymbol{Y}^*_{L,L-1,M} + Y^*_{LM}\boldsymbol{Y}_{L,L-1,M})$$
$$\times(\varepsilon_-\boldsymbol{p}_+ + \varepsilon_+\boldsymbol{p}_-) + (2L+1)\frac{\omega^2}{q^2}\,[(\boldsymbol{Y}_{L,L-1,M}\boldsymbol{p}_-)(\boldsymbol{Y}^*_{L,L-1,M}\boldsymbol{p}_+)$$
$$+(\boldsymbol{Y}_{L,L-1,M}\boldsymbol{p}_-)(\boldsymbol{Y}_{L,L-1,M}\boldsymbol{p}_+)].$$

The products $(\boldsymbol{Y}_{L,L-1,M}\boldsymbol{p}_+)$ appearing in this expression can be easily evaluated if we use the expansion of $\boldsymbol{Y}_{L,L-1,M}$ into a longitudinal and a transverse vector spherical harmonic. For the longitudinal vector we have

$$\boldsymbol{Y}_{LM}^{(-1)}\boldsymbol{p}_+ = p_+ Y_{LM}\cos\vartheta,$$
$$\boldsymbol{Y}_{LM}^{(-1)}\boldsymbol{p}_- = (q - p_+\cos\vartheta)\, Y_{LM},$$

and for the transverse one

$$\boldsymbol{Y}_{LM}^{(1)}\boldsymbol{p}_+ = -\boldsymbol{Y}_{LM}^{(1)}\boldsymbol{p}_-,$$
$$|\boldsymbol{Y}_{LM}^{(1)}\boldsymbol{p}_+|^2 = |\boldsymbol{Y}_{LM}^{(1)}|^2 p_+^2 \sin^2\vartheta.$$

Substituting these expressions into (41.11) and integrating over $d\varphi_+$ and $do_q$ we finally obtain

$$d\beta_L^{(1)}(\varepsilon_+, q) = \frac{2\alpha}{\pi(L+1)}\left\{\frac{q^{2L-1}}{\omega^{2L+1}}\left[\frac{L}{2} + \left(\frac{(3L+1)\omega^2}{2} - 2L\varepsilon_+\varepsilon_-\right)\frac{1}{\omega^2-q^2}\right.\right.$$
$$\left. - \frac{2L\omega^2\varepsilon_+\varepsilon_-}{(\omega^2-q^2)^2}\right] + \frac{q^{2L-3}}{\omega^{2L-1}}\left[\frac{3L+1}{3} - \big((L+1)\varepsilon_+\varepsilon_- + L\omega^2\right.$$
$$\left.\left. + (L+1)m^2\big)\frac{1}{\omega^2-q^2} + \omega^2\big(2L\varepsilon_+\varepsilon_- + (L+1)m^2\big)\frac{1}{(\omega^2-q^2)^2}\right]\right\} d\varepsilon_+\, dq. \qquad (41.12)$$

## TABLE 13

*Values of total conversion coefficients* $\beta_L^{(0)}$

| L \ $\frac{\omega}{m}$ | 3 | 4 | 5 | 6 | 7 | 10 | 15 | 20 |
|---|---|---|---|---|---|---|---|---|
| 1 | $0.710 \cdot 10^{-4}$ | $0.257 \cdot 10^{-3}$ | $0.469 \cdot 10^{-3}$ | $0.675 \cdot 10^{-3}$ | $0.864 \cdot 10^{-3}$ | $0.134 \cdot 10^{-2}$ | $0.193 \cdot 10^{-2}$ | $0.236 \cdot 10^{-2}$ |
| 2 | $0.273 \cdot 10^{-4}$ | $0.141 \cdot 10^{-3}$ | $0.301 \cdot 10^{-3}$ | $0.471 \cdot 10^{-3}$ | $0.637 \cdot 10^{-3}$ | $0.108 \cdot 10^{-2}$ | $0.164 \cdot 10^{-2}$ | $0.206 \cdot 10^{-2}$ |
| 3 | $0.114 \cdot 10^{-4}$ | $0.833 \cdot 10^{-4}$ | $0.204 \cdot 10^{-3}$ | $0.346 \cdot 10^{-3}$ | $0.491 \cdot 10^{-3}$ | $0.902 \cdot 10^{-3}$ | $0.144 \cdot 10^{-2}$ | $0.186 \cdot 10^{-2}$ |
| 4 | $0.503 \cdot 10^{-5}$ | $0.511 \cdot 10^{-4}$ | $0.144 \cdot 10^{-3}$ | $0.262 \cdot 10^{-3}$ | $0.390 \cdot 10^{-3}$ | $0.770 \cdot 10^{-3}$ | $0.129 \cdot 10^{-2}$ | $0.169 \cdot 10^{-2}$ |
| 5 | $0.229 \cdot 10^{-5}$ | $0.323 \cdot 10^{-4}$ | $0.103 \cdot 10^{-3}$ | $0.203 \cdot 10^{-3}$ | $0.297 \cdot 10^{-3}$ | $0.668 \cdot 10^{-3}$ | $0.117 \cdot 10^{-2}$ | $0.157 \cdot 10^{-2}$ |
| 10 | $0.58 \cdot 10^{-7}$ | $0.409 \cdot 10^{-5}$ | $0.248 \cdot 10^{-4}$ | $0.680 \cdot 10^{-4}$ | $0.130 \cdot 10^{-3}$ | $0.372 \cdot 10^{-3}$ | $0.792 \cdot 10^{-3}$ | $0.115 \cdot 10^{-2}$ |

*Values of total conversion coefficients* $\beta_L^{(1)}$

| L \ $\frac{\omega}{m}$ | 3 | 4 | 5 | 6 | 7 | 10 | 15 | 20 |
|---|---|---|---|---|---|---|---|---|
| 1 | $0.291 \cdot 10^{-3}$ | $0.671 \cdot 10^{-3}$ | $0.998 \cdot 10^{-3}$ | $0.127 \cdot 10^{-2}$ | $0.150 \cdot 10^{-2}$ | $0.205 \cdot 10^{-2}$ | $0.267 \cdot 10^{-2}$ | $0.312 \cdot 10^{-2}$ |
| 2 | $0.100 \cdot 10^{-3}$ | $0.333 \cdot 10^{-3}$ | $0.581 \cdot 10^{-3}$ | $0.811 \cdot 10^{-3}$ | $0.101 \cdot 10^{-2}$ | $0.152 \cdot 10^{-2}$ | $0.212 \cdot 10^{-2}$ | $0.256 \cdot 10^{-2}$ |
| 3 | $0.393 \cdot 10^{-4}$ | $0.185 \cdot 10^{-3}$ | $0.373 \cdot 10^{-3}$ | $0.565 \cdot 10^{-3}$ | $0.746 \cdot 10^{-3}$ | $0.121 \cdot 10^{-2}$ | $0.179 \cdot 10^{-2}$ | $0.222 \cdot 10^{-2}$ |
| 4 | $0.165 \cdot 10^{-4}$ | $0.109 \cdot 10^{-3}$ | $0.253 \cdot 10^{-3}$ | $0.413 \cdot 10^{-3}$ | $0.573 \cdot 10^{-3}$ | $0.100 \cdot 10^{-2}$ | $0.156 \cdot 10^{-2}$ | $0.198 \cdot 10^{-2}$ |
| 5 | $0.732 \cdot 10^{-5}$ | $0.668 \cdot 10^{-4}$ | $0.177 \cdot 10^{-3}$ | $0.312 \cdot 10^{-3}$ | $0.485 \cdot 10^{-3}$ | $0.856 \cdot 10^{-3}$ | $0.139 \cdot 10^{-2}$ | $0.180 \cdot 10^{-2}$ |
| 10 | $0.17 \cdot 10^{-6}$ | $0.784 \cdot 10^{-5}$ | $0.393 \cdot 10^{-4}$ | $0.978 \cdot 10^{-4}$ | $0.174 \cdot 10^{-3}$ | $0.451 \cdot 10^{-3}$ | $0.900 \cdot 10^{-3}$ | $0.127 \cdot 10^{-2}$ |

By integrating over $q$ we obtain the energy distribution of the converted positrons. For $L \neq 1$ we obtain

$$\begin{aligned} d\beta_L^{(1)}(\varepsilon_+) = \frac{a}{\pi\omega^{2L+1}} \Big\{ & \omega^{2(L-1)} [\varepsilon_+^2+\varepsilon_-^2-2(L-1)m^2] \ln \frac{m^2+\varepsilon_+\varepsilon_-+p_+p_-}{m\omega} \\ & + \frac{1}{2(L+1)} [(p_++p_-)^{2L}-(p_+-p_-)^{2L}] - \left[\frac{(\varepsilon_+-\varepsilon_-)^2}{4(L+1)}+m^2\right] [(p_++p_-)^{2(L-1)} \\ & -(p_+-p_-)^{2(L-1)}] + \frac{\omega^2}{2} [(\varepsilon_+\varepsilon_-+p_+p_--m^2)(p_++p_-)^{2(L-2)} \\ & -(\varepsilon_+\varepsilon_--p_+p_--m^2)(p_+-p_-)^{2(L-2)}] - \frac{1}{2} [\varepsilon_+^2+\varepsilon_-^2-2(L-1)m^2] \\ & \times \sum_{n=1}^{L-1} \frac{\omega^{2(L-n-1)}}{n} [(p_++p_-)^{2n}-(p_+-p_-)^{2n}] \Big\} d\varepsilon_+ \qquad (L \neq 1). \end{aligned} \tag{41.13}$$

For $L = 1$ (electric dipole) the differential conversion coefficient is equal to

$$d\beta_1^{(1)}(\varepsilon_+) = \frac{a}{\pi\omega^3} \left\{ (\varepsilon_+^2+\varepsilon_-^2) \ln \frac{m^2+\varepsilon_+\varepsilon_-+p_+p_-}{m\omega} + 2p_+p_- \right\} d\varepsilon_+. \tag{41.14}$$

We also give the expressions for the conversion coefficients for electric quadrupole and octopole radiation which follow from (41.13):

$$\begin{aligned} d\beta_2^{(1)}(\varepsilon_+) &= \frac{a}{\pi\omega^5} \Big\{ \omega^2(\varepsilon_+^2+\varepsilon_-^2-2m^2) \ln \frac{m^2+\varepsilon_+\varepsilon_-+p_+p_-}{m\omega} \\ &\qquad + \frac{8}{3} p_+p_-(\varepsilon_+\varepsilon_--m^2) \Big\} d\varepsilon_+ ; \\ d\beta_3^{(1)}(\varepsilon_+) &= \frac{a}{\pi\omega^7} \Big\{ \omega^4(\varepsilon_+^2+\varepsilon_-^2-4m^2) \ln \frac{m^2+\varepsilon_+\varepsilon_-+p_+p_-}{m\omega} \\ &\qquad - p_+p_- [2\omega^4-\omega^2m^2-8m^4-(9\omega^2+4m^2)\varepsilon_+\varepsilon_-+4\varepsilon_+^2\varepsilon_-^2] \Big\} d\varepsilon_+ . \end{aligned} \tag{41.15}$$

For the total internal conversion coefficient we give only the expressions which are obtained in the limiting case $\omega \ll m$. In this case if $(\omega/m)^2 \gg L$ we have

$$\beta_L^{(1)} = \frac{2a}{3\pi} \left\{ \ln \frac{2\omega}{m} - \frac{23}{12} + \frac{5L+1}{4(L+1)(2L+1)} - \sum_{n=1}^{L-1} \frac{4n+5}{2(2n+1)(2n+3)} \right\}. \tag{41.16}$$

This formula holds for $L \geqslant 2$. For $L = 1$ we have

$$\beta_1^{(1)} = \frac{2\alpha}{3\pi}\left(\ln\frac{2\omega}{m} - \frac{5}{3}\right). \tag{41.17}$$

In the opposite limiting case $L \gg (\omega/m)^2 \gg 1$ we obtain for $\beta_L^{(1)}$ the same expression as for the conversion coefficient for a magnetic multipole (41.10).

Table 13 gives the values of the coefficients for internal conversion accompanied by pair production for the cases of electric and magnetic multipoles obtained by numerical integration of expressions (41.7) and (41.13).

### 41.3 *Excitation of Nuclei by Electrons*

The process which is the inverse of internal conversion of $\gamma$ rays is the excitation of a nucleus by an electron. In this case the energy of the final state of the electron is less than that of the initial state, $\varepsilon_2 < \varepsilon_1$, while the final energy of the nucleus is greater than the initial energy $E_2 > E_1$. By utilizing the results of § 40 we can write the following expression for the probability of excitation of the nucleus resulting in its angular momentum being changed by $L$ and its parity being changed by $(-1)^{L+\lambda+1}$, and with the electron making a transition from the initial state (1) to the final state (2):

$$w_L^{(\lambda)} = w'^{(\lambda)}_{L\,\text{rad}}\beta_L^{(\lambda)}, \tag{41.18}$$

where $w'^{(\lambda)}_{L\,\text{rad}}$ is the probability of emission of radiation in the transition of the nucleus from an excited (final) state into the initial state, while $\beta_L^{(\lambda)}$ is the "conversion coefficient" equal to

$$\beta_L^{(\lambda)} = \frac{e^2}{4}\,\omega\left|\int \psi_2^* B_{LM}^{(\lambda)*}\psi_1\,dr\right|^2. \tag{41.19}$$

Just as in the case of the investigation of conversion accompanied by pair production we confine ourselves here to the free electron approximation (211), (194). We denote by $p_1$ and $p_2$ the values of the electron momentum in the initial and final states, and by $u_1$ and $u_2$ the amplitudes of the plane waves corresponding to them. In accordance with (41.18) and (41.19) the differential cross section for the process in which we are interested is equal to

$$d\sigma_L^{(\lambda)} = \frac{\alpha\omega}{4v_1} w'^{(\lambda)}_{L\,\text{rad}} \frac{dp_2}{(2\pi)^3}\,\delta(\varepsilon_1-\varepsilon_2-\omega)\,\frac{1}{2}\sum_{\mu}\left|\int e^{iqr}u_2^* B_{LM}^{(\lambda)}u_1\,dr\right|^2, \tag{41.20}$$

where $v_1$ is the initial electron velocity

$$q = p_1 - p_2$$

and $\sum_\mu$ denotes summation over the initial and final electron polarizations (the factor $\frac{1}{2}$ in front of the sum corresponds to averaging over the initial polarizations). On comparing formula (41.20) with formula (41.2) for the coefficient of conversion accompanied by pair production we can easily show that the last factor in these formulas, viz., the factor

$$\sum_\mu \left| \int e^{i(p_1-p_2)r} u_2^* B_{LM}^{(\lambda)*} u_1 \, dr \right|^2$$

in (41.20) and the factor

$$\sum_\mu \left| \int e^{i(p_+ + p_-)r} u^* B_{LM}^{(\lambda)} v \, dr \right|^2$$

in (41.2) differ only by the following replacement of parameters: $p_- \to p_1$, $-p_+ \to p_2$, $-\varepsilon_+ \to \varepsilon_2$, and $\varepsilon_- \to \varepsilon_1$. Therefore (41.20) can be written in the following form:

$$d\sigma_L^{(\lambda)} = \frac{1}{2} w_{L\,\mathrm{rad}}^{(\lambda)} \delta(\varepsilon_2 - \varepsilon_1 - \omega) \frac{dp_2}{v_1} b_L^{(\lambda)}(p_1 - p_2), \tag{41.21}$$

where $b_L^{(\lambda)}$ is a function which can be determined from the expressions for the differential conversion coefficient (41.3), (41.11) if we write them in the form

$$d\beta_L^{(\lambda)} = b(p_-, p_+)\delta(\varepsilon_+ + \varepsilon_- - \omega) dp_+ dp_- .$$

On going over in (41.21) from the variables $p_2$ to the variables $q$ and on integrating over the angles we obtain

$$d\sigma_L^{(\lambda)} = \frac{1}{2} w_{L\,\mathrm{rad}}^{(\lambda)} b_L^{(\lambda)}(p_+, q) \frac{dq}{p_1^2}, \tag{41.22}$$

where

$$b_L^{(\lambda)}(p_+, q) = \frac{d\beta_L^{(\lambda)}(\varepsilon_+, q)}{d\varepsilon_+ dq}$$

and $d\beta_L^{(\lambda)}$ is defined by expressions (41.6) and (41.12).

In order to obtain the total cross section for nuclear excitation $\sigma_L^{(\lambda)}$ it is necessary to integrate (41.22) over $q$ between the limits $q_{\min}$ and $q_{\max}$ determined by the conservation laws. Since the total cross section cannot depend on the absolute direction of the initial electron momentum, (41.22) can also be averaged over the directions of the vector $p_+$:

$$\sigma_L^{(\lambda)} = \frac{1}{2} w_{L\,\mathrm{rad}}^{(\lambda)} \frac{1}{p_1^2} \frac{1}{4\pi} \int \frac{do_1}{4\pi} \int_{q_{\min}}^{q_{\max}} b(p_+, q)\, dq .$$

This formula can be written in the form

$$\sigma_L^{(\lambda)} = \frac{w_{L\,\mathrm{rad}}^{(\lambda)}}{8\pi p_1^2} b_L^{(\lambda)}(\varepsilon_1) \tag{41.23}$$

where

$$b_L^{(\lambda)}(\varepsilon) = \frac{d\beta_L^{(\lambda)}(\varepsilon)}{d\varepsilon}$$

and $d\beta_L^{(\lambda)}$ is determined by formulas (41.7) and (41.13) in which we must replace $p_+$ and $p_-$ by $p_1$ and $p_2$.

### 41.4. *Monoenergetic Positrons*

If there is an unoccupied state in one of the atomic shells, then a pair can be created with the electron occupying this bound state, and the positron having a sharply defined energy. We are here dealing with the phenomenon of internal conversion involving the emission of monoenergetic positrons. In order to evaluate the conversion coefficient in this case we can make direct use of formula (40.13) where we must interpret $\psi_2$ as the electron wave function belonging to the discrete spectrum of the shell, and $\psi_1$ as a negative frequency wave function belonging to the continuous spectrum normalized per unit energy. For the $K$-shell formulas (40.25) and (40.28) will hold in which the continuous spectrum radial functions $g_\varkappa$ and $f_\varkappa$ must refer to negative frequencies. In the approximation of small $Z$ and of energies large compared to the binding energy of the $K$-shell we obtain the expressions for the conversion coefficients

$$\beta_L^{(1)} = 2\alpha(Z\alpha)^3 \frac{(L+1)\,\omega^2 + 4Lm^2}{(L+1)\,\omega^2} \left(1 - \frac{2m}{\omega}\right)^{L-\frac{1}{2}},$$

$$\beta_L^{(0)} = 2\alpha(Z\alpha)^3 \left(1 - \frac{2m}{\omega}\right)^{L+\frac{1}{2}} .$$

As an example we give the value of the ratio of the probability of conversion accompanied by the emission of a monoenergetic positron (with the electron being produced in the $K$-shell) to the total probability of conversion accompanied by pair production (182). Calculations carried out with exact radial functions for $\omega = 1.4$ MeV in the case of an electric dipole give for this ratio the value 1/3. As $\omega$ decreases this ratio increases.

The process which is inverse with respect to the production of monoenergetic positrons is the following one: when a positron collides with an electron belonging to the $K$-shell of the atom this pair is absorbed by the nucleus with no photons being emitted, and the nucleus becoming excited. The probability of such a process can be written as

$$w = w_\gamma \beta, \tag{41.24}$$

where $\beta$ is the coefficient of internal conversion accompanied by the production of monoenergetic positrons, and $w_\gamma$ is the probability of excitation of the nucleus by a photon of the corresponding energy.

Since the excitation energy acquired by the nucleus is sufficiently great, then as a result of the absorption of the pair the nucleus can disintegrate. For example, the effective cross section for the fission of uranium caused by such a process may be of the order of $10^{-31}$ cm$^2$ (157).

## § 42. Coulomb (Monopole) Transitions

### 42.1. *Reduction to the Static Interaction*

The general formulas for the probability of conversion or excitation obtained in § 40 become inapplicable if the angular momenta of the initial and final states of the nucleus are both equal to zero. For $L = 0$ expression (40.11) vanishes in accordance with the fact that there exist no photon states of angular momentum equal to zero. Nevertheless, the matrix element of $S^{(2)}$ for a transition between two such states is different from zero. Such transitions are called monopole or $E0$-transitions.

As we have already pointed out in § 40, the integrals in (40.11) extend over the region outside the nucleus, while the general expression for the matrix element also contains, in accordance with (40.32), an integral over the region occupied by the nucleus, which, generally speaking, differs from zero for $L = 0$. Thus, the matrix element of the effective interaction energy has the form for $L = 0$

$$U_{i\to f} = -\int\limits_{\Omega} j_{21}(\boldsymbol{r}_1) J_{21}(\boldsymbol{r}_2) \frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} d\boldsymbol{r}_1\, d\boldsymbol{r}_2,$$

where $j_{21}$ and $J_{21}$ are the four-vectors of the electron and the nuclear transition current density, and the integration is carried out over the volume $\Omega$ occupied by the nucleus. This matrix element can also be written in the form (37.16)

$$U_{i\to f} = -\int\limits_{\Omega} j_{21}(\boldsymbol{r}_1) A_{21}(\boldsymbol{r}_1)\, d\boldsymbol{r}_1, \tag{42.1}$$

where

$$A_{21}(\boldsymbol{r}_1) = \int J_{21}(\boldsymbol{r}_2) \frac{e^{i\omega|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|} d\boldsymbol{r}_2. \tag{42.2}$$

The case $L = 0$ implies a spherically symmetric distribution of nuclear transition current, whose spatial ($\boldsymbol{J}_{21}$) and time ($J_{21}^0$) components have the structure

$$\boldsymbol{J}_{21}(\boldsymbol{r}_2) = f(r_2)\boldsymbol{r}_2,$$
$$J_{21}^0(\boldsymbol{r}_2) = g(r_2),$$

where $f(r)$ and $g(r)$ are certain functions which are independent of the angles. The potentials associated with these currents have an analogous structure

$$\begin{aligned} \boldsymbol{A}_{21}(\boldsymbol{r}_2) &= a_0(r_2)\boldsymbol{r}_2 = -\nabla\lambda(r_2), \\ A_{21}^0(\boldsymbol{r}_2) &= \varphi_0(r_2). \end{aligned} \tag{42.3}$$

On substituting (42.3) into (42.1) we obtain

$$U_{i\to f} = +\int (j_{21}^0 \varphi + \boldsymbol{j}_{21}\nabla\lambda)\, d\boldsymbol{r}. \tag{42.4}$$

Since in accordance with (42.3) the vector potential is the gradient of a scalar function, it can be eliminated by means of a gauge transformation.

We introduce the transformed potential $\varphi_0 = \varphi - i\omega\lambda$. On integrating by parts the term containing $\nabla\lambda$ in (42.4) and on utilizing the equation of continuity $\operatorname{div}\boldsymbol{j}_{21} = -i\omega j_{21}^0$, we obtain

$$U_{i\to f} = \int j_{21}^0 \varphi\, d\boldsymbol{r} = e\int \psi_2^* \varphi \psi_1\, d\boldsymbol{r}. \tag{42.5}$$

In order to obtain the potential $\varphi$ we utilize the fact that $\varphi_0$ satisfies the equation $\Delta\varphi_0+\omega^2\varphi_0 = -g(\boldsymbol{r})$, and therefore $\Delta\varphi = \Delta(\varphi_0+i\omega\lambda) = -g-\omega^2\varphi_0+i\omega\Delta\lambda$. Since $A_{21} = -\nabla\lambda$, it follows from the Lorentz condition for the potentials that $\Delta\lambda = -i\omega\varphi_0$ and therefore

$$\Delta\varphi = -g = e\Psi_2^*\Psi_1. \tag{42.6}$$

Formulas (42.5) and (42.6) can also be written in the form

$$U_{i\to f} = -a\int \Psi_2^*(\boldsymbol{r}_1)\,\psi_2^*(\boldsymbol{r}_2)\,\Psi_1(\boldsymbol{r}_1)\,\psi_1(\boldsymbol{r}_2)\,\frac{d\boldsymbol{r}_1\,d\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}. \tag{42.7}$$

We see that for $L = 0$ the transition is brought about by the electrostatic (Coulomb) interaction between the charges.

Formula (42.5) can be conveniently written in the form

$$U_{i\to f} = \int J_{21}^0\Phi\,d\boldsymbol{r}, \tag{42.8}$$

where $\Phi$ is the electrostatic potential due to the electron density distribution and satisfies the equation

$$\Delta\Phi = -e\psi_2^*\psi_1. \tag{42.9}$$

In this equation the right hand side may turn out not to have spherical symmetry. However, in virtue of the spherical symmetry of the charge distribution in (42.8) only the symmetric part of the potential will give a nonvanishing result. Therefore, the right hand side of (42.9) can be averaged over the angles.

### 42.2. *Conversion and Nuclear Excitation in the Case of an E0-transition*

We now obtain the probability of conversion and nuclear excitation in the case of an $E0$-transition (68), (216), (194).

The electron wave function does not vary appreciably over the volume occupied by the nucleus. Therefore, the right hand side of (42.9) can be regarded constant, and we can start from the equation

$$\frac{1}{r}\frac{d^2(r\Phi)}{dr^2} = -e\varrho_0,$$

where

$$\varrho_0 = \psi_2^*(0)\psi_1(0).$$

The general solution of this equation is given by $\Phi = c_1+c_2/r-e\varrho_0(r^2/6)$. The constant $c_1$ is unimportant, since the integral (42.8) vanishes for constant $\Phi$ in virtue of the orthogonality of the nuclear wave functions. The constant $c_2$ is equal to zero, this being a consequence of the condition that $\Phi$ is finite at $r = 0$. Therefore

$$U_{i\to f} = -\frac{2\pi}{3}\varrho_0 Q_0, \tag{42.10}$$

where

$$Q_0 = \frac{e^2}{4\pi}\int \Psi_2^* \Psi_1 r^2 dr. \tag{42.11}$$

The selection rules for $E0$-transitions follcw from (42.11). They consist of the statements that $\Delta J = 0$, and that the parity remains the same. Thus, $E0$-transitions can take place not only in the case when the nuclear angular momenta in the initial and the final states are both zero, but also in the case when $J_1 = J_2 \neq 0$ with the parity of the two states being the same (40). In the latter case in addition to the $E0$-transition M1-and $E2$-transitions are also possible, and they, in contrast to the $E0$-transition, can take place not only by means of the conversion process, but also by means of photon emission. Transitions between states of $J_1 = J_2 = 0$, but of different parity, are possible only by means of emission of two photons, or by means of still more complicated processes.

The quantity $Q_0$ is the "zero moment" of the nucleus similar and in order of magnitude equal to the quadrupole moment. Since for $L = 0$ there is no photon emission, it is impossible to introduce a conversion coefficient. In order of magnitude $Q_0 \sim R^2 e^2$, where $R$ is the nuclear radius.

The probability of conversion in the $K$-shell can be expressed as

$$w = 2\cdot 2\pi\left(\frac{2\pi}{3}\right)^2 |Q_0|^2 |\psi_K(0)|^2 \, |\psi_\varepsilon(0)|^2, \tag{42.12}$$

where $\psi_K(0)$ is the value of the $K$-electron wave function in the nucleus, while $\psi_\varepsilon(0)$ is the value of the continuous spectrum electron wave function in the nucleus normalized per unit energy and evaluated for energy $\varepsilon = \omega + |\varepsilon_K|$ and angular momentum $j = \frac{1}{2}$.

In the case of conversion accompanied by pair creation the differential probability is equal to

$$dw = \frac{(2\pi)^3}{9}|Q_0|^2 \sum_{\mu_1, \mu_2} |\varrho_0|^2 \frac{dp_+ dp_-}{(2\pi)^6}\,\delta(\omega - \varepsilon_- - \varepsilon_+).$$

In the free electron approximation we have $\varrho_0 = \bar{u}\beta v$ where $u$ and $v$ are the amplitudes of the plane waves of positive and negative frequencies; $\sum_{\mu_1, \mu_2}$ denotes summation over the polarizations of the electron and the positron.

On carrying out the summation we obtain

$$\sum_{\mu_1, \mu_2} |\varrho_0|^2 = \frac{1}{4\varepsilon_-\varepsilon_+} \operatorname{Sp}\beta(i\hat{p}_- - m)\beta(i\hat{p}_+ + m) = \frac{\varepsilon_+\varepsilon_- + p_+p_- - m^2}{\varepsilon_-\varepsilon_+}$$

and, consequently,

$$dw = \frac{1}{9}\frac{1}{(2\pi)^3}|Q_0|^2|p_-||p_+|d\varepsilon_+ do_+ do_-(\varepsilon_-\varepsilon_+ - m^2 + p_-p_+). \quad (42.13)$$

In a similar manner the differential cross section for nuclear excitation is determined by the formula:

$$d\sigma = \frac{1}{18}|Q_0|^2\frac{|p_2|}{|p_1|}do_2(\varepsilon_1\varepsilon_2 + m^2 + p_1p_2). \quad (42.14)$$

The total cross section for nuclear excitation is equal to

$$\sigma = \frac{2\pi}{9}|Q_0|^2\frac{|p_2|}{|p_1|}(\varepsilon_1\varepsilon_2 + m^2). \quad (42.15)$$

CHAPTER VII

# Investigation of the Scattering Matrix

## § 43. Properties of Exact Solutions of the Equations of Quantum Electrodynamics. Propagators

### 43.1. *Stationary States of a System of Interacting Fields*

In the two preceding chapters we have investigated various electrodynamic processes in the first nonvanishing approximation. For the investigation of higher order approximations, it is necessary to use subsequent terms in the expansion of the scattering matrix. In the language of graphical representation of the scattering matrix this means that for a given process we must consider more complicated diagrams with the same external lines.

In investigating higher order approximations it is very useful to make maximum use of those properties of the system of interacting fields which can be formulated without reference to perturbation theory. In this section we shall establish some of these properties.

The system of interacting fields (the electromagnetic and the electron-positron fields) which we are studying can be characterized, like any other quantum mechanical system, by a set of stationary states.

We denote the wave functions (the state vectors) of the stationary states of this system by $\Phi_r$ where $r$ is the set of quantum numbers characterizing this state. The fact that the state is stationary means, by definition, that the wave function $\Phi_r$ is an eigenfunction of the energy operator. Since our system is a closed system, it follows from considerations of invariance that $\Phi_r$ is also an eigenfunction of the energy-momentum operator $\mathrm{P}_\mu$

$$\mathrm{P}_\mu \Phi_r = p_\mu \Phi_r. \qquad (43.1)$$

The four-vector $p_\mu(\boldsymbol{p}, ip_0)$ is obviously contained in the set of quantum numbers $r$.

The lowest energy state of the system is called the *vacuum* state. We denote it by $\Phi_0$ and assume that in the vacuum state

$$p_\mu = 0.$$

This value is the only one which guarantees relativistic invariance of the ground state of the system.

We assume that the electric charge of the vacuum is equal to zero.

For excited states the energy of the system must be positive, $p_0 > 0$. In order for this property to be relativistically invariant obviously the following relation must hold: $p^2 < 0$.

We consider a certain Heisenberg operator $\mathbf{F}(x)$ which is, in particular, a function of the coordinates and the time. It is defined by its matrix elements

$$\left(r\,|\,\mathbf{F}(x)\,|\,r'\right) = \left(\Phi_r^0,\ \mathbf{F}(x)\,\Phi_{r'}^0\right),$$

where $\Phi_r^0$ is a time-independent wave function in the Heisenberg picture.

We can easily obtain the dependence on the coordinates of the matrix element $(0\,|\mathbf{F}(x)|\,r)$ which connects the vacuum state with some stationary state $r$. In order to do this we utilize the result of subsection 22.3. It follows from (22.18) that

$$\left(0\,|\,\mathbf{F}(x)\,|\,r\right) = f_r e^{ipx}, \tag{43.2}$$

where $f_r$ is independent of $x$. Similarly

$$\left(r\,|\,\mathbf{F}(x)\,|\,0\right) = f_r' e^{-ipx}. \tag{43.3}$$

If $\mathbf{F}(x)$ is a Hermitian operator, we have $f_r' = f_r^*$.

The operators of the electron-positron and the electromagnetic fields $\psi(x)$ and $\mathbf{A}(x)$ are of the greatest importance from our point of view.

In accordance with (43.2) the matrix element of $\psi(x)$ can be written as

$$\left(0\,|\,\psi(x)\,|\,r\right) = \xi(r)\,u(p)\,e^{ipx}, \tag{43.4}$$

where $\xi(r)$ is some invariant function of the quantum numbers $r$, while $u(p)$ is a normalized bispinor.

Since the operator $\psi(x)$ (cf. § 18) has matrix elements corresponding only to electron annihilation or to positron creation, and the charge is an integral of the motion, then $\xi(r)$ in formula (43.4) differs from zero only if the state $r$ has a single electron charge $Q = e$.

Moreover, for a given $p$ the state $r$ is fourfold degenerate corresponding to the existence of four linearly independent bispinors. We can choose them in the form of eigenfunctions of the matrices $i\hat{p}$ and $\Sigma p$. As was shown in § 10, the operator $i\hat{p}$ has the two eigenvalues $\mp M$, where $M = \sqrt{-p^2}$, corresponding to the different parities $I = \pm i$. For a given $I$ the matrix $\Sigma f$ has the two eigenvalues $2|p|\mu$ ($\mu = \pm\frac{1}{2}$) corresponding to different polarizations.

Thus, for a state $r$ for which $(0|\psi(x)|r)$ differs from zero, the set of quantum numbers includes $p$; $Q = e$; $\mu = \pm\frac{1}{2}$; $I = \pm i$. Moreover the state may possess additional invariant quantum numbers. We shall call these states *single-charge* states. It is known from experiment that single-charge states cannot have an energy smaller than $m_c$, where $m_c$ is the electron mass. Therefore, we shall assume, although we cannot give a rigorous proof of this assertion, that the four-momentum of single-charge states satisfies the condition $p^2 \leqslant -m_c^2$.

We identify the state $p^2 = -m_c^2$ with the state of a free electron. Therefore we also assume that the parity in this state can have only the one value $I = i$, and that the set of quantum numbers: $p$, $Q = e$, $I = i$, $\mu = \pm\frac{1}{2}$ is a complete set.

Single-charge states for which $p^2 < -m_c^2$ can be interpreted as combinations of one electron and a certain number of photons and electron-positron pairs. Such a system must be characterized by additional quantum numbers and, in particular, may have arbitrary parity.

In analogy with (43.4) we can write the matrix element $(r|\bar{\psi}(x)|0)$ of the operator $\bar{\psi}(x)$ in the form

$$(r|\bar{\psi}(x)|0) = \xi'(r)\bar{u}(p)e^{-ipx}, \tag{43.5}$$

where $r$ stands for the same set of quantum numbers as in (43.4).

The matrix elements $(0|\bar{\psi}(x)|r')$ and $(r'|\psi(x)|0)$ have nonvanishing values for $Q = -e$. They can be obtained from (43.4) and (43.5) with the aid of the transformation of charge conjugation

$$\begin{aligned}(0|\bar{\psi}(x)|r') &= \mathrm{C}^+(0|\psi(x)|r),\\ (r'|\psi(x)|0) &= \mathrm{C}(r|\bar{\psi}(x)|0),\end{aligned} \tag{43.6}$$

where $r'$ differs from $r$ by a change in the signs of the quantum numbers $Q$ and $\mu$. Indeed, (43.6) follows from the relations (cf. subsection 23.3) $\Phi^0_{r'} = \Lambda\Phi^0_r$; $\Phi^0_0 = \Lambda\Phi^0_0$; $\Lambda\psi(x)\Lambda^+ = \mathrm{C}\bar{\psi}(x)$.

We now proceed to investigate the matrix elements of the operator of the electromagnetic field $\mathbf{A}(x)$. On the basis of (43.2) and (43.3) we write them in the form

$$\begin{aligned} (0|\mathbf{A}_\mu(x)|r) &= \eta(r)a_\mu(k)e^{ikx}, \\ (r|\mathbf{A}_\mu(x)|0) &= \eta'(r)a_\mu(k)e^{-ikx}. \end{aligned} \tag{43.7}$$

Here $a_\mu$ is a four-vector which we assume to be normalized just as in the case of free fields (cf. subsection 16.3), i.e., $a_\mu = e_\mu/(2k_0\Omega)^{1/2}$, where $e_\mu$ is a unit vector, and $\Omega$ is the normalization volume. The factor $\eta(r)$ is an invariant; it can depend on the invariants $k^2$ and $k_\mu a_\mu$, so that $\eta(r)$ has different values in the case of transverse ($k_\mu a_\mu = 0$) and longitudinal polarizations. With respect to the transverse states the operator $\mathbf{A}(x)$ is Hermitian so that

$$\eta'(r) = \eta^*(r) \qquad (k_\mu a_\mu = 0). \tag{43.8}$$

We note that for $k^2 \neq 0$ all the matrix elements of the operator $\mathbf{A}(x)$ have the property of transversality. Indeed, from the equations of motion $\Box\, \mathbf{A}_\mu(x) = -\mathbf{j}_\mu(x)$ and from relations (43.2), (43.3) it follows that

$$\begin{aligned} k^2(0|\mathbf{A}_\mu(x)|r) &= (0|\mathbf{j}_\mu(x)|r), \\ k^2(r|\mathbf{A}_\mu(x)|0) &= (r|\mathbf{j}_\mu(x)|0). \end{aligned}$$

Since the current operator $\mathbf{j}(x)$ satisfies the condition of continuity $[\partial \mathbf{j}_\mu(x)/\partial x_\mu] = 0$, we have $k_\mu(0|\mathbf{j}_\mu(x)|r) = k_\mu(r|\mathbf{j}_\mu(x)|0) = 0$, whence

$$\left.\begin{aligned} k_\mu(0|\mathbf{A}_\mu(x)|r) &= 0, \\ k_\mu(r|\mathbf{A}_\mu(x)|0) &= 0 \end{aligned}\right. \quad (k^2 \neq 0). \tag{43.9}$$

For $k^2 = 0$ we identify the state $r$ with a free photon. For $k^2 < 0$ the state $r$ can be interpreted as the combination of a certain number of photons and electron-positron pairs. All the states $r$ for which the matrix element (43.7) is different from zero are uncharged ($Q = 0$).

We now emphasize the principal points of difference between the matrix elements of the operators $\psi$ and $\mathbf{A}$ which correspond to exact solutions of the equations of quantum electrodynamics, and the matrix elements for noninteracting fields discussed in Chapter III.

First of all, the exact solutions contain matrix elements of the operators $\psi$ and $\mathbf{A}$ which connect the vacuum state with states for which $p^2 < -m_c^2$ and $k^2 < 0$, while in the case of noninteracting fields the matrix elements of $\boldsymbol{\psi}$ and $\mathbf{A}$ connect the vacuum state only with one-particle states, for which $p^2 = -m_c^2$ and $k^2 = 0$.

Secondly, for $p^2 = -m_c^2$ and $k^2 = 0$ the matrix elements of $\boldsymbol{\psi}$ and $\mathbf{A}$ differ from the matrix elements of $\psi$ and A by the factors $\xi(r)$ and $\eta(r)$. We write these matrix elements in the form

$$\begin{aligned} \left(0\right|\boldsymbol{\psi}(x)\left|r\right) &= Z_1^{1/2}u(p)e^{ipx}, \quad p^2 = -m_c^2, \\ \left(0\right|\mathbf{A}(x)\left|r\right) &= Z^{1/2}a(k)e^{ikx}, \quad k^2 = 0, \end{aligned} \tag{43.10}$$

where $Z_1^{1/2}$ and $Z^{1/2}$ stand for the values of $\xi(r)$ and $\eta(r)$ for $p^2 = -m_c^2$ and $k^2 = 0$.

### 43.2. *Propagators and Their Spectral Representation*

In Chapter III we have defined for noninteracting fields the photon propagator $D^c_{\mu\nu}$ and the electron propagator $S^c_{\mu\nu}$ as the expectation values of the chronological product of the field operators in the vacuum state. These functions were the Green's functions for the d'Alembert wave equation and the Dirac equation.

We now consider similar vacuum expectation values for interacting fields. They are also called *Green's functions* or *propagators.*

The photon propagator which we denote by $G^{\gamma}_{\mu\nu}$ is defined as

$$G^{\gamma}_{\mu\nu}(x-y) = \left(0\right|T\left(\mathbf{A}_\mu(x)\mathbf{A}_\nu(y)\right)\left|0\right). \tag{43.11}$$

The electron propagator which we denote by $G^e_{\alpha\beta}$ is defined in a similar manner:

$$G^e_{\alpha\beta}(x-y) = \left(0\right|T\left(\boldsymbol{\psi}_\alpha(x)\bar{\boldsymbol{\psi}}_\beta(y)\right)\left|0\right). \tag{43.12}$$

The functions $G^{\gamma}$ and $G^e$ can obviously depend only on the difference of coordinates $x-y$ in view of homogeneity of space-time.

We shall show that the electron and the photon Green's functions $G^e$ and $G^{\gamma}$ can be represented in the form of expansions in terms of Green's functions for free particles of different masses (99), (124), (76). These expansions will enable us to investigate a number of general properties of the propagators.

We begin with the photon Green's function. Let $x_0 > y_0$. Then in accordance with (43.11) we have

$$G^{\gamma}_{\mu\nu}(x-y) = \left(0\right|\mathbf{A}_\mu(x)\mathbf{A}_\nu(y)\left|0\right), \quad \text{where} \quad x_0 > y_0.$$

By utilizing the rule of matrix multiplication we write $G^{\gamma}$ in the form

$$G^{\gamma}_{\mu\nu}(x-y) = \sum_{r} \left(0|\mathbf{A}_{\mu}(x)|r\right)\left(r|\mathbf{A}_{\nu}(y)|0\right),$$

where $r$ is the complete set of quantum numbers characterizing the stationary states. On substituting into this equation expressions (43.7) for the matrix elements we obtain

$$G^{\gamma}_{\mu\nu}(x-y) = \sum_{r} \eta(r)\,\eta'(r)\,a_{\mu}(k)\,a_{\nu}(k)\,e^{ik(x-y)}. \tag{43.13}$$

We note that in accordance with (43.9) the longitudinal components of the matrix elements of the operator $\mathbf{A}$, i.e., those which do not satisfy the relation $a_{\mu}k_{\mu} = 0$, differ from zero only in the case $k^2 = 0$. In Chapter III we have obtained an expression for the longitudinal part $D^{c(||)}_{\mu\nu}$ of the photon propagator in the case of the free field ($k^2 = 0$)

$$D^{c(||)}_{\mu\nu}(x-y) = \frac{\partial^2 \chi(x-y)}{\partial x_{\mu}\partial x_{\nu}},$$

where $\chi$ is an arbitrary function. The longitudinal part of the function $G^{\gamma}$ will obviously have the same form. Therefore, expression (43.13) can be rewritten in the form

$$G^{\gamma}_{\mu\nu}(x-y) = \sum_{k,\,s} |\eta(k^2,\,s)|^2 \sum_{\lambda} a^{(\lambda)}_{\mu}(k)\,a^{(\lambda)}_{\nu}(k)\,e^{ik(x-y)} + \frac{\partial^2\chi(x-y)}{\partial x_{\mu}\partial x_{\nu}}, \tag{43.13$'$}$$

where the quantum number $\lambda$ defines the polarization state satisfying the condition of transversality, while $s$ is the set of the remaining quantum numbers. Further, for the sake of brevity we omit the arbitrary term $\partial^2\chi/\partial x_{\mu}\partial x_{\nu}$.

The four-fold summation over $k$ in (43.13′) can be carried out in two stages: first the three-fold summation over $\mathbf{k}$ is carried out for a fixed $k_0 = \sqrt{\mathbf{k}^2+M^2}$, and then summation over $M$ is performed. In this way we obtain

$$\begin{aligned} G^{\gamma}_{\mu\nu}(x-y) &= Z \sum_{k,\,\lambda(k^2=0)} a^{(\lambda)}_{\mu}(k)\,a^{(\lambda)}_{\nu}(k)\,e^{ik(x-y)} \\ &+ \sum_{M,s} |\eta(-M^2,\,s)|^2 \sum_{k,\,\lambda(k^2=-M^2)} a^{(\lambda)}_{\mu}(k)\,a^{(\lambda)}_{\nu}(k)\,e^{ik(x-y)}. \end{aligned} \tag{43.13$''$}$$

(The index $s$ is absent from the first sum, since single photon states are characterized only by their momentum and polarization).

In a similar manner we obtain for $x_0 < y_0$ an expression which differs from (43.13″) by the replacement of $k$ by $-k$.

On taking into account the fact that the summations over $\boldsymbol{k}$ and over $-\boldsymbol{k}$ are identical, we can write for arbitrary times $x_0$ and $y_0$

$$G^{\lambda}_{\mu\nu}(x-y) = Z \sum_{\boldsymbol{k},\lambda} a^{(\lambda)}_{\mu}(k) a^{(\lambda)}_{\nu}(k) e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})-i|\boldsymbol{k}|\,|x_0-y_0|}$$

$$+ \sum_{M,s} |\eta(-M^2, s)|^2 \sum_{\boldsymbol{k},\lambda\,(k^2=-\mu^2)} a^{(\lambda)}_{\mu}(k) a^{(\lambda)}_{\nu}(k) e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})-i\sqrt{\boldsymbol{k}^2+M^2}\,|x_0-y_0|}. \tag{43.14}$$

We know the results of evaluating the sums over $\boldsymbol{k}$ and $\lambda$ for given values of $M$ which appear in expression (43.14). We have evaluated them in § 17, and they are expressed in terms of the functions $D^c$ and $\Delta^c_{(M)}$

$$\sum_{\boldsymbol{k},\lambda} a^{(\lambda)}_{\mu}(k) a^{(\lambda)}_{\nu}(k) e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})-i\sqrt{\boldsymbol{k}^2+M^2}\,|x_0-y_0|} = \Delta^c_{(M)\mu\nu}(x-y),$$

where $\Delta^c_{(M)\mu\nu}(x-y)$ is determined by the four-fold Fourier integral

$$\Delta^c_{(M)\mu\nu}(x-y) = \frac{1}{(2\pi)^4} \int \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \Delta^c_M(k) e^{ik(x-y)}\, d^4k$$

and

$$\Delta^c_M(k) = \frac{-i}{k^2+M^2-i0}$$

(of. subsection 17.3). For $M=0$ we have $\Delta^c_M(k) = D^c(k)$.

The final expression for the photon Green's function can be conveniently represented in the form of the four-fold Fourier integral

$$G^{\gamma}_{\mu\nu}(x-y) = \frac{1}{(2\pi)^4} \int \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) G^{\gamma}(k) e^{ik(x-y)}\, d^4k.$$

On introducing the notation

$$\sum_{M}^{M+dM} \sum_{s} |\eta(-M^2, s)|^2 = Z\varrho(M^2)\, dM^2,$$

we can rewrite $G^{\gamma}(k)$ in the form

$$G^{\gamma}(k) = Z\left\{D^c(k) + \int_0^\infty \varrho(M^2) \Delta^c_M(k)\, dM^2\right\}. \tag{43.15}$$

or

$$G^{\gamma}(k) = \frac{Z}{i}\left\{\frac{1}{k^2-i0} + \int_0^{\infty} \frac{\varrho(M^2)\,dM^2}{k^2+M^2-i0}\right\}. \tag{43.15'}$$

Thus, we have obtained a representation of the photon Green's function in the form of an expansion in terms of the Green's functions for free particles of different masses. The quantity $\varrho(M^2)$ is essentially positive and can be interpreted as the density of states of a given mass.

We see from (43.15) that $G^{\gamma}(k)$ has a pole at $k^2 = 0$, i.e., at a point corresponding to a real photon.

The representation (43.15) implies that the function $G^{\gamma}(k)$ has definite analytical properties. In order to establish these properties we utilize the fact that the function $\Delta^c_M(k)$ can, in accordance with (17.41) and (17.19'), be separated into a real and an imaginary part

$$\Delta^c_M(k) = \pi\,\delta(k^2+M^2) - iP\frac{1}{k^2+M^2}.$$

On substituting this expression into (43.15) we obtain a simple relation between the real part of $G^{\gamma}(k)$ and the function $\varrho$

$$\operatorname{Re} G^{\gamma}(k) = \pi Z\{\delta(k^2)+\varrho(-k^2)\,\theta(-k^2)\}, \tag{43.16}$$

where

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

From this it follows, in particular, that

$$\operatorname{Re} G^{\gamma}(k) \begin{cases} = 0, & k^2 > 0, \\ > 0, & k^2 < 0. \end{cases} \tag{43.16'}$$

Further, we obtain

$$\operatorname{Im} G^{\gamma}(k) = -ZP\left\{\frac{1}{k^2} + \int_0^{\infty} \frac{\varrho(M^2)\,dM^2}{k^2+M^2}\right\}. \tag{43.16''}$$

Equations (43.16) and (43.16'') imply the following relation between the real and the imaginary parts of the photon propagator:

$$\operatorname{Im} G^{\gamma}(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} G^{\gamma}(k')\,dk'^2}{k'^2-k^2} \tag{43.17}$$

From (43.15′) we can also draw some conclusions with respect to the asymptotic behavior of $G^{\gamma}(k)$ for large $k^2$. Since $\varrho(M^2) > 0$, and if the integral $\int \varrho(M^2)dM^2 \equiv J$ converges, then $G^{\gamma}(k) \sim (J/k^2)$, where $k^2 \to \infty$. However, if the integral $J$ does not exist, then $G^{\gamma}(k)$ falls off as $k^2 \to \infty$ slower than $1/k^2$.

We now proceed to obtain the expansion of the electron Green's function $G^e(p)$ in terms of the free particle Green's functions.

In accordance with (43.12) we have for $x_0 > y_0$

$$G^e_{\alpha\beta}(x-y) = (0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0).$$

On utilizing (43.4) and (43.5) we obtain

$$G^e_{\alpha\beta}(x-y) = \sum_r{}' \xi(r)\xi'(r)u_\alpha(p)\overline{u}_\beta(p)e^{ip(x-y)}, \qquad x_0 > y_0$$

or

$$G^e_{\alpha\beta}(x-y) = \sum_{p,s,I}{}' \xi(p^2, s, I)\,\xi'(p^2, s, I) \sum_\mu u_\alpha^{(\mu,I)}(p)\overline{u}_\beta^{(\mu,I)}(p)e^{ip(x-y)}, \tag{43.18}$$

where $\mu$ is the quantum number specifying the polarization, $I$ is the parity, and $s$ is the set of the remaining quantum numbers.

Just as in the investigation of the photon Green's function, we replace the four-fold summation over $p$ by the three-fold summation over $\mathbf{p}$ for a given $p_0 = \sqrt{\mathbf{p}^2+M^2}$, and by a summation over $M$, i.e., we write (43.18) in the form

$$G^e_{\alpha\beta}(x-y) = \sum_{M,s,I}{}' \xi\xi' \sum_{\mu,\mathbf{p}} u_\alpha^{(\mu,I)}\overline{u}_\beta^{(\mu,I)}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})-i\sqrt{\mathbf{p}^2+M^2}(x_0-y_0)}.$$

For $x_0 < y_0$ we obtain, on utilizing (43.6), an expression which differs from (43.18) by the replacement of $x-y$ by $y-x$. Therefore, in the general case of arbitrary $x_0$ and $y_0$ the function $G^e$ has the form

$$\begin{aligned} G^e_{\alpha\beta}(x-y) = Z_1 &\sum_{\mathbf{p},\mu} u_\alpha^{(\mu,I)}(p)\overline{u}_\beta^{(\mu,I)}(p)\,e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})-i\sqrt{\mathbf{p}^2+m_c^2}|x_0-y_0|} \\ &+ \sum_{M,s,I}{}' \xi(-M^2, s, I)\,\xi'(-M^2, s, I) \\ &\times \sum_{\mathbf{p},\mu} u_\alpha^{(\mu,I)}(p)\overline{u}_\beta^{(\mu,I)}\,e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})-i\sqrt{\mathbf{p}^2+M^2}|x_0-y_0|}. \end{aligned} \tag{43.18'}$$

Here the first sum refers to the state of a real free electron, so that the index $s$ is absent from it, and $I = +i$.

The sums over $\mathbf{p}$ and $\mu$ appearing in (43.18′) were evaluated in § 19. Indeed, for $I = i$ we have

$$\sum_{\mathbf{p},\mu} u_{\alpha}^{(\mu)}(p)\bar{u}_{\beta}^{(\mu)}(p)\, e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})-i\sqrt{\mathbf{p}^2+M^2}\,|x_0-y_0|} = S^{c}_{(M)\alpha\beta}(x-y),$$

where

$$S^{c}_{(M)}(x-y) = \frac{1}{(2\pi)^4}\int S^{c}_{(M)}(p)\, e^{ip(x-y)}\, d^4p$$

and

$$S^{c}_{(M)}(p) = -\frac{1}{\hat{p}-iM-0}.$$

The states with $I = -i$ differ by the fact that for them the amplitudes $u$ satisfy the equation $(i\hat{p}-M)u = 0$, Therefore, the values of the sum for $I = -i$ differ from the values for $I = i$ by the replacement of $S^{c}_{(M)}$ by $S^{c}_{(-M)}$, where

$$S^{c}_{(-M)}(p) = -\frac{1}{\hat{p}+iM+0}.$$

Thus, on introducing the notation

$$\sum_{M}^{M+dM}{}' \sum_{s} \xi(-M^2, s, \pm i)\, \xi'(-M^2, s, \pm i) = Z_1 \sigma(\pm M)\, dM,$$

we can write the electron Green's function in the form

$$G^{e}(x-y) = Z_1 \Big\{ S^{c}_{(m_c)}(x-y) + \int_{m_c}^{\infty} \sigma(M) S^{c}_{(M)}(x-y)\, dM + \int_{m_c}^{\infty} \sigma(-M) S^{c}_{(-M)}(x-y)\, dM \Big\}. \tag{43.19}$$

The same expansion can obviously also be written for the Green's function in the momentum representation, i.e., for the Fourier coefficients in the expansion

$$G^{e}(x-y) = \frac{1}{(2\pi)^4}\int G^{e}(p)\, e^{ip(x-y)}\, d^4p,$$

viz.:

$$G^{e}(p) = -Z_1 \left\{ \frac{1}{\hat{p}-im_c-0} + \int_{m_c}^{\infty} \frac{\sigma(M)\, dM}{\hat{p}-iM-0} + \int_{m_c}^{\infty} \frac{\sigma(-M)\, dM}{\hat{p}+iM+0} \right\}. \tag{43.20}$$

We see from (43.20) that $G^e(p)$ has a pole at $p^2+m_c^2=0$ corresponding to the state of a real electron.

Expansions of the type (43.15) and (43.20) can also be obtained for the vacuum expectation values of the commutators $(0|[\mathbf{A}_\mu(x), \mathbf{A}_\nu(y)]|0)$ and the anticommutators $(0|\{\psi_\alpha(x), \bar{\psi}_\beta(y)\}|0)$. By proceeding in the same fashion as in the derivation of formulas (43.15) and (43.20) we obtain

$$(0|[\mathbf{A}_\mu(x), \mathbf{A}_\nu(y)]|0) = Z\left\{D_{\mu\nu}(x-y)+\int_0^\infty \varrho(M^2)\Delta_{(M)\mu\nu}(x-y)\,dM^2\right\},$$

$$(0|\{\psi_\alpha(x), \bar{\psi}_\beta(y)\}|0) = Z_1\left\{S_{\alpha\beta}(x-y)+\int_{m_c}^\infty [\sigma(M)S_{(M)\alpha\beta}(x-y) + \sigma(-M)S_{(-M)\alpha\beta}(x-y)]\,dM\right\}, \qquad (43.21)$$

where the functions $\Delta_{(M)}$ and $S_{(M)}$ are defined by formulas (17.37) and (19.5). In particular, for $x_0 = y_0$ the following commutation relations follow from the above expressions:

$$(0|[\mathbf{A}_\mu(\mathbf{x}, t), \mathbf{A}_\nu(\mathbf{y}, t)]|0) = 0,$$

$$\left(0\left|\left[\mathbf{A}_\mu(\mathbf{x}, t), \frac{\partial \mathbf{A}_\nu(\mathbf{y}, t)}{\partial t}\right]\right|0\right) = -i\delta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y})Z\left(1+\int_0^\infty \varrho(M^2)\,dM^2\right),$$

$$(0|\{\psi_\alpha(\mathbf{x}, t), \bar{\psi}_\beta(\mathbf{y}, t)\}|0) = \gamma_4\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})\,Z_1\left\{1+\int_{m_c}^\infty [\sigma(M)+\sigma(-M)]\,dM\right\}.$$

On comparing these formulas with the commutation relations (22.5) we obtain expressions for the constants $Z$ and $Z_1$

$$Z^{-1} = 1+\int_0^\infty \varrho(M^2)\,dM^2, \qquad (43.22)$$

$$Z_1^{-1} = 1+\int_{m_c}^\infty [\sigma(M)+\sigma(-M)]\,dM. \qquad (43.22')$$

Since $\varrho(M^2)$ is an essentially positive quantity, it follows from this that $0 \leqslant Z \leqslant 1$.

Equation (43.22) can be interpreted as the condition that the system of wave functions for a given value of the momentum $k$ be complete. Indeed, in the case of a noninteracting electromagnetic field there exists only one value of the energy $k_0 = |k|$, and therefore $Z = 1$. The interaction adds states of energies $k_0 = \sqrt{k^2+M^2}$, with their density being given by $Z\varrho(M^2)$. Therefore the condition for the completeness of the system assumes the form (43.22).

Let us also consider the problem of the transformation of the electron Green's function under the gauge transformation $A_\mu \to A_\mu + \partial\chi/\partial x_\mu$ (116), where $\chi$ is an arbitrary scalar function.

The operator of the electron-positron field $\psi$ transforms in this case in accordance with $\psi \to \psi e^{i\chi}$. Therefore, the gauge transformation for the Green's function has the form

$$G^{(e)}(x, x') = G_0^{(e)}(x-x')\left(0|T(e^{ie\chi(x)} - e^{-ie\chi(x')})|0\right)$$

where $G_0^{(e)}(x-x')$ is the Green's function corresponding to the transverse gauge for the electromagnetic potentials, i.e., $d_l = 0$ (cf. (17.21)).

On expanding $\chi(x)$ in plane waves and on utilizing relations (17.12′) we can easily show that the last factor in the expression for $G^{(e)}(x, x')$ is equal to $\exp\left(ie^2[B(0)-B(x-x')]\right)$ where

$$B(x-x') = \int \frac{d_l(k^2)}{k^2} e^{ik(x-x')}\, d^4k.$$

Therefore, the gauge transformation for the electron Green's function can finally be written in the form

$$G^{(e)}(x-x') = G_0^{(e)}(x-x') \exp\left(ie^2\, [B(0)-B(x-x')]\right) \qquad (43.23)$$

### 43.3. *Connection between Propagators and the Scattering Matrix. Integral Equations for Propagators*

The foregoing matrix elements of the field operators $\mathbf{A}, \psi, \bar{\psi}$ and their products were defined in the Heisenberg picture. They can be determined most simply in a unique and relativistically invariant manner in this particular picture, since only the operators depend on the time, while the wave functions (the state vectors) are constant.

We now express these matrix elements in the interaction picture. In order to do this we use the transformation formulas

$$\Phi(t) = \mathrm{S}(t, -\infty)\Phi^0,$$
$$\mathrm{F}(x) = \mathrm{S}^{-1}(x_0, -\infty)\,\mathrm{F}(x)\,\mathrm{S}(x_0, -\infty),$$

or

$$\mathrm{F}(x) = \mathrm{S}^{-1}T\big(\mathrm{F}(x)\mathrm{S}\big), \tag{43.24}$$

where $\mathrm{S} = S(\infty, -\infty)$ is the scattering matrix and $T$ is the chronological operator. From this it follows that the matrix element $\big(r|\mathbf{F}(x)|r'\big)$ has the following form in the interaction picture:

$$\big(r|\mathbf{F}(x)|r'\big) = \big(\Phi_r^0, \mathrm{S}^{-1}T\big(\mathrm{F}(x)\mathrm{S}\big)\Phi_{r'}^0\big) \tag{43.25}$$

where $\Phi_r^0$ denotes the constant wave function for the system of fields in the stationary state $r$. Since it is assumed that at time $t = -\infty$ there is no interaction between the fields, the state vector $\Phi_r^0$ can be regarded as coinciding with $\Phi_r(-\infty)$, where $\Phi_r(t)$ is the wave function for the system of fields in the interaction picture describing the state $r$.

In the case of stationary states $\Phi_r(\infty)$ may differ from $\Phi_r(-\infty) = \Phi_r^0$ only by an inessential phase factor, i.e.,

$$\Phi_r(\infty) = \mathrm{S}\Phi_r(-\infty) = \lambda_r\Phi_r(-\infty) \tag{43.26}$$

where the absolute value of the quantity $\lambda_r = (\Phi_r^0, \mathrm{S}\Phi_r^0) = (r|\mathrm{S}|r)$ is equal to unity, $|\lambda_r| = 1$.

Thus, we have

$$\big(r|\mathbf{F}(x)|r'\big) = \frac{1}{(r|\mathrm{S}|r)}\big(r|T\big(\mathrm{F}(x)\mathrm{S}\big)|r'\big) \tag{43.27}$$

In the derivation of this formula we have made no distinction between the instant at which the interaction between the fields is switched on, and the instant when the wave function of the system in the Heisenberg picture coincides with the wave function in the interaction picture. However, a more consistent approach is a somewhat different one, based on the adiabatic hypothesis, according to which at $t = -\infty$ the interaction between the fields is switched on infinitely slowly, as a result of which the wave functions in the two pictures — the Heisenberg and the interaction picture—coincide not at $t = -\infty$, but at some finite value of the time, when the interaction has been completely switched on. However, such an approach does not change the result (43.27). We must only keep in mind that for the evaluation of the matrix element $\big(r|\mathrm{F}(x)|r'\big)$ in the Heisenberg picture we must utilize the eigenfunctions of the total Hamiltonian for the system of fields H, which describe the states for $r$ and $r'$, while for the evaluation of the matrix element $\left(r\middle|T\big(\mathrm{F}(x)\mathrm{S}\big)\middle|r'\right)$ in the interaction picture we must utilize the eigenfunctions of the unperturbed Hamiltonian $\mathrm{H}_0$ which also refer to the states $r$ and $r'$.

Similarly, it can be easily shown that for a product of two operators $\mathbf{F}_1(x)$ and $\mathbf{F}_2(y)$ the matrix element connecting the two stationary states $r$ and $r'$ has the form

$$\left(r\middle|T\big(\mathbf{F}_1(x)\,\mathbf{F}_2(y)\big)\middle|r'\right) = \left(r\middle|T\big(\mathrm{F}(x)\,\mathrm{F}(y)\,\mathrm{S}\big)\middle|r'\right)(r|\mathrm{S}|r)^{-1}. \qquad (43.28)$$

In particular, the propagators defined earlier can be expressed in the interaction picture in the following manner:

$$G^{e}_{\alpha\beta}(x-y) = \left(0\middle|T\big(\psi_\alpha(x)\,\bar{\psi}_\beta(y)\,\mathrm{S}\big)\middle|0\right), \qquad (43.29)$$

$$G^{\gamma}_{\mu\nu}(x-y) = \left(0\middle|T\big(A_\mu(x)A_\nu(y)\,\mathrm{S}\big)\middle|0\right). \qquad (43.30)$$

(Here, and in future, we omit the factor $(0|\mathrm{S}|0)^{-1}$ where the index 0 denotes the vacuum state for the free fields.)

We can use these formulas to establish the relation between the propagators and the scattering matrix.

In Chapter IV we have obtained a general expression for the scattering matrix in the form of a chronological product of field operators. But every specific electrodynamic problem is associated with expanding the scattering matrix in terms of normal products. The only way of doing this is to take the series expansion of the scattering matrix and to expand its terms in normal products one at a time. However, in principle we can consider an exact expansion of the scattering matrix in terms of normal products. Such an expansion must be of the form

$$\mathrm{S} = \sum_{n,\nu} K^{(n,\nu)}(x_1, \ldots, x_n;\ y_1, \ldots, y_n;\ \xi_1, \ldots, \xi_\nu)$$
$$\times N\big(\bar{\psi}(x_1), \ldots, \bar{\psi}(x_n); \psi(y_1), \ldots, \psi(y_n);\ \mathrm{A}(\xi_1), \ldots, \mathrm{A}(\xi_\nu)\big). \qquad (43.31)$$

For the sake of brevity we have omitted here the spinor indices on $\psi$ and $\bar{\psi}$ and the vector indices on A; the quantity $K^{(n,\nu)}$ is a spin tensor of the $2n$th rank and a tensor of the $\nu$th rank, i.e., a quantity having $2n$ spinor indices and $\nu$ vector indices. (The number of times that $\psi$ appears as a factor in each term is, in accordance with expression (22.4) for the current, obviously equal to the number of times that $\bar{\psi}$ appears as a factor.)

The expansion coefficient $K^{(n,\nu)}$ obviously defines the exact expression for the matrix element of a process in which the total number of electrons and positrons in the initial and final states is equal to $2n$, while the number of photons is equal to $\nu$.

As an example we consider the Compton effect. By utilizing the expressions for the matrix elements of the operators $\psi$, $\overline{\psi}$ and A we obtain, in accordance with (43.31), the following expressions for the matrix element:

$$S_{i\to f}=a_\alpha(k_1)a_\beta(k_2)\overline{u}(p_2)\{K^{(1,2)}_{\alpha\beta}(-p_2,p_1,k_1,-k_2)\\ +K^{(1,2)}_{\alpha\beta}(-p_2,p_1,-k_2,k_1)\}\,u(p_1), \quad (43.32)$$

where $K^{(1,2)}(q_1,q_2,f_1,f_2)$ is the corresponding Fourier component of $K^{(1,2)}$

$$K^{(1,2)}(q_1,q_2,f_1,f_2)\\ =\int K^{(1,2)}(x,y,\xi_1,\xi_2)\,e^{i(q_1x+q_2y+f_1\xi_1+f_1\xi_2)}\,d^4x\,d^4y\,d^4\xi_1\,d^4\xi_2,$$

while $u(p)$ and $a(k)$ are the amplitudes of the electron and photon states.

The matrix element for any process can be expressed in a similar manner. Let there be $n_1$ electrons and $n_2$ positrons in the initial state, and $n-n_2$ electrons and $n-n_1$ positrons in the final state. Further, we denote the total number of photons in the initial and the final states by $\nu$. Then we have

$$S_{i\to f}=\int K^{(n,\nu)}(x_1,\ldots,\xi_\nu)\,\overline{\Psi}'(x_1,\ldots,x_n)\\ \times\Psi(y_1,\ldots,y_n)\Phi(\xi_1,\ldots,\xi_\nu)\,d^4x_1\ldots d^4\xi_\nu,$$

where $\Psi$ is the antisymmetrized wave function (in configuration space) for the electrons in the initial state and for the positrons in the final state, $\Psi'$ is the antisymmetrized wave function for the electrons in the final state and the positrons in the initial state, and $\Phi$ is the symmetrized wave function for the photons in both the initial and final states

$$\Psi(y_1,\ldots,y_n)=u(p_1)\ldots u(p_{n_1})v(-p_{n_1+1})\ldots v(-p_n)\sum(\pm)e^{i\sum\limits_j p_jy_j},$$

$$\overline{\Psi}'(x_1,\ldots,x_n)=\overline{v}(-p_1')\ldots\overline{v}(-p_{n_2}')\overline{u}(p_{n_2+1}')\ldots\overline{u}(p_n')\sum(\pm)e^{-i\sum\limits_j p'_jx_j}$$

$$\Phi=\sum a(k_1)\ldots a(k_\nu)e^{i\sum k_j\xi_j}.$$

The quantities $K^{(1,0)}$ and $K^{(0,2)}$ which are called the electron and the photon *self-energy functions* will be of particular importance for us. We denote them in the following manner:

$$K^{(1,0)}_{\alpha\beta}(x, y) = \Sigma_{\alpha\beta}(x, y)\,; \qquad K^{(0,2)}_{\mu\nu}(\xi_1, \xi_2) = \Pi_{\mu\nu}(\xi_1, \xi_2). \quad (43.33)$$

The quantity $\Sigma$ depends on two spinor indices $\alpha$, $\beta$ and on two four-dimensional coordinates, and in the absence of external fields it obviously depends only on their difference: $\Sigma(x, y) = \Sigma(x-y)$. Its Fourier components, therefore, depend on only one variable (momentum). We denote them by $\Sigma(p)$.

The quantity $\Pi$ depends on two vector indices $\mu$, $\nu$, and also on the difference of coordinates $\Pi_{\mu\nu}(x,y) = \Pi_{\mu\nu}(x-y)$. Its Fourier components are functions of one momentum. We denote these components by $\Pi_{\mu\nu}(k)$.

The function $\Sigma$ is simply related to the electron propagators $G^e$. In order to establish this important relation we substitute into the expression (43.29) for the function $G^e$ the expansion for the scattering matrix (43.31):

$$G^e_{\alpha\beta}(x-y)$$
$$= \left(0\middle|T\left\{\psi_\alpha(x)\bar{\psi}_\beta(y)N\left[1-\int\Sigma_{\gamma\delta}(x'-y')\psi_\delta(y')\bar{\psi}_\gamma(x')\,d^4x'\,d^4y'+\cdots\right]\right\}\middle|0\right).$$

In accordance with the rules of § 24 the mixed $T$-products appearing in the right hand side of this equation can be expanded in terms of $N$-products. Since the vacuum expectation value of any $N$-product is equal to zero, all the terms will vanish, except the first two for which explicit expressions have been given earlier.

Further, by utilizing the equations

$$\left(0\middle|T\big(\psi_\alpha(x)\bar{\psi}_\beta(y)\big)\middle|0\right) = S^c_{\alpha\beta}(x-y),$$
$$\left(0\middle|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)N[\psi_\delta(y')\bar{\psi}_\gamma(x')]\}\middle|0\right) = -S^c_{\alpha\gamma}(x-x')\,S^c_{\delta\beta}(y'-y),$$

we obtain

$$G^e(x-y) = S^c(x-y)+\int S^c(x-x')\,\Sigma(x'-y')\,S^c(y'-y)\,d^4x'\,d^4y'. \quad (43.34)$$

In the momentum representation expression (43.34) assumes the form

$$G^e(p) = S^c(p)+S^c(p)\Sigma(p)\,S^c(p). \quad (43.35)$$

In a similar manner we can obtain the relation between $G^\gamma$ and $\Pi$. On substituting (43.31) into (43.30), we obtain

$$G^\gamma_{\mu\nu}(x-y) = D^c_{\mu\nu}(x-y)+\int D^c_{\mu\lambda}(x-x')\Pi_{\lambda\gamma}(x'-y')\,D^c_{\gamma\nu}(y'-y)\,d^4x'\,d^4y'. \tag{43.36}$$

In the momentum representation relation (43.36) assumes the form

$$G^\gamma_{\mu\nu}(k) = D^c_{\mu\nu}k+D^c_{\mu\lambda}(k)\Pi_{\lambda\sigma}(k)\,D^c_{\sigma\nu}(k). \tag{43.36$'$}$$

We have shown earlier (cf. (43.14)) that we can confine our discussion only to the transverse part of the function $G^\gamma(k)$, which satisfies the relation $G^\gamma_{\mu\nu}k_\nu = 0$, since the longitudinal part is an arbitrary function which coincides with the corresponding function in the absence of interaction. Therefore, the photon self-energy function must also satisfy the condition of transversality, or, in other words, the quantities $D^c_{\mu\nu}(k)$, $G^\gamma_{\mu\nu}(k)$ and $\Pi_{\mu\nu}(k)$ are proportional to the tensor $J_{\mu\nu} = \delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}$,

$$G^\gamma_{\mu\nu}(k) = G^\gamma(k)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right),$$

$$D^c_{\mu\nu}(k) = D^c(k)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right),$$

$$\Pi_{\mu\nu}(k) = \Pi(k)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right).$$

Since $J_{\alpha\beta}J_{\beta\gamma} = J_{\alpha\gamma}$ equations (43.36) and (43.36′) can be rewritten in the form of relations between scalar functions

$$G^\gamma(k) = D^c(k)+D^c(k)\,\Pi(k)\,D^c(k), \tag{43.37}$$

or in the coordinate representation

$$G^\gamma(x-y) = D^c(x-y)+\int D^c(x-x')\Pi(x'-y')D^c(y'-y)\,d^4x'd^4y'. \tag{43.37$'$}$$

We now define the quantities $M(p)$ and $P(k)$ related to $\Sigma(p)$ and $\Pi(k)$ in the following manner:

$$1+S^c(p)\Sigma(p) = \left(1+iS^c(p)\,M(p)\right)^{-1},$$

$$1+D^c(k)\Pi(k) = \left(1+iD^c(k)\,P(k)\right)^{-1}$$

or

$$\left.\begin{aligned} \Sigma(p) &= -iM(p)\big(1+iS^c(p)M(p)\big)^{-1}, \\ \Pi(k) &= -iP(k)\big(1+iD^c(k)P(k)\big)^{-1}. \end{aligned}\right\} \tag{43.38}$$

Then relations (43.35) and (43.37) assume the form

$$\begin{aligned} G^e(p) &= S^c(p)-iS^c(p)M(p)G^e(p), \\ G^\gamma(k) &= D^c(k)-iD^c(k)P(k)G^\gamma(k), \end{aligned} \tag{43.39}$$

or

$$\begin{aligned} \frac{1}{G^e(p)} &= \frac{1}{S^c(p)}+iM(p), \\ \frac{1}{G^\gamma(k)} &= \frac{1}{D^c(k)}+iP(k). \end{aligned} \tag{43.40}$$

Equations (43.39) for the Green's functions were obtained by Dyson (53). In the coordinate representation equations (43.39) assume the form of integral equations

$$\begin{aligned} G^e(x-y) &= S^c(x-y)-i\int S^c(x-x')M(x'-y')G^e(y'-y)\,d^4x'\,d^4y', \\ G^\gamma(x-y) &= D^c(x-y)-i\int D^c(x-x')P(x'-y')G^\gamma(y'-y)\,d^4x'\,d^4y'. \end{aligned} \tag{43.41}$$

These integral equations can be rewritten in integro-differential form, which enables us to establish their connection with the Dirac and the d'Alembert equation. In order to do this we multiply the first of equations (43.39) by $-1/S^c(p) = \hat{p}-im$. We then obtain

$$\big(\hat{p}-im-iM(p)\big)G^e(p) = -1 \tag{43.42'}$$

In order to obtain the corresponding equation in the coordinate representation we apply to the first of equations (43.41) the differential operator $\hat{p}-im = -i\gamma_\mu\partial/\partial x_\mu - im$. Since (cf. § 18) $(\hat{p}-im)S^c(x-y) = -\delta(x-y)$, then

$$(\hat{\mathrm{p}}-im)G^e(x-y)-i\int M(x-y')G^e(y'-y)\,d^1y' = -\delta(x-y). \tag{43.42''}$$

If we denote by M the integral operator whose kernel is the function $M(x-y)$, i. e., $\mathrm{M}G^e(x-y) = \int M(x-y')G^e(y'-y)\;d^4y'$ then equation (43.42'') can be rewritten in the following form

$$(\hat{\mathrm{p}}-im-i\mathrm{M})G^e(x-y) = -\delta(x-y). \tag{43.42}$$

Equation (43.42) shows that the electron propagator is the Green's function for an equation similar to the Dirac equation $(\hat{p}-im-i\mathrm{M})\psi=0$ but differing from the latter by the addition to the mass $m$ of the operator M. For this reason the operator M is called the *mass operator.*

We transform the equation for the photon propagator in an analogous manner. On multiplying the second of equations (43.39) by $1/D^c(k)=ik^2$ we obtain

$$i(k^2+P(k))G^\gamma(k)=1. \tag{43.43'}$$

On applying to the second of equations (43.41) the operator $i\,\Box = i\partial^2/\partial x_\mu^2$ and on utilizing the equation (cf. § 17) $-i\,\Box\, D^c(x-y)=\delta(x-y)$ we obtain

$$i(-\Box+\mathrm{P})G^\gamma(x-y)=\delta(x-y), \tag{43.43}$$

where P is an integral operator whose kernel is the function $P(x-y)$, i.e.,

$$\mathrm{P}G^\gamma(x-y)=\int P(x-y')G^\gamma(y'-y)d^4y'.$$

We see that $iG^\gamma(x-y)$ is the Green's function for the equation $(-\Box+\mathrm{P})A=0$ which in macroscopic electrodynamics describes the propagation of electromagnetic waves in a medium whose polarization properties are described by the operator P. Therefore the operator P is called the *vacuum polarization operator* or the *polarization operator.*

We can say that the mass operator M describes the interaction of the electron with its own electromagnetic field. This interaction consists of emission and absorption of virtual photons. In a similar way we can say that the polarization operator P describes the interaction of a photon with the electron-positron field. This interaction consists of creation and annihilation of virtual pairs.

In conclusion, let us also derive the equation satisfied by the matrix elements of the Heisenberg field operators corresponding to a transition from vacuum to a single electron or a single photon state. We begin with matrix element $(0|\boldsymbol{\psi}(x)|r)=(0|T(\psi(x)S)|r)$. On substituting for S the expansion (43.31), and on carrying out the same operations which lead to the equation (43.34) for the function $G^e$ we obtain

$$(0|\boldsymbol{\psi}(x)|r)=(0|\psi(x)|r)+\int S^c(x-x')\Sigma(x'-y')(0|\psi(y')|r)\,d^4x'\,d^4y'.$$

Going over to the momentum representation and utilizing the notations

$$(0|\boldsymbol{\psi}(x)|r)=u(p)e^{ipx},\qquad (0|\psi(x)|r)=u(p)e^{ipx},$$

we obtain

$$\boldsymbol{u}(p) = u(p) + S^c(p)\Sigma(p)u(p). \tag{43.44}$$

We obtain an analogous relation also for the matrix elements

$$(r|\bar{\psi}(x)|0) = \bar{\boldsymbol{u}}e^{-ipx} \quad \text{and} \quad (0|A(x)|r) = \boldsymbol{a}(k)e^{ikx}$$

$$\begin{aligned} \bar{\boldsymbol{u}}(p) &= \bar{u}(p) + \bar{\boldsymbol{u}}(p)\Sigma(p)S^c(p), \\ \boldsymbol{a}(k) &= a(k) + D^c(k)\Pi(k)\boldsymbol{a}(k). \end{aligned} \tag{43.44'}$$

On introducing the mass and the polarization operators M and P we obviously get the equations for $\boldsymbol{u}(p)$ and $\boldsymbol{a}(k)$

$$\begin{aligned} \boldsymbol{u}(p) &= u(p) - iS^c(p)M(p)\boldsymbol{u}(p), \\ \boldsymbol{a}(k) &= a(k) - iD^c(k)P(k)\boldsymbol{a}(k). \end{aligned} \tag{43.45'}$$

In the coordinate representation these equations are integral equations. Since $(i\hat{p}+m)\psi = 0$ and $\Box\mathrm{A} = 0$ they are equivalent to the integro-differential equations

$$\begin{aligned} (i\hat{\mathrm{p}} + m + \mathrm{M})(0|\psi(x)|r) &= 0, \\ (-\Box + \mathrm{P})(0|\mathrm{A}(x)|r) &= 0. \end{aligned} \tag{43.45}$$

We see that these equations differ from the equations for the corresponding propagators by the fact that they are homogeneous, while the equations for the propagators contain inhomogeneities in the form of $\delta$-functions. In other words, the propagators are the Green's functions for the equations for the matrix elements.

### 43.4. *Electromagnetic Mass of the Electron*

We have previously (cf. subsection 43.1) defined the electron mass $m_c$ as the energy of the lowest state of the system of interacting fields having a unit charge and zero momentum. Obviously, the quantity $m_c$ need not coincide with the "mechanical" mass $m$ of an electron which is not interacting with the electromagnetic field. Indeed, the mechanical electron mass $m$ plays the role only of a constant which appears in the equations of quantum electrodynamics and which does not directly determine the energy levels of the system of interacting fields. The difference between $m_c$ and $m$ may be called the *electromagnetic electron mass.*

We now establish the relation between $m_c$ and $m$. In order to do this we make use of the spectral representation of the electron propagator (43.20). For values of $p^2$ close to $-m_c^2$ we can retain in (43.20) only the first term, i.e.,

$$-\frac{1}{G^e(p)} = Z_1^{-1}(\hat{p}-im_c), \quad p^2+m_c^2 \to 0. \tag{43.46}$$

We compare this expression with (43.40)

$$-\frac{1}{G^e(p)} = \hat{p}-im-iM(p). \tag{43.47}$$

The mass operator in the momentum representation $M(p)$ is a matrix which depends on the four-vector $p$. Such a matrix can be written as $M(p) = c_1+c_2\hat{p}$, where $c_1$ and $c_2$ are scalar functions depending only on $p^2$.

The preceding expression can be written in the convenient form

$$-iM(p) = a(p^2)(\hat{p}-im_c)-ib(p^2). \tag{43.48}$$

When $p^2+m_c^2 \to 0$ we can replace the functions $a(p^2)$ and $b(p^2)$ by their values for $p^2 = -m_c^2$. On introducing the notation $a(-m_c^2) = a_0$; $b(-m_c^2) = \delta m$, we obtain from (43.46), (43.48)

$$Z_1^{-1}(\hat{p}-im_c) = \hat{p}-im-i\,\delta m+a_0(\hat{p}-im_c)$$

or

$$\begin{aligned} m_c &= m+\delta m, \\ Z_1^{-1} &= a_0+1. \end{aligned} \tag{43.49}$$

These formulas express the electromagnetic electron mass $m_c-m$, and also the constant $Z_1$ in terms of the mass operator $M(p)$.

We now investigate the expansion of the photon propagator in the neighborhood of the point $k^2 = 0$. On retaining in (43.15′) only the first term we obtain

$$\frac{1}{G^\gamma(k)} = iZ^{-1}k^2. \tag{43.50}$$

On the other hand, it follows from (43.40) that

$$\frac{1}{G^\gamma(k)} = i\big(k^2+P(k)\big). \tag{43.51}$$

In the neighborhood of the point $k^2 = 0$ the polarization operator can be written in the following form: $P(k) = P(0) + Bk^2$. On substituting this expansion into (43.51) and on comparing with (43.50) we obtain

$$Z^{-1} = 1+B = 1+\left(\frac{\partial P}{\partial k^2}\right)_{k^2=0}, \qquad (43.52)$$
$$P(0) = 0.$$

If the second of relations (43.52) were not satisfied, then the expansion (43.50) would also not hold, and the photon Green's functions would have a pole not at $k^2 = 0$, but at $k^2 + P(0) = 0$, i.e., the photon would have a rest mass. The equation $P(0) = 0$ is associated with the gauge invariance of the theory, whose significance lies in the fact that all physically observable quantities can depend only on the electromagnetic field tensor $F_{\mu\nu}(k) = k_\mu A_\nu(k) - k_\nu A_\mu(k)$, and not on the potentials $A_\mu(k)$ themselves. Therefore, physical quantities must vanish when $k_\mu = 0$. Since $P(k)$ depends only on $k^2$, we have $P(0) = 0$.

We can also associate this property of $P(k)$ directly with the transversality of the self-energy function $\Pi_{\mu\nu}(k)$.

On assuming that the transverse tensor $P_{\mu\nu} = P(k)\left(\delta_{\mu\nu} - (k_\mu k_\nu / k^2)\right)$ has no singularities at $k = 0$, we can write it in the form

$$P_{\mu\nu}(k) = P_1(k^2)\,(k^2\,\delta_{\mu\nu} - k_\mu k_\nu),$$

where the function $P_1(k^2)$ is finite at $k^2 = 0$. From this it follows that

$$P(k^2) = k^2 P_1(k^2) \to 0 \qquad \text{for} \qquad k^2 = 0.$$

In conclusion we return to the investigation of the matrix elements $(0|\psi(x)|r)$ and $(0|\mathrm{A}(x)|r)$. We have seen that they satisfy equations (43.45). On utilizing the properties of operators M and P for $k^2 = 0$ and $p^2 + m_c^2 = 0$, we obtain

$$(i\hat{\mathrm{p}} + m_c)(0|\psi(x)|r) = 0,$$
$$\Box(0|\mathbf{A}(x)|r) = 0,$$

i.e., the equations for noninteracting fields. Since these equations are homogeneous we cannot, of course, obtain from them relations between the amplitudes $\boldsymbol{u}$, $\bar{\boldsymbol{u}}$, $\boldsymbol{a}$ and the corresponding amplitudes for free fields $u$, $\bar{u}$, $a$. However, we have already obtained such relations earlier (cf. (43.10)). These relations contain the universal constants $Z_1$ and $Z$ whose meaning and significance will be elucidated later.

## § 44. Structure of the Scattering Matrix

### 44.1. *Self-Energy Parts of Diagrams*

The higher order diagrams for the scattering matrix have a complex structure. Therefore, it is very useful to break up these diagrams into parts, and to investigate successively those more or less complex parts which may appear in diagrams for different processes. A study of the properties of such parts and the evaluation of quantities corresponding to them enable us to obtain solutions for a large number of electrodynamic problems in higher order approximations.

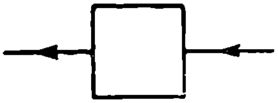

Fig. 55.

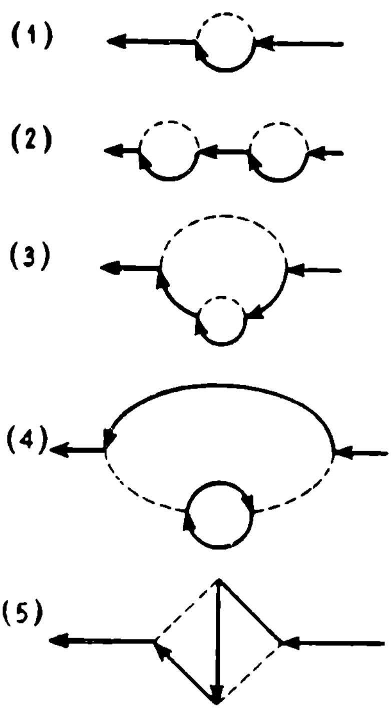

Fig. 56.

In doing this we must keep in mind that in higher order diagrams we may completely leave out of consideration those parts which contain no external lines, and which are not connected by electron or photon lines with the principal part of the diagram. Indeed the totality of all such parts, which are called *vacuum* parts, will contribute to the matrix element the factor $(0|S|0) \equiv (\Phi_0^0, S\Phi_0^0)$, of absolute value equal to unity by virtue of (43.26).

Of the greatest importance are those parts of the diagram which are called the electron self-energy, the photon self-energy, and the vertex parts. We now give their definitions.

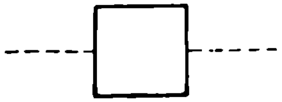

Fig. 57.

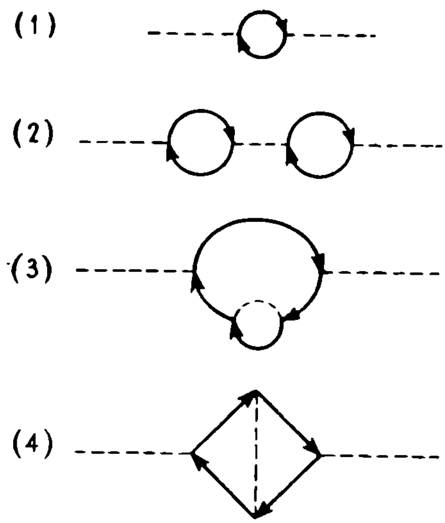

Fig.58.

We call a part of the diagram which is connected to other parts by two electron lines only *an electron self-energy part*.

An electron self-energy part is shown schematically in the form of a square in Fig. 55. Fig. 56 shows electron self-energy parts of the second and fourth orders.

We call a part of the diagram which is connected to other parts by two photon lines only *a photon self-energy part*.

A photon self-energy part is schematically shown in the form of a square in Fig. 57. Fig. 58 shows photon self-energy parts of the second and fourth orders.

We denote by $\Sigma_W$ the quantity in the matrix element which corresponds to the electron self-energy part $W$. This quantity is a second rank spinor (a four-rowed matrix in the spin variables) which depends on the four-momentum $p$ of the incoming or the outgoing electron line (the momenta corresponding to the incoming and the outgoing lines are equal by virtue of the law of conservation of momentum) $\Sigma_W \equiv \Sigma_W(p)$.

It can be easily shown that the totality of quantites corresponding to all the self-energy parts $W$ coincides with the self-energy function which we have defined in the preceding section $\Sigma(p) = \sum_W \Sigma_W(p)$. Indeed, the term containing $\Sigma$ in the expansion of the scattering matrix (43.31) corresponds exactly to the totality of all diagrams terminated by two electron lines.

We consider a diagram containing an internal electron line joining two points $A$ and $B$. In higher order approximations diagrams will appear in which there are inserted between $A$ and $B$ various self-energy parts $W$. We can describe the whole set of such parts joining $A$ and $B$ by means of a single effective electron line. We shall represent it by a thick solid line. By definition such an effective line is determined by the graphical equation shown in Fig. 59 where the square represents the self-energy function $\Sigma$. From this equation we see that the effective electron line represents the electron propagator $G^e$. Indeed, Fig. 59 implies the relation $G^e(p) = S^c(p) + S^c(p)\Sigma(p)S^c(p)$, which coincides with (43.35). The same result can also be obtained with the aid of the definition of the propagator $G^e(x-y) = \left(0\middle|T\left(\psi(x)\bar{\psi}(y)S\right)\middle|0\right)$, since if we expand the scattering matrix S into a perturbation theory series we will obtain from this definition a set of expressions corresponding to all possible diagrams which begin with an electron line at the point $A$ and terminate with an electron line at the point $B$.

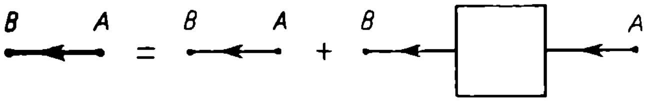

Fig. 59.

At this point, let us consider the external electron line of a diagram which joins the rest of the diagram at the point $A$ (Fig. 60). In higher order approximations diagrams will appear in which between the point $A$ and the external line there will be inserted various self-energy parts. The set of all such diagram parts can be described by an effective external electron line defined by the graphical equation shown in Fig. 61. It can easily be shown that the external electron line represents

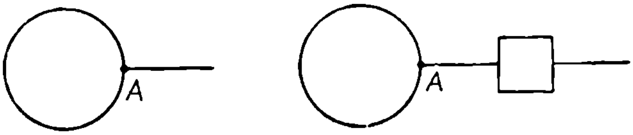

Fig. 60.

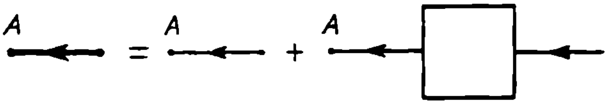

Fig. 61.

the matrix element of the Heisenberg operator $\psi$ or $\bar{\psi}$. Indeed, Fig. 61 implies the relation $\boldsymbol{u}(p) = u(p)+S_0(p)\Sigma(p)u(p)$, which coincides with (43.44). We can also obtain the same result by considering the graphical representation of the matrix element $(0|\psi(x)|r) = (0|T\psi S|r)$.

This result is very important since it eliminates the necessity of considering those diagram parts which can be replaced by effective external electron lines. The matrix elements of the operators $\psi$ and $\bar{\psi}$ which correspond to effective external lines are defined by formulas (43.10).

If we cannot break up an electron self-energy part into smaller parts interconnected by only a single electron line we call it a *compact* electron self-energy part. Examples of such parts are shown in Fig. 56 (1, 3, 4, 5). The electron self-energy part shown in Fig. 56 (2) is not compact, since it can be broken up into two compact parts.

We denote by $\Sigma^*_W$ the quantity which corresponds to a compact electron self-energy part $W$. We also define the entire compact self-energy part as the sum of all the compact parts $\Sigma^* = \sum\limits_W \Sigma^*_W$.

The function $\Sigma^*(p)$ is simply related to the mass operator $M(p)$ which we have defined earlier. In order to establish this relation we construct the self-energy function $\Sigma$ from the functions $\Sigma^*$ correspond ing to the compact parts. This can be done in accordance with the schematic diagram of Fig. 62, where the square corresponds to $\Sigma$ and a rectangle to $\Sigma^*$. This schematic diagram corresponds to the infinite series $\Sigma = \Sigma^* + \Sigma^* S^c \Sigma^* + \Sigma^* S^c \Sigma^* S^c \Sigma^* + \ldots$, on summing which we obtain $\Sigma = \Sigma^*(1 - S^c \Sigma^*)^{-1}$. On comparing this last equation with (43.38) we see that $\Sigma^*(p) = -iM(p)$.

The definitions given above and the resultant relations may be transferred completely to the case of the photon self-energy parts and photon lines.

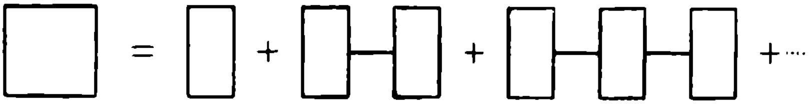

Fig. 62.

We denote by $\Pi_W$ the quantity which corresponds to a definite diagram of a photon self-energy part $W$. The quantity $\Pi_W$ is a four-dimensional second-rank tensor, and is a function of the momentum of the incoming and outgoing photon lines. We call a photon self-energy part *compact* if it cannot be broken up into parts joined by only a single photon line. Figure 58 shows photon self-energy parts of the second and fourth orders. The self-energy parts of Fig. 58 (1, 3, 4) are compact, while the self-energy part of Fig. 58 (2) consists of two compact parts. We denote by $\Pi_W^*$ the quantity which corresponds to a compact photon self-energy part $W$.

Just as in the case of electron self-energy parts, the totality of all photon self-energy parts corresponds to the function $\Pi(k) = \sum_W \Pi_W(k)$, which coincides with the photon self-energy function defined in the preceding section.

The totality of all the compact photon self-energy parts corresponds to the function $\Pi^*(k) = \sum_W \Pi_W^*(k)$, which is related to $\Pi(k)$ by $\Pi = \Pi^* + \Pi^* D^c \Pi^* + \ldots = \Pi^*(1 - D^c \Pi^*)^{-1}$, i.e., on the basis of (43.38) we have $\Pi^*(k) = -iP(k)$, where $P(k)$ is the polarization operator.

We also define an effective photon line joining points $A$ and $B$ of a diagram as the totality of all the photon self-energy parts together with the two photon lines joining them to the points $A$ and $B$. We represent an effective photon line in our diagrams by means of a thick broken line.

The quantity in the matrix element which corresponds to an effective internal photon line is the photon Green's function $G^{\gamma}(k)$. Fig. 63 shows the graphical relations between $G^{\gamma}$, $\Pi$ and $\Pi^*$, corresponding to equations (43.37) and (43.39).

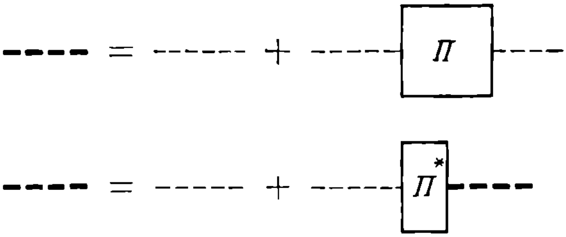

Fig. 63.

The external effective photon line is related to the self-energy parts by the same relations which are shown graphically in Fig. 63 and which are equivalent to equation (43.44′). Because of this we can leave out of consideration those parts of the diagram which are equivalent to external photon lines, since the latter correspond to the matrix elements of the operator **A** which are determined by formula (43.10).

### 44.2. *Vertex Parts of Diagrams*

We shall call a part of a diagram which is connected to the other parts by two electron lines and one photon line *a vertex part*. In Fig. 64 the vertex part is shown schematically in the form of a triangle.

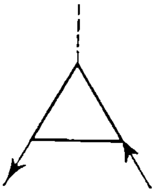

Fig. 64.

Vertex parts can also be divided into *compact and noncompact* parts. The former are those diagrams which cannot be divided into parts interconnected by only one electron or photon line. Noncompact vertex parts can be regarded as combinations of a compact vertex part with effective electron or photon lines. Therefore, in future we shall use the term vertex part to refer only to compact vertex parts. Figure 65 shows vertex parts of the third and fifth orders.

We denote by $\Lambda_Y$ the quantity which corresponds in the matrix element to the vertex part $Y$, and we denote the sum of all the compact vertex parts by $\Lambda = \sum_Y \Lambda_Y$. We call the quantity corresponding to the totality of vertex parts and to an ordinary vertex diagram (which corresponds to the factor $\gamma$) *the vertex function*, and we denote it by $\Gamma$, where $\Gamma = \gamma + \Lambda$.

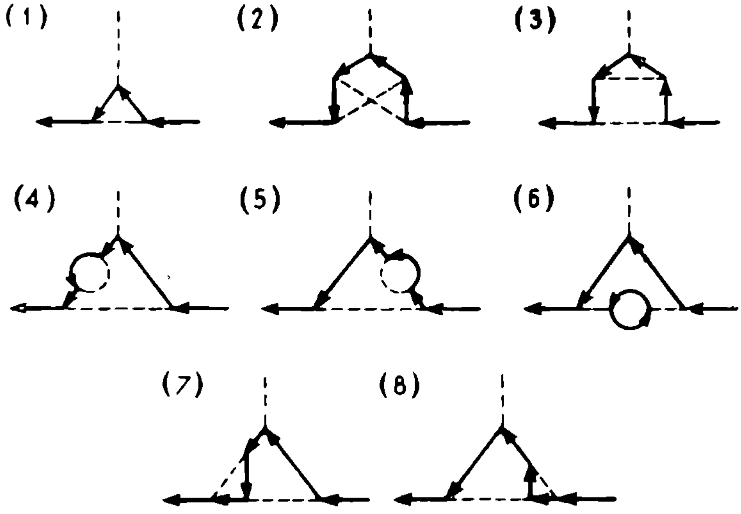

Fig. 65.

The matrix $\Lambda$, like the matrix $\gamma$ corresponding to a simple vertex of the diagram, has one vector index ($\mu$) and two spinor indices ($\alpha, \beta$): $\Lambda \equiv \Lambda_\mu \equiv \Lambda_{\mu,\alpha\beta}$ and is a function of two electron momenta ($p_1$, $p_2$) and one photon momentum ($k$) corresponding either to external or to internal lines, by means of which the vertex part is joined to the rest of the diagram: $\Lambda = \Lambda(p_1, p_2\,; k)$. In virtue of the law of conservation of momentum these quantities are related by $p_1 - p_2 = k$.

In the coordinate representation $\Lambda$ is a function of two differences of coordinates of the three points defining the vertex part of the diagram.

It is useful to divide the self-energy and the vertex parts defined earlier into two further classes: reducible and irreducible. We call a part which does not contain within itself any self-energy or vertex parts *irreducible*; in the opposite case we call it *reducible*.

There exists only one irreducible electron self-energy part, viz., the second order self-energy part shown in Fig. 56 (*1*). It can be easily shown, for example, that the remaining diagrams of Fig. 56 are reducible.

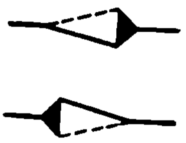

Fig. 66.

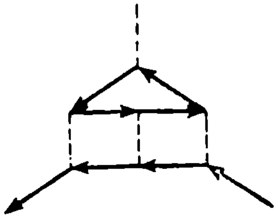

Fig. 67.

Thus, diagram *2* is noncompact, diagrams *3*, *4* contain self-energy parts, while diagram *5* contains a vertex part. Diagram *5* can be represented in one of the forms shown in Fig. 66 where the triangles denote vertex parts.

Similarly there exists only one irreducible photon self-energy part, viz., the second order self-energy part shown in Fig. 58 (*1*).

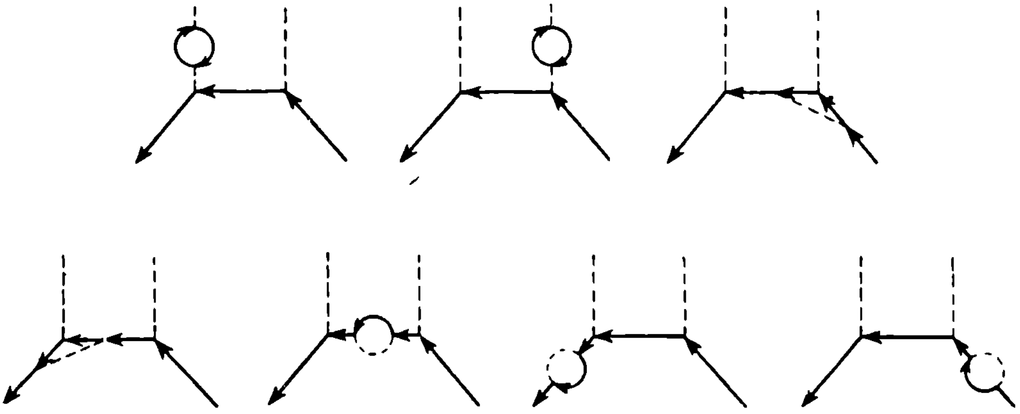

Fig. 68.

There exists an infinite number of irreducible vertex parts. Of the diagrams shown in Fig. 65 diagrams (*1*) and (*2*) are irreducible. An example of a more complicated irreducible vertex part is shown in Fig. 67.

By utilizing the concepts of irreducible parts and effective external lines we can replace complex diagrams which we encounter in the investigation of higher order approximations by skeleton diagrams in which effective lines vertex and self-energy parts have been segregated. Each one of these parts represents a set of diagrams of different orders.

As an example we consider the fourth order Compton effect diagrams shown in Fig. 68. The totality of these diagrams and of the second order diagram shown in Fig. 21 reduces to the single effective skeleton diagram shown in Fig. 69. This diagram corresponds to the matrix element

$$M = Z_1 Z e^2 a_{2_\mu} a_{1_\nu} \bar{u}_2 \Gamma_\mu(p_2+k_2, p_2; k_2) G^e(p_2+k_2) \Gamma_\nu(p_1, p_1+k_1; k_1) u_1. \tag{44.1}$$

If we have second order approximation expressions for $\Gamma$ and $G^e$ it is sufficient to substitute these expressions into $M$ and to neglect higher order terms in the product in order to obtain the sum of matrix elements corresponding to the diagrams of Figs. 21 and 68.

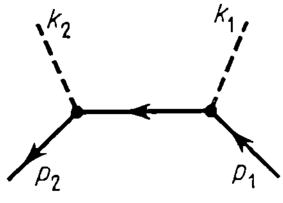

Fig. 69.

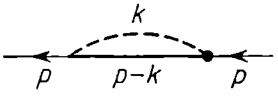

Fig. 70.

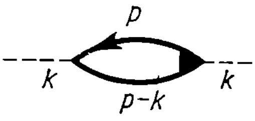

Fig. 71.

With the aid of the vertex function we can construct a general expression for the mass and the polarization operators. In order to do this we investigate the internal structure of the exact compact self-energy part. It can be represented by means of the skeleton diagram shown in Fig. 71. On the basis of this diagram we can write $\Sigma^*(p)$ in the form

$$\Sigma^*(p) = \frac{e^2}{(2\pi)^4} \int \gamma_\mu G^e(p-k) \Gamma_\nu(p, p-k; k) G^\gamma_{\mu\nu}(k)\, d^4k. \tag{44.2}$$

Similarly the compact photon self-energy part can be represented by means of the skeleton diagram shown in Fig. 71. On the basis of this diagram we can write the expression for $\Pi^*(k)$:

$$\Pi^*(k) = \frac{e^2}{(2\pi)^4} \frac{1}{3} \operatorname{Sp} \int \gamma_\mu G^e(p) \Gamma_\mu(p, p-k; k) G^e(p-k)\, d^4p. \tag{44.3}$$

On substituting (44.2) and (44.3) into (43.39) we obtain two integral equations relating the three quantites $G^e$, $G^\gamma$ and $\Gamma$

$$\begin{aligned} G^e(p) &= S^c(p) + \frac{e^2}{(2\pi)^4} S^c(p) \int \gamma_\mu G^e(p-k) \Gamma_\nu(p, p-k; k) G^e(p) \\ &\qquad \times G^\gamma_{\mu\nu}(k)\, d^4k, \\ G^\gamma(k) &= D^c(k) + \frac{e^2}{3(2\pi)^4} D^c(k) \operatorname{Sp} \gamma_\mu \int G^e(p) \\ &\qquad \times \Gamma_\mu(p, p-k; k) G^e(p-k)\, d^4p\, G^\gamma(k). \end{aligned} \tag{44.4}$$

If there existed a third equation defining the vertex part we would have a closed system of three exact equations of quantum electrodynamics. However, there is no such exact equation, since the exact equations of quantum electrodynamics are, in fact, not integral equations but functional equations (cf. § 49).

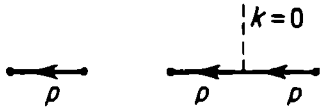

Fig. 72.

However, we note that for $k = 0$ there exists a simple relation between the vertex part and the electron self-energy part known as Ward's identity (202)

$$\Lambda_\mu(p, p; 0) = \frac{\partial \Sigma^*(p)}{\partial p_\mu}, \tag{44.5}$$

from which on the basis of (43.40) it follows that

$$\Gamma_\mu(p, p; 0) = -\frac{\partial}{\partial p_\mu}\left(\frac{1}{G^e(p)}\right). \tag{44.6}$$

In order to prove equation (44.5) we use the identity

$$\frac{\partial S^c(p)}{\partial p_\mu} = -S^c(p) \frac{\partial}{\partial p_\mu}\left(\frac{1}{S^c(p)}\right) S^c(p) = S^c(p) \gamma_\mu S^c(p). \tag{44.7}$$

The function $S^c(p)$ is represented graphically by an electron line. The right hand side of this identity corresponds to a diagram consisting of two lines with identical momenta and of one vertex (Fig. 72). This establishes the relation between the diagrams corresponding to the function $S^c(p)$ and its derivative.

We now consider a compact self-energy part $W$, for example the one shown in Fig. 73. We choose the momenta corresponding to the variables of integration in such a way that the momentum $p$ should appear in each segment of the solid line joining the initial and the final points of the diagram (Fig. 73). Differentiation of the factors corresponding to different segments of this line will give us the complete set of vertex parts which can be constructed by adding photon lines to the different electron lines forming the diagram $W$ (Fig. 74). The electron lines forming closed loops will not take part in the differentiation. But such loops cannot lead to vertex parts. Indeed, on the basis of Furry's theorem (cf. subsection 25.4) we need take into account only closed loops with an even number of vertices. But differentiation adds an extra vertex, and, therefore, all such diagrams may be neglected on the basis of Furry's theorem.

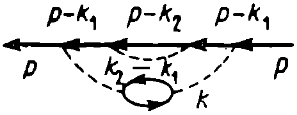

Fig. 73.

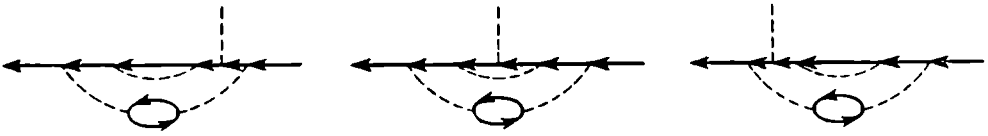

Fig. 74.

We have thus shown that the set of all the compact self-energy parts gives on differentiation the set of all the nonvanishing vertex parts, i.e., relation (44.5).

Relation (44.6) also enables us to obtain an expression for $\Gamma_\mu(p, p; 0)$ in the case $p^2+m_c^2 = 0$, i.e., in the case when the vertex part terminates by an external electron line. In this case $G^e(p)$ is equal to expression (43.46), on differentiating which we obtain

$$\Gamma_\mu(p, p; 0) = Z_1^{-1}\gamma_\mu, \quad p^2+m_c^2 = 0. \tag{44.8}$$

### 44.3. *Renormalization of Electron Mass*

The difference between the energy spectra of the free electron-positron field and of the electron-positron field interacting with the electromagnetic field, which manifests itself in the difference between the masses $m$ and $m_c$, introduces difficulties into the solution of problems of quantum electrodynamics, since the usual scattering theory methods assume that the energy spectrum of the system is independent of the perturbation.

However, in spite of the difference between the masses $m$ and $m_c$ it is possible to redefine the unperturbed system in such a way that the noninteracting electron will also have the mass $m_c$. This is carried out by the following simple method, which is called *mass renormalization.*

We write the Lagrangian density for the system of interacting fields, as before, in the form $L = L_e + L_\gamma + L_i$, but introducing the following change of notation. We interpret $L_e$ as the Lagrangian density for a free electron of mass $m_c$

$$L_e = \overline{\psi}(i\hat{\mathrm{p}}+m_c)\psi \equiv \overline{\psi}(i\hat{\mathrm{p}}+m)\psi+\delta m\overline{\psi}\psi. \tag{44.9}$$

We subtract from $L_i$ the term $\delta m\overline{\psi}\psi$ added to $L_e$

$$L_i = ie\overline{\psi}\hat{\mathrm{A}}\psi-\delta m\overline{\psi}\psi. \tag{44.10}$$

Then the scattering matrix assumes the form

$$\mathrm{S} = T(e^{i\int L_i d^4x}) = T(e^{-e\int\overline{\psi}\hat{\mathrm{A}}\psi d^4x-i\delta m\int\overline{\psi}\psi d^4x}). \tag{44.11}$$

The matrix elements of S will contain in addition to the ordinary three line vertices corresponding to the first term of (44.10), in which two electron lines and one photon line meet, also two line vertices (in Fig. 75 such a vertex is shown by a cross). In such vertices corresponding to the second term of (44.10) only two electron lines meet. The additional term in $L_i$ obviously leads to only one additional irreducible diagram representing an electron self-energy part. Every diagram containing an irreducible electron self-energy part now has a similar diagram added to it in which the irreducible self-energy part is replaced

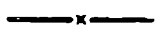

Fig. 75.

by the self-energy part shown in Fig. 75. Therefore, we can introduce an irreducible self-energy part renormalized with respect to its mass which consists of a sum of self-energy parts corresponding to Fig. 56 (*1*) and Fig. 75. It corresponds to the function $\Sigma^{(2)\prime}(p) = \Sigma^{(2)}(p) - i\delta m$.

We now consider an arbitrary reducible compact diagram $W$ for an electron self-energy part. We can evaluate $\Sigma_w$ according to the former by identifying $m$ with $m_c$, and by making each irreducible compact self-energy part correspond to the function $\Sigma^{(2)\prime}$, and not to the function $\Sigma^{(2)}$. After this has been done, the self-energy part of Fig. 75 must be added to the diagram as a whole.

The renormalization of the mass leads to the replacement of the mass operator $M(p)$ by $M_1(p) = M(p) - \delta m$. For $p = p_0$, where $p_0^2 = -m_c^2$ we have in accordance with (43.48) $M_1(p_0) = a_0(i\hat{p}_0 + m_c)$, where $a_0 = Z_1^{-1} - 1$.

The quantity $M_1(p)$ can be expressed in terms of the vertex part $\Lambda(p, p; 0) \equiv \Lambda(p)$. On utilizing expressions (44.5), we obtain

$$M_1(p) = i \int_{p_0}^{p} \Lambda_\mu(q)\, dq_\mu + a_0(i\hat{p}_0 + m_c), \tag{44.12}$$

where the integration is taken along an arbitrary curve joining the points $p_0$ and $p$.

## § 45. Renormalization of Electron Charge

### 45.1. *Physical Charge of the Electron*

We have already seen that because of the interaction between the electron and the electromagnetic field the electron mass $m_c$ does not coincide with the constant $m$ appearing in the equations of quantum electrodynamics. We shall now show that neither is the constant $e$ which appears in the equations of quantum electrodynamics and in the scattering matrix the actual electron charge, and we shall obtain the relation between the constant $e$ and the charge of the electron.

The physical charge of the electron can be determined by investigating the scattering of electromagnetic waves of low frequency $k$ by an electron at rest. This process is a purely classical one and no quantum corrections should be necessary in the limiting case $k \to 0$. It is, therefore, natural to define the physical charge of the electron $e_c$ as the coefficient appearing in the Thomson formula for the effective scattering cross section for a photon of $k = 0$

$$\sigma = \frac{8\pi}{3}\left(\frac{e_c^2}{4\pi mc^2}\right)^2.$$

We shall show that the exact solution of this problem in quantum electrodynamics leads to the Thomson formula with the following value of the charge $e_c$ expressed in terms of the constant $e$:

$$e_c = Z^{1/2}e. \tag{45.1}$$

In order to prove (195) this fundamental relation we consider all possible diagrams describing the scattering of a photon by an electron. We divide such diagrams into three classes (Fig. 76): 1) diagrams which can be reduced to the skeleton diagram of Fig. 69. 2) diagrams which represent compact electron self-energy parts, to the electron lines of which two external photon lines are connected, 3) diagrams in which the external photon lines are connected to closed electron loops.

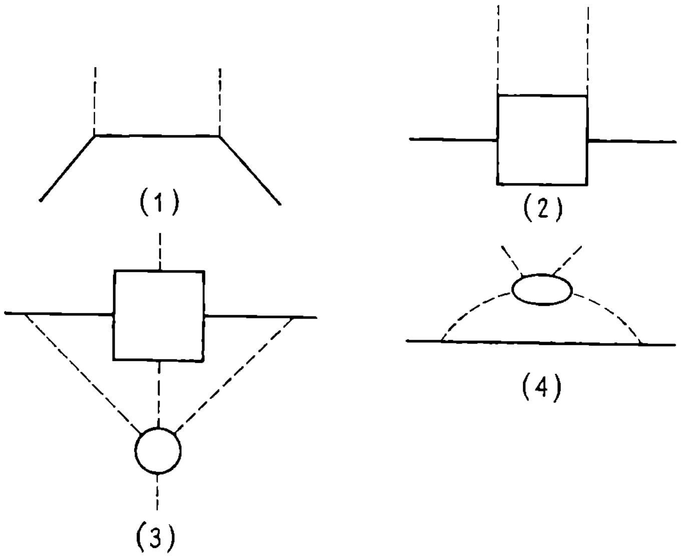

Fig. 76.

Diagrams of the latter type give no contribution to the matrix element. Indeed, diagrams of this type containing loops with an odd number of vertices may be left out of consideration in virtue of Furry's theorem. Other diagrams of this type containing an even number of vertices (cf., for example, Fig. 76 (3, 4)) are related for $k = 0$ by expression (44.5) to diagrams containing loops with an odd number of vertices, and, consequently, also give no contribution to the matrix element.

We now consider matrix elements corresponding to diagrams of the first and second type. The matrix element corresponding to a diagram of the first type can obviously be written in the form

$$M_1 = ZZ_1 e^2 \bar{u}_2 (\Gamma_\mu G^e \Gamma_\nu + \Gamma_\nu G^e \Gamma_\mu) u_1 a_{1\mu} a_{2\nu}.$$

Since we are considering the case $k = 0$, then on the basis of Ward's identity the matrix element corresponding to a diagram of the second kind is of the form

$$M_2 = ZZ_1 e^2 \bar{u}_2 \frac{\partial^2 \Sigma^*(p)}{\partial p_\mu \partial p_\nu} u_1 a_{1\mu} a_{2\nu}.$$

By utilizing expressions (44.5), (44.6) it can be easily shown that

$$\frac{\partial^2 \Sigma^*(p)}{\partial p_\mu \partial p_\nu} = -\frac{\partial^2 G^{e^{-1}}(p)}{\partial p_\mu \partial p_\nu}.$$

Further, on utilizing the expression

$$\frac{\partial G^e(p)}{\partial p_\mu} = G^e(p) \Gamma_\mu (p, p, 0) G^e(p),$$

we obtain

$$\frac{\partial^2 G^e}{\partial p_\mu \partial p_\nu} = G^e \Gamma_\nu G^e \Gamma_\mu G^e + G^e \Gamma_\mu G^e \Gamma_\nu G^e - G^e \frac{\partial^2 G^{e^{-1}}}{\partial p_\mu \partial p_\nu} G^e.$$

From this it follows that

$$\frac{\partial^2 \Sigma^*}{\partial p_\mu \partial p_\nu} + \Gamma_\mu G^e \Gamma_\nu + \Gamma_\nu G^e_\mu \Gamma_\mu = G^{e^{-1}} \frac{\partial^2 G^e}{\partial p_\mu \partial p_\nu} G^{e^{-1}}.$$

Therefore, the sum of the matrix elements $M_1$ and $M_2$ can be written in the form

$$M = M_1 + M_2 = ZZ_1 e^2 \bar{u}_2 G^{e^{-1}} \frac{\partial^2 G^e}{\partial p_\mu \partial p_\nu} G^{e^{-1}} u_1 a_{1\mu} a_{2\nu}.$$

On utilizing the expression

$$G^e(p) = Z_1 S^c(p) [1 + c_1(p^2) + \hat{p} c_2(p^2)], \tag{45.2}$$

which follows from (43.20), and also the fact that

$$c_1(-m^2) = c_2(-m^2) = 0,$$

we obtain for $M$ the expression

$$M = Ze^2 u_2 S^{c^{-1}} \frac{\partial^2 S^c}{\partial p_\mu \partial p_\nu} S^{c^{-1}} u_1 a_{1_\mu} a_{2_\nu}$$

$$+2\left(\frac{\partial c_1}{\partial p^2}\right)_{p^2=-m^2} Ze^2 \bar{u}_2 (\gamma_\mu p_\nu + \gamma_\nu p_\mu) u_1 a_{1_\mu} a_{2_\nu}.$$

If we choose the potentials $a_1$ and $a_2$ in such a way that the following conditions are satisfied

$$a_{1_\mu} p_\mu = 0, \quad a_{2_\nu} p_\nu = 0$$

then the matrix element will assume the form

$$M = Ze^2 \bar{u}_2 S^{c^{-1}}(p) \frac{\partial^2 S^c(p)}{\partial p_\mu \partial p_\nu} S^{c^{-1}}(p) u_1 a_{1_\mu} a_{2_\nu}.$$

On noting that

$$\frac{\partial^2 S^c(p)}{\partial p_\mu \partial p_\nu} = S^c(p)\gamma_\mu S^c(p)\gamma_\nu + S^c(p)\gamma_\nu S^c(p)\gamma_\mu,$$

we finally rewrite $M$ in the form

$$M = Ze^2 \bar{u}_2 [\gamma_\mu S^c(p)\gamma_\nu + \gamma_\nu S^c(p)\gamma_\mu] u_1 a_{1_\mu} a_{2_\nu}.$$

On comparing this expression with expression (28.2), (28.3), we see that for $k = 0$ the exact matrix element differs from its first order approximation only by the factor $Z$. Since with the aid of the first order approximation we have obtained in § 28 the Thomson formula for $k = 0$ in which the quantity $e$ played the role of the charge, we shall now obtain the same formula in which, in accordance with (45.1), the charge will be the quantity $e_c$.

### 45.2. *Renormalization of Propagators and Vertex Parts*

The foregoing formula (45.1) elucidates the meaning of the constant $Z$. Of great importance is the fact that in the expression for the matrix elements of any arbitrary processes the constant $Z$ can appear only in the form of the combination $Ze^2 = e_c^2$. Our immediate problem is to prove this assertion. We shall also prove that the constant $Z_1$

does not appear at all in the expressions for the matrix elements (53), (203). With this aim in view we introduce the functions

$$G_c^e = Z_1^{-1} G^e, \tag{45.3}$$

$$G_c^\gamma = Z^{-1} G^\gamma, \tag{45.4}$$

which we shall call the *renormalized propagators.* In addition, we introduce the renormalized vertex function $\Gamma_c$

$$\Gamma_c = Z_1 \Gamma, \tag{45.5}$$

and also the function $\Lambda_1$

$$\Lambda_1 = Z_1 \Lambda. \tag{45.6}$$

Between the renormalized propagators and vertex parts we can establish a number of relations analogous to the relations which hold between the original (unrenormalized) quantities.

First of all, we prove the relation

$$\Gamma_c(p_1, p_2; k) = \gamma + \Lambda_R(p_1, p_2; k), \tag{45.7}$$

where

$$\Lambda_R(p_1, p_2; k) = \Lambda_1(p_1, p_2; k) - \Lambda_1(p_0, p_0; 0)$$

and $p_0$ is the momentum of the free electron which satisfies the condition $p_0^2 + m_c^2 = 0$. The quantity $\Lambda_R$ is known as the *regularized vertex part.*

For our proof we use the definition (45.6) from which it follows that

$$\Lambda_1(p_0, p_0; 0) = Z_1(\Gamma(p_0, p_0; 0) - \gamma).$$

On substituting into this equation expression (44.8) we obtain $\Lambda_1(p_0, p_0; 0) = (1 - Z_1)\gamma$. Thus formula (45.7) assumes the form

$$\Gamma_c(p_1, p_2; k) = Z_1 \gamma + \Lambda_1(p_1, p_2; k), \tag{45.8}$$

i.e., it coincides with the definitions (45.5), (45.6).

Further, we show that the renormalized function $G_c^e$ satisfies the equation

$$G_c^e(p) = S^c(p) - iS^c(p) M_R(p) G_c^e(p), \tag{45.9}$$

where

$$M_R(p) \equiv i\Sigma_R^* = i \int_{p_0}^{p} \left(\Lambda_{1\mu}(q, q; 0) - \Lambda_{1\mu}(p_0, p_0; 0)\right) dq_\mu. \tag{45.10}$$

The quantity $\Sigma^*_R$ is called the *regularized electron self-energy part*, while $M_R$ is called the *regularized mass operator*.

In order to prove (45.9) we start with equation (43.39)

$$G^e(p) = S^c(p) - iS^c(p)M(p)G^e(p)$$

and assume that the mass has been renormalized, i.e., we interpret $S^c(p)$ as $S^c_{m_e}(p)$ and $M(p)$ as $M_1(p)$.

On substituting for $M$ expression (44.12) we obtain from the preceding expression

$$Z_1 G^e_c(p) = S^c(p) + S^c(p) \int_{p_0}^{p} \Lambda_1(q, q; 0)\, dq\, G^e_c(p) - iS^c(p)\, M(p_0) Z_1 G^e_c(p).$$

In order to show that this equation coincides with (45.9) we subtract it from (45.9). Making use of the definition (45.10) we obtain

$$(1-Z_1) G^e_c(p) = S^c(p) \left\{ -\int_{p_0}^{p} \Lambda_1(p_0, p_0; 0)\, dq + iZ_1 M(p_0) \right\} G^e_c(p).$$

Since $\Lambda_1(p_0, p_0;\ 0) = (1-Z_1)\gamma$ and $M(p_-) = (Z_1^{-1}-1)(i\hat{p}_0 + m_c)$(cf. (44.12)), then it follows that the expression in the braces is equal to $-(1-Z_1)(p-im_c)$. Since $S^c(p)(\hat{p}-im_c) = -1$, we obtain an identity which proves the validity of equation (45.9).

The renormalized function $G^\gamma_c$ satisfies the equation

$$G^\gamma_c(k) = D^c(k) - iD^c(k) P_R(k) G^\gamma_c(k), \tag{45.11}$$

where

$$P_R(k) = -i \int_0^k V_{R_\mu}(q)\, dq_\mu,$$

$$V_{R_\mu}(q) = V_{1_\mu}(q) - \left( \frac{\partial V_{1\mu}}{\partial q_\nu} \right)_{q=0} q_\nu, \tag{45.12}$$

$$V_{1_\mu}(q) = ZV_\mu(q); \qquad V_\mu(q) = i\frac{\partial P(q)}{\partial q_\mu}.$$

The quantity $P_R$ is called the *regularized polarization operator*.

In order to prove the validity of equation (45.11) we substitute into it the definitions (45.12). On noting that on the basis of (43.53) we have

$$\left( \frac{\partial V_{1\mu}}{\partial k_\nu} \right)_{k=0} = iZ \left( \frac{\partial^2 P(k)}{\partial k_\mu\, \partial k_\nu} \right)_{k=0} = 2i(1-Z)\delta_{\mu\nu}, \qquad P(0) = 0,$$

it can be easily shown that equation (45.11) is a consequence of equation (43.39).

### 45.3. *Three-Photon Vertex Parts*

For subsequent development it is important to note that the quantity $V_\mu(k) = i[\partial P(k)/\partial k^2]2k_\mu = -[\partial \Pi^*(k)/\partial k_\mu]$ is analogous to the vertex part $\Lambda_\mu(p) \equiv \Lambda_\mu(p, p; 0) = -i[\partial M(p)/\partial p_\mu]$ and that rules can be formulated for evaluating it in accordance with corresponding diagrams.

We consider a part of a diagram which is joined to the other parts by three photon lines. We call it the three-photon vertex part, and denote the quantity corresponding to it by $V_{\mu\alpha\beta}(k_1, k_2, k_3)$, where $k_1$, $k_2$, $k_3$ are the momenta of the photon lines leaving the diagram, with $k_1+k_2+k_3 = 0$. The simplest example of a third order three-photon vertex part $V^{(3)}(k_1, k_2, k_3)$ is shown in Fig. 77. We recall that such parts have no real significance for finding the elements of the scattering matrix, since together with a diagram of the type of Fig. 77 we must always take into account a similar diagram but with the opposite sense of circulation along the closed electron loop, and in accordance with Furry's theorem the sum of such diagrams gives no contribution to the matrix element. Nevertheless, the quantity $V_{\mu\alpha\beta}(k_1, k_2, k_3)$ defined in accordance with the diagram of Fig. 77 is itself different from zero, and we can consider it formally.

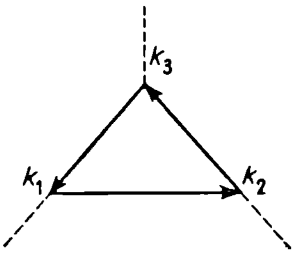

Fig. 77.

Let $k_1 = -k_2 = k$, $k_3 = 0$. We can easily show by utilizing expression analogous to (44.7) that in this case $V_{\mu\alpha\beta}(k, -k, 0) = -\partial \Pi^{(2)}_{\alpha\beta}(k)/\partial k_\mu$ where $\Pi^{(2)}_{\alpha\beta}(k)$ is the second order photon self-energy part shown in Fig. 58 (*1*). Since $\Pi_{\alpha\beta}(k) = \Pi(k)(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)$, the vector $V_\mu(k)$ defined by $V_\mu(k) = \frac{1}{3} V_{\mu\alpha\beta}$ can be expressed in terms of $\Pi^{(2)}(k)$ in the manner

$$V^{(3)}_\mu(k) = -\frac{\partial \Pi^{(2)}(k)}{\partial k_\mu}.$$

In order to generalize this relation to three-photon vertex parts and to compact photon self-energy parts of arbitrary structure, we introduce one more definition, viz., in analogy with the first order vertex part, i.e., with the ordinary vertex diagram which corresponds to the factor $\gamma_\mu = -\partial(S^c(p))^{-1}/\partial p_\mu$, we formally define the first order three-photon vertex part

$$\frac{\partial[D^c(k)]^{-1}}{\partial k_\mu} = 2ik_\mu$$

and represent it by the fictitious diagram of Fig. 78 in which three photon lines emerge from one point.

We now consider an arbitrary compact photon self-energy part. We must distinguish between two types of corresponding diagrams. In diagrams of the first type (for example, in the diagram shown in Fig. 79) both external photon lines emerge from the same electron loop.

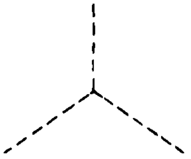

Fig. 78.

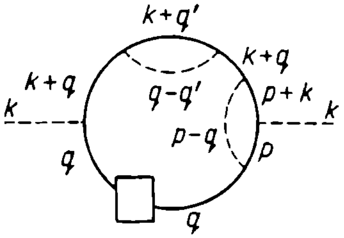

Fig. 79.

Therefore, the external momentum $k$ refers only to the electron lines of one half of the loop (for example, of the upper half-loop of Fig. 79). Differentiation of the self-energy part of Fig. 79 gives the set of three-photon vertices corresponding to the diagrams of Fig. 80.

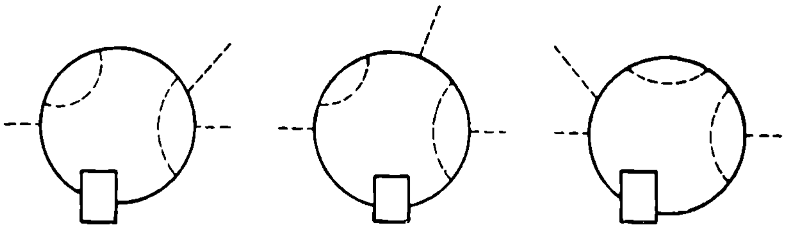

Fig. 80.

In diagrams of the second type (for example, in the diagram shown in Fig. 81) the external photon lines emerge from different electron loops. In this case the external momentum must also refer to the photon lines (for example, in Fig. 81 the upper dotted line joining the two electron loops). In differentiating such self-energy parts terms will appear containing $\partial D^c(k)/\partial k_\mu = D^c(k) 2ik_\mu D^c(k)$. In accordance with the definition of the first order three-photon vertex part introduced earlier we can likewise represent such terms by diagrams (Fig. 82).

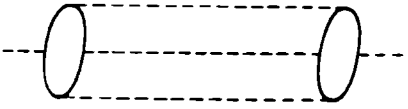

Fig. 81.

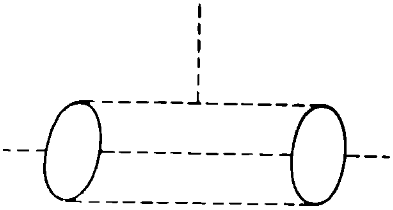

Fig. 82.

Therefore we can make each diagram of a photon self-energy part correspond to a set of diagrams of three-photon vertex parts $V_{Y\mu}$. Further, on introducing $V_\mu = \sum_Y V_{Y\mu}$, we obtain $V_\mu(k) = -[\partial \Pi^*(k)/\partial k_\mu] = i[\partial P(k)/\partial k_\mu]$ in accordance with (45.12).

On defining, by analogy with the quantity $\Gamma_\mu(k,k,0)$, the three-photon vertex function

$$\Delta_\mu(k) = 2ik_\mu + V_\mu(k), \tag{45.13}$$

we obtain a relation analogous to (44.6)

$$\Delta_\mu(k) = \frac{\partial [G^\gamma(k)]^{-1}}{\partial k_\mu}. \tag{45.14}$$

We also introduce the renormalized three-photon vertex function

$$\Delta_{c\mu} = Z\Delta_\mu. \tag{45.15}$$

By utilizing (45.15) and (45.13) we can easily show that

$$\Delta_{c\mu} = 2ik_\mu + V_{R\mu}. \tag{45.16}$$

Formulas (45.15) and (45.16) are analogous to the corresponding formulas (45.5) and (45.7) for the vertex function $\Gamma$.

### 45.4. *Renormalization of Matrix Elements*

To evaluate the matrix element for a process it is sufficient to consider the set of the corresponding irreducible diagrams, i.e., diagrams which within themselves contain no self-energy or vertex parts. All the reducible diagrams will be taken into account if we treat each irreducible diagram as a skeleton diagram in which the internal electron and photon lines and the vertices correspond to the factors $G^e$, $G^\gamma$ and $\Gamma$, while the external lines correspond to the amplitudes $\boldsymbol{u}$, $\boldsymbol{a}$.

Let a given irreducible diagram contain $n$ vertices, $F_e$ internal electron lines, $F_\gamma$ internal photon lines, $N_e$ external electron lines, and $N_\gamma$ external photon lines. Since two electron lines and one photon line meet in each vertex, and each internal line appears in two vertices, while each external line appears in one vertex, we have

$$n = F_e + \frac{N_e}{2} = 2F_\gamma + N_\gamma. \tag{45.17}$$

We can write the expression for the matrix element in the following schematic form:

$$M \sim e^n \int (\Gamma)^n (G^e)^{F_e} (G^\gamma)^{F_\gamma} (u)^{N_e} (a)^{N_\gamma}, \tag{45.18}$$

which merely indicates the number of factors of each type appearing in $M$. On substituting into (45.18) the renormalized quantities

$$\Gamma = Z_1^{-1}\Gamma_c; \qquad G^e = Z_1 G_c^e; \qquad G^\gamma = Z G_c^\gamma;$$

$$u = Z_1^{1/2} u; \qquad a = Z^{1/2} a; \qquad e = Z^{-1/2} e_c,$$

we obtain on taking into account (45.17)

$$M \sim e_c^n \int (\Gamma_c)^n (G_c^e)^{F_e} (G_c^\gamma)^{F_\gamma} (u)^{N_e} (a)^{N_\gamma}. \tag{45.19}$$

We see that the constants $Z$ and $Z_1$ do not appear in (45.19), i.e., the matrix element can be evaluated with the aid of the renormalized quantities $\Gamma_c$, $G_c^e$, $G_c^\gamma$ in accordance with the same rules as in terms of the original quantities $\Gamma$, $G^e$, $G^\gamma$, if we at the same time replace the constant $e$ (which we can call the primary charge) by the true electron charge $e_c$.

In evaluating matrix elements corresponding to various processes it is desirable to utilize the renormalized quantities from the outset. In accordance with the results obtained in subsection 45.3, it is sufficient for this to have rules for the evaluation of $\Lambda_1$ and $V_1$, since the quantities $\Gamma_c$, $G_c^e$ and $G_c^\gamma$ can be expressed in terms of $\Lambda_1$ and $V_1$ by means of formulas (45.7)–(45.16).

In order to establish such rules we consider first of all an irreducible skeleton diagram $\Lambda$. Let this diagram contain $n$ vertices ($n$ is an odd number, for examples cf. Fig. 83). The expression for $\Lambda$ can be schematically written as

$$\Lambda \sim e^{n-1} \int (\Gamma)^n (G)^{n-1} (G^\gamma)^{\frac{n-1}{2}}. \tag{45.20'}$$

On replacing all the quantities by the corresponding renormalized quantities we obtain

$$\Lambda_1 \sim e_c^{n-1} \int (\Gamma_c)^n (G_c^e)^{n-1} (G_c^\gamma)^{\frac{n-1}{2}}, \tag{45.20}$$

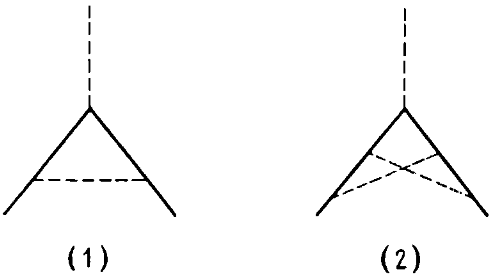

Fig. 83.

i.e., $\Lambda_1$ can be evaluated by means of the same rules as $\Lambda$ merely by replacing the original functions and the quantity $e$ by the renormalized functions and the true electron charge $e_c$.

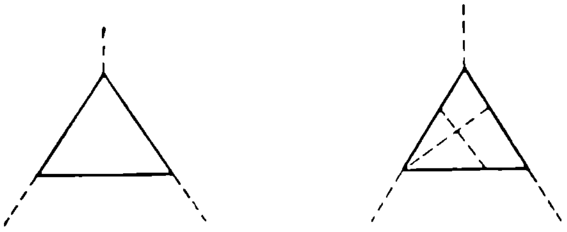

Fig. 84.

In the same way we can easily show by considering the irreducible skeleton diagrams of the three-photon vertex function (for example, cf. Fig. 84) that the renormalized quantity $V_1 = ZV$ is related to the renormalized functions $G_c^e$, $G_c^\gamma$, $\Gamma_c$, $\Delta_c$ by the same formulas which relate $V$ to the functions $G^e$, $G^\gamma$, $\Gamma$ and $\Delta$ with $e$ replaced by $e_c$.

### 45.5. *Formulation of Perturbation Theory as an Expansion in Powers of $e_c$.*

The foregoing analysis enables us to evaluate the elements of the scattering matrix in the form of an expansion in powers of the true electron charge $e_c$.

Each physical process can be made to correspond to a set of irreducible skeleton diagrams $W$ with different numbers of vertices $N$.

A skeleton diagram $W$ consists of effective external lines and of an internal part. We denote the expression corresponding to the internal part of the diagram $W$ by $K(W)$. Then the exact value of the matrix element will be determined by the quantity $K = \sum_W K(W)$.

In accordance with (45.19) $K(W)$ contains the factor $e_c^N$ and can be expressed in terms of the functions $\Gamma_c$, $G^e$ and $G_c^\gamma$, each of which can in turn be represented in the form of a series in powers of $e_c^2$. If we confine ourselves in these expansions to terms proportional to any given power of $e_c$, then we shall finally go over from skeleton diagrams to elementary irreducible diagrams, i.e., to diagrams in which the vertices and the lines correspond to the factors $\gamma_\mu$, $S^c$, $D^c$ which no longer contain $e_c$.

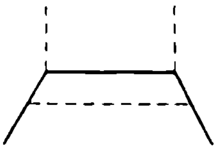

Fig. 85.

As an example we consider the evaluation of the matrix element for the scattering of a photon by an electron. In the first approximation i.e., up to terms in $e_c^2$, obviously only the elementary diagram of Fig. 69 is retained, and we obtain the formulas of § 28.

In the second approximation, i.e., up to terms in $e_c^4$, firstly the elementary diagram of Fig. 85 is added; secondly, the diagram of Fig. 69 must be treated as a skeleton diagram in which $\Gamma_c$ and $G_c$ must be evaluated up to terms of order $e_c^2$, i.e., $\Gamma_c = \gamma + \Lambda_R^{(3)} = \gamma + \Lambda_1^{(1)} - \Lambda_{10}^{(3)}$ and $G_c = S^c + S^c \Sigma_R^{(2)} S^c$. The vertex part $\Lambda_1^{(3)}$ is evaluated in accordance with the elementary diagram of Fig. 83 ($l$). The regularized self-energy part $\Sigma_R^{(2)}$ can be directly expressed in terms of the self-energy part $\Sigma^{(2)}$

evaluated in accordance with the diagram of Fig. 56 (*1*), by means of the formula

$$\Sigma_R^{(2)}(p) = \Sigma^{(2)}(p) - \Sigma^{(2)}(p_0) - \frac{\partial \Sigma^{(2)}(p_0)}{\partial p_\mu}(p_\mu - p_{0\mu}). \qquad (45.21)$$

Indeed, it follows from the definition (45.10) that

$$\Sigma_R^{(2)}(p) = \int\limits_{p_0}^{p} [\Lambda_{1\mu}(q) - \Lambda_{1\mu}(p_0)]\, dq_\mu$$

$$= \int\limits_{p_0}^{p} \frac{\partial \Sigma^{(2)}(q)}{\partial q_\mu}\, dq_\mu - \left(\frac{\partial \Sigma^{(2)}(p)}{\partial p_\mu}\right)_{p=p_0} (p_\mu - p_{0\mu}).$$

In the third approximation (up to terms of order $e_c^6$) we must evaluate the diagrams of Fig. 86 treated as elementary diagrams. The diagram of Fig. 85 should be treated as a skeleton diagram, in which the quantities $\Gamma_c$, $G_c^e$ and $G_c^\gamma$ are evaluated up to terms of order $e_c^2$. The first two have been discussed earlier. For the photon Green's function $G_c^\gamma$ we have similarly in this approximation $G_c^\gamma = D^c + D^c \Pi_R^{(2)} D^c$. The quantity $\Pi_R^{(2)}$ can also be expressed directly in terms of the photon self-energy part $\Pi^{(2)}$ evaluated directly on the basis of the diagram of Fig. 58 (*1*) by means of

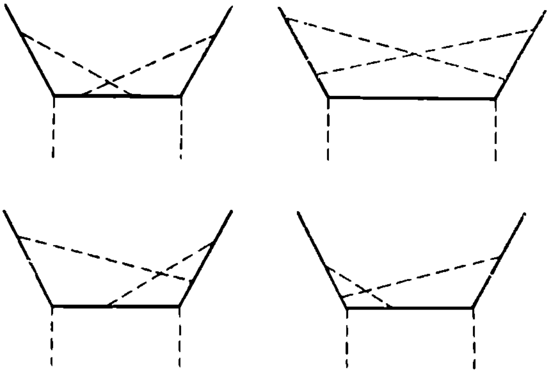

Fig. 86.

$$\Pi_R^{(2)}(k) = \Pi^{(2)}(k) - \Pi^{(2)}(0) - \left(\frac{\partial \Pi^{(2)}}{\partial k^2}\right)_{k^3=0} k^2, \qquad (45.22)$$

which follows from the definitions (45.12).

In the diagram of Fig. 69 when calculations up to terms of order $e_c^6$ are being performed we must take into account in the expressions for $\Gamma_c$ and $G_c$ terms of order $e_c^4$. The expression for $\Gamma_c$ has the form $\Gamma_c = \gamma + \Lambda_R^{(3)} + \Lambda_R^{(5)}$. The vertex part $\Lambda_1^{(5)}$ contains the factor $e_c^4$ and, therefore, must be evaluated directly in accordance with the elementary irreducible diagram of Fig. 83 (2). The vertex part $\Lambda_1^{(3)}$ contains the factor $e_c^2$. Therefore, it must be evaluated on the basis of the skeleton diagram of Fig. 83 (1) into which expressions for $\Gamma_c$, $G_c^e$ and $G_c^\gamma$ accurate to terms of order $e_c^2$ are substituted.

In order to evaluate $M_R$ up to terms of order $e_c^4$ we can utilize formula (45.10) after substituting into it the function $\Lambda_R$ on the basis of the diagrams of Fig. 83 up to terms of order $e_c^4$. From this we can easily establish the following rules for the evaluation of $\Sigma_R^{(4)}$ in terms of the reducible elementary diagrams for the fourth order electron self-energy parts $\Sigma_I^{(4)}$, $\Sigma_{II}^{(4)}$ and $\Sigma_{III}^{(4)}$ shown in Fig. 87.

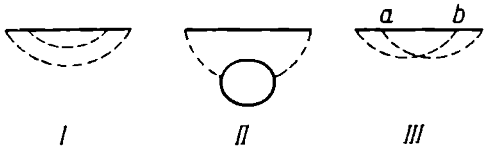

Fig. 87.

In the case of diagram $I$ which contains within itself the second order self-energy part $\Sigma^{(2)}$ it is necessary, first of all, to regularize the latter, i.e., to replace it by the quantity $\Sigma_R^{(2)}$ in accordance with formula (45.21). Then, after calculating, with this replacement kept in mind, the quantity corresponding to the whole diagram which we denote by $\Sigma_I^{(4)\prime}$, we must once again perform a regularization, i.e., replace $\Sigma_I^{(4)\prime}$ by the quantity $\Sigma_{IR}^{(4)}$ in accordance with a formula analogous to (45.21)

$$\Sigma_R^{(4)}(p) = \Sigma^{(4)\prime}(p) - \Sigma^{(4)\prime}(p_0) - \left(\frac{\partial \Sigma^{(4)\prime}(p)}{\partial p_\mu}\right)_{p_0} (p_\mu - p_{0\mu}). \qquad (45.23)$$

For diagram $II$ containing the photon self-energy part of the second order $\Pi^{(2)}$ it is first of all necessary to regularize the latter, i.e., to replace $\Pi^{(2)}$ by $\Pi_R^{(2)}$ in accordance with formula (45.22), and then, after having evaluated on the basis of such modified diagrams the corresponding quantity $\Sigma_{II}^{(4)\prime}$, to regularize it by means of formula (45.23).

In the case of diagram *III* containing the vertex part $\Lambda^{(3)}$ the following more complicated expression is obtained:

$$\Sigma_{IIIR}^{(4)} = e_c^4 \int \Big\{ R(p, t_1, t_2) - R(p_0, t_1, t_2) - (p_\mu - p_{0\mu}) \times \left( \frac{\partial R(p, t_1, t_2)}{\partial p_\mu} \right)_{p_0} \Big\} d^4 t_1 \, d^4 t_2, \tag{45.24}$$

where

$$\begin{aligned} R(p, t_1, t_2) = {} & F_\mu(p, t_1) G_{\nu\mu}(p, t_1, t_2) H_\nu(p, t_2) \\ & - F_\mu(p_0, t_1) G_{\nu\mu}(p_0, t_1, 0) H_\nu(p, t_2) \\ & - F_\mu(p, t_1) G_{\nu\mu}(p_0, 0, t_2) H_\nu(p_0, t_2), \end{aligned}$$

$$F_\mu(p, t_1) = D^c(t_1) \gamma_\mu S^c(p - t_1),$$

$$G_{\nu\mu}(p, t_1, t_2) = i\gamma_\nu S^c(\gamma - t_1 - t_2) \gamma_\mu,$$

$$H_\nu(p, t_2) = S^c(p - t_2) \gamma_\nu D^c(t_2).$$

## § 46. Divergences in the Scattering Matrix and Their Removal

### 46.1. *Divergences in Irreducible Diagrams*

In the evaluation of higher order approximations to the elements of the scattering matrix we encounter fundamental difficulties which for many years gave rise to the opinion that the applicability of quantum electrodynamics is, in general, limited to the first approximation. These difficulties consist of the fact that the higher order approximations to the matrix elements contain integrals which diverge for large values of the momenta of the virtual particles.

In this section we shall examine in detail the problem of these divergences and the methods of removing them. In the course of this discussion we shall not investigate the divergences in the region of low momenta ("infrared catastrophe") which are encountered in a number of cases. The latter are associated with the inapplicability of perturbation theory at low photon frequencies, and can be removed comparatively simply by the method described in § 30.

We consider an irreducible part of a diagram. In the general case it corresponds to a multiple integral over the momenta of the virtual particles $I = \int F(q_1 \ldots q_n) d^4 q_1 . \ldots d^4 q_n$. The integrand in this integral is a rational fraction. Let us determine the degree of this fraction. In order to do this we note that each internal electron line of the diagram corresponds to the factor $S^c(p)$ which contains the inverse first power of the momentum, while each internal photon line corresponds to the factor $D^c(p)$ containing the inverse second power of the momentum. Therefore, the integrand is a fraction of degree $-(F_e+2F_\gamma)$, where $F_e$ and $F_\gamma$ are the numbers of internal electron and photon lines. We now determine the number of variables of integration. The number of different momenta is equal to the number of internal lines, i.e., $F_e+F_\gamma$. But they are not independent, since the three momenta of the lines meeting at each one of the $n$ vertices of the diagram are related by the law of conservation of momentum. One of the conservation laws may be applied to the external lines, so that the total number of independent momenta over which the integration is carried out is equal to $F_e+F_\gamma-n+1$. If we write the integral schematically in the form

$$I \sim \int \frac{(d^4q)^{K_1}}{(q)^{K_2}},$$

then on taking into account the foregoing calculation and utilizing relations (45.17) we obtain

$$K_1 = F_e+F_\gamma-n+1 = \tfrac{1}{2}(n-N_e-N_\gamma)+1,$$

$$K_2 = 2n-\frac{N_e}{2}-N_\gamma,$$

where $N_e$ and $N_\gamma$ are the numbers of external electron and photon lines.

Since we are considering an irreducible diagram, the integrand cannot be decomposed into factors containing independent variables. Therefore, the convergence of the integral $I$ is determined by the difference

$$K = K_2-4K_1 = \tfrac{3}{2}N_e+N_\gamma-4. \tag{46.1}$$

For $K > 0$ the integral is convergent, while for $K \leqslant 0$ it is divergent. It is noteworthy that the quantity $K$ depends only on the number of external lines.

It follows from (46.1) that there exists a limited number of types of divergent integrals corresponding to the following values of the numbers $N_e$ and $N_\gamma$:

$$\begin{array}{llll}
1) & N_e = 2, & N_\gamma = 1, & K = 0, \\
2) & N_e = 2, & N_\gamma = 0, & K = -1, \\
3) & N_e = 0, & N_\gamma = 4, & K = 0, \\
4) & N_e = 0, & N_\gamma = 3, & K = -1, \\
5) & N_e = 0, & N_\gamma = 2, & K = -2, \\
6) & N_e = 0, & N_\gamma = 1, & K = -3, \\
7) & N_e = 0, & N_\gamma = 0, & K = -4.
\end{array} \tag{46.1'}$$

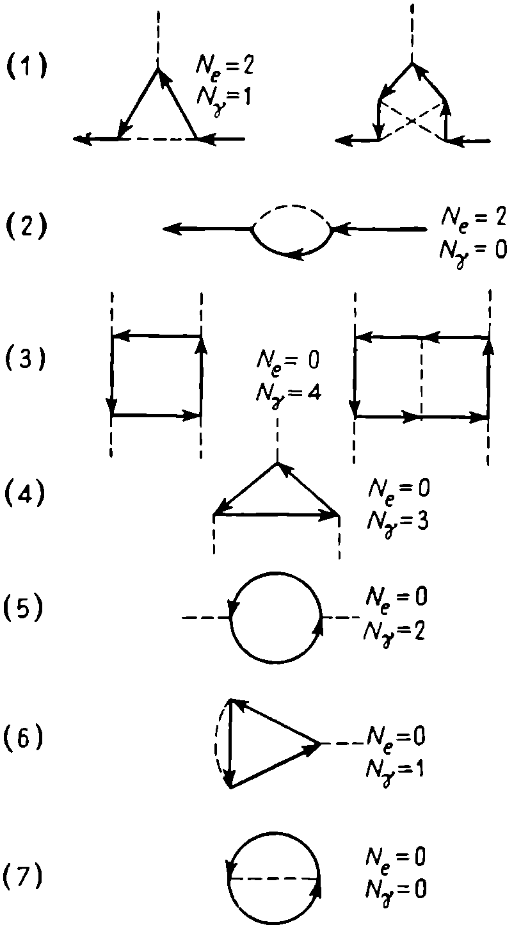

Fig. 88.

Figure 88 shows the simplest irreducible diagrams corresponding to these divergences. Not all of them can, in fact, appear in the expressions for the matrix elements. Thus, for example, the diagram of Fig. 88 (7) is a vacuum loop which, as has been explained in subsection 44.1, need not be taken into consideration. Diagrams of the type of Fig. 88

(*4*, *6*) which represent closed electron loops with an odd number of vertices, can also be left out of account in accordance with Furry's theorem (cf. § 25). However, we retain diagrams of the type of Fig. 88 (*4*) since they determine the auxiliary quantity—the three-photon vertex $V_\mu$—which we have introduced previously in subsection 45.6.

We see that the principal irreducible parts, from which the expression for the scattering matrix in higher order approximations can be constructed, are divergent expressions with values of $K = 0, -1, -2$. Among them are included the irreducible self-energy parts $\Sigma^{(2)}$ and $\Pi^{(2)}$, the vertex parts $\Lambda^{(2n+1)}$, and also the part corresponding to photon-photon scattering (diagram of Fig. 88 (*3*)).

### 46.2. *Introduction of a Cut-Off Momentum*

The appearance of divergences in the scattering matrix shows that the theory is unsatisfactory. It is clear that the perturbation theory series is, strictly speaking, meaningless if the second term of the series is infinite. Thus, the following problem arises which is of the greatest fundamental significance.

On the one hand, the results of applying first order perturbation theory, which were described in Chapters V and VI, are in excellent agreement with experiment. On the other hand, in order that the first approximation should have some theoretical significance, the subsequent approximations should lead only to small corrections.

The idea of an escape from this contradiction is already suggested by classical electrodynamics. As is well known, classical electrodynamics is not a logically closed theory. Its consistent application leads to contradictions, which, for example, manifest themselves in the infinite electromagnetic mass of the electron. The content of these contradictions can be expressed as the inapplicability of the equations of classical electrodynamics at distances smaller than the classical electron radius $e^2/mc^2$. (In fact, because of quantum effects classical electrodynamics is already inapplicable at distances of the order of $\hbar/mc$.)

If quantum electrodynamics, like classical electrodynamics, is also inapplicable at sufficiently small distances, then without entering here into a discussion of the problem whether this limitation is associated with deficiencies of the fundamental ideas and equations of electrodynamics, or merely with the use of perturbation theory methods, we may, nevertheless, assume that our theory takes the interaction into account correctly within the domain of space-time intervals $|x^2| > 1/L^2$, where $L$ is a sufficiently large quantity. Since in Fourier expansions large momenta correspond to small $|x^2|$, this means that the formulas of quantum electrodynamics are valid only in the domain of momenta smaller than some limiting momentum $L$. For $|p| \geqslant L$ the formulas of quantum electrodynamics are incorrect, either because of the inapplicability of perturbation theory, or as a result of the fact that the fundamental equations of quantum electrodynamics are inapplicable to processes for which the domain of small space-time intervals plays an essential role.

We shall further assume that the domain of very large virtual momenta $|p| > L$ actually plays no role at all in any process in which the particles in colliding transfer to each other momenta $\Delta p$ small compared to $L$. With this assumption we can carry out integration in momentum space over a finite region $|p| < L$.

We shall show later that the cut-off momentum $L$ can be chosen in such a way that, on the one hand, the condition $\Delta p \ll L$ is satisfied, and, on the other hand, the use of perturbation theory is justified. We shall see that the expression for the matrix elements evaluated by methods presented in § 45 does not depend on the cut-off momentum $L$ which appears only in the expressions for mass and charge renormalization. We shall also see that the use of perturbation theory in quantum electrodynamics can be justified over a very wide range.

Thus, we shall assume that the integrals of interest to us, which determine $\Lambda$, $\Sigma$ and $\Pi$ and other quantities, are evaluated over a finite region in four-dimensional momentum space $|p| \leqslant L$. This region must obviously be invariant under Lorentz transformations.

Instead of introducing a finite region of integration we can also use another equivalent method. It consists of introducing into the integrand a "cut-off factor", i.e., a certain factor $f(q, L)$ which has the properties $f(q, L) = 1$ for $q \ll L$ and $f(q, L) \to 0$ for $q > L$, where $f$ is made to approach zero in such way that the integral converges. For the cut-off factor we can choose, for example, a function of the form (second of references (62))

$$f(q, L) = \left(\frac{L^2}{q^2+L^2}\right)^n,$$

where $n > 1$.

### 46.3. *Convergence of Regularized Expressions for Irreducible Vertex Parts and Self-Energy Parts*

We shall now show that elements of the scattering matrix expressed in accordance with formulas of sections 45 in terms of the true electron charge $e_c$ and the true electron mass $m_c$ do not depend on the cut-off momentum $L$, i.e., do not contain any divergences.

As we have seen, the matrix elements contain the renormalized propagators and vertex functions $G^e_c$, $G^\gamma_c$, $\Gamma_c$, which can be expressed in terms of the regularized vertex parts $\Lambda_R$ and $V_R$.

We begin by considering the simplest third order vertex part

$$\Lambda^{(3)}(p_1, p_2\,; k) = \int R(p_1, p_2\,; q)\, d^4 q, \tag{46.2}$$

where

$$R(p_1, p_2\,; q) = \frac{e^2}{(2\pi)^4}\, \gamma_\alpha S^c(p_1-q)\, \gamma_\mu S^c(p_2-q)\, \gamma_\beta D^c_{\alpha\beta}(q).$$

The regularized expression for the vertex part is, in accordance with (45.7), given by

$$\Lambda^{(3)}_R(p_1, p_2\,; k) = \Lambda^{(3)}_1(p_1, p_2\,; k) - \Lambda^{(3)}_1(p_0, p_0\,; 0),$$

where $p_0$ is the momentum of the free electron: $p_0^2 + m_c^2 = 0$.

In accordance with the general results (46.1) the integral (46.2) diverges logarithmically. Since the momenta $p_1$ and $p_2$ appear in the integrand $R(p_1, p_2\,; q)$ in the form of the differences $p_1 - q \equiv t_1$ and $p_2 - q \equiv t_2$

$$R(p_1, p_2\,; q) \equiv f(t_1, t_2, q),$$

the expression for $R(p_1, p_2\,; q)$ can be written in the form

$$R(p_1, p_2\,; q) = R(p_0, p_0\,; q) + \left(\frac{\partial R}{\partial p_{1\mu}}\right)_{p_1 = p'} (p_{1_\mu} - p_{0_\mu}) + \left(\frac{\partial R}{\partial p_{2_\mu}}\right)_{p_2 = p''} (p_{2_\mu} - p_{0_\mu}) = R(p_0, p_0\,; q) + \left(\frac{\partial f}{\partial t_{1_\mu}}\right)_{t_1 = p' + q} (p_{1_\mu} - p_{0_\mu}) + \left(\frac{\partial f}{\partial t_{2_\mu}}\right)_{t_2 = p'' + q} (p_{2_\mu} - p_{0_\mu}),$$

where the derivatives are evaluated at some points $p'$ and $p''$ intermediate between $p_0$ and $p_1$ or $p_0$ and $p_2$. Since the function $R$ is a rational fraction of degree $-4$ with respect to $q$, then $\partial f/\partial t_1$ and $\partial f/\partial t_2$ will be rational fractions of degree –5, and the integral defining the regularized vertex part

$$\Lambda_R^{(3)} = \int \{R(p_1, p_2; q) - R(p_0, p_0; q)\} d^4q \tag{46.2'}$$

will converge. (This fact justifies the use of the term "regularized.")

The foregoing proof of the convergence of $\Lambda_R^{(3)}$ can be easily generalized to the case of an arbitrary irreducible vertex part.

An irreducible vertex part of $(2n+1)$th order $\Lambda^{(2n+1)}(p_1, p_2; k)$ (cf., for example, Fig. 83 for $n = 2$ and 4) can be written in the form of a logarithmically divergent integral

$$\Lambda^{(2n+1)}(p_1, p_2; k) = \int R(p_1, p_2, q_1, \dots, q_n) d^4q_1 \dots d^4q_n.$$

The variables $p_1$ and $p_2$ appear here only in the form of linear combinations with the variables of integration $t_1 = p_1 - q_1$; $t_2 = p_1 - q_1 - q_2$; .... Therefore the difference is given by

$$R(p_1, p_2, q_1, \dots, q_n) - R(p_0, p_0, q_1, \dots q_n) = \sum_i \left(\frac{\partial R}{\partial t_i}\right)_{t_i'} f(p_1, p_2),$$

where the derivatives $\partial R/\partial t_1$ are evaluated at some intermediate points $t_i'$, and, consequently, the expression for

$$\Lambda_R^{(2n+1)}(p_1, p_2; k)$$
$$= \int \{R(p_1, p_2, q_1, \dots, q_n) - R(p_0, p_0, q_1, \dots, q_n)\} d^4q_1 \dots d^4q_n$$

will be convergent.

We note that if we choose the transverse gauge for $D^c$, i.e., $D^c_{\alpha\beta}(q) = D^c(q)(\delta_{\alpha\beta} - [q_\alpha q_\beta/q^2])$, then the integral (46.2) will be convergent, since the terms of highest degree in $q$ will cancel in the integrand (second of references (115)). However, such a simplification occurs only for the third order vertex part $\Lambda^{(3)}$, and does not occur for other irreducible vertex parts. Therefore, in future we shall use the expression for $D^c$ in the following form $D^c_{\alpha\beta}(q) = \delta_{\alpha\beta} D^c(q)$, since it leads to simpler algebraic expressions.[1]

We now consider the simplest irreducible third order three-photon vertex

$$V_\mu^{(3)}(k) = \int Q_\mu(k, q)\, d^4q, \tag{46.3}$$

---

[1] For each approximation the gauge of the propagator can be so chosen that the expression for the vertex part will converge (24a).

where

$$Q_\mu(k, q) = \frac{e^2}{3(2\pi)^4} \operatorname{Sp} \gamma_\alpha S^c(q-k)\gamma_\mu S^c(q-k)\gamma_\alpha S^c(q).$$

In accordance with (46.1) this integral contains a linearly divergent part, i.e., it depends on $L$ linearly. However, in view of the invariance of the region of integration the terms proportional to $L$ must vanish. Indeed, we can assume for the investigation of the divergent part of the integral (46.3) that $k_\mu \ll q_\mu$. If we omit $k_\mu$ in the integrand, then after the integration and the evaluation of the trace have been carried out, there will remain no vector parameter in terms of which the vector quantity $V_\mu$ might be expressed. A nonvanishing result can come only from the terms of $Q_\mu$ linear in $k_\mu$, but they will lead only to a logarithmic divergence.

From the foregoing it follows, in particular, that

$$V^{(3)}(0) = \int Q(0, q)\, d^4q = 0.$$

On substituting $Q(k, q)$ in

$$Q(k, q) = Q(0, q) + \left(\frac{\partial Q}{\partial k_\alpha}\right)_0 k_\alpha + \left(\frac{\partial^2 Q}{\partial k_\alpha \partial k_\beta}\right)_{k'} k_\alpha k_\beta$$

we can easily show that the regularized expression for the three-photon vertex part

$$V_R^{(3)} = V^{(3)} - \left(\frac{\partial V^{(3)}}{\partial k_\mu}\right)_0 k_\mu = \int \left\{ Q(k, q) - Q(0, q) - \left(\frac{\partial Q}{\partial k_\alpha}\right)_0 k_\alpha \right\} d^4q$$

is convergent, since the integrand which is proportional to the second derivative $\partial^2 Q/\partial q_\alpha \partial q_\beta$ is a rational fraction of degree two units lower than the degree of $Q$.

A similar proof can also be given in the case of an arbitrary irreducible three-photon vertex part.

From the fact that the expressions for the regularized irreducible vertex parts $\Lambda_R^{(3)}$ and $V_R^{(3)}$ are convergent it follows that the expressions for the regularized irreducible self-energy parts $\Sigma_R^{(2)}$ and $\Pi_R^{(2)}$ are also convergent.

This can also be shown directly by using the explicit expressions for $\Sigma_R^{(2)}(p)$ and $\Pi_R^{(2)}(p)$. Indeed, in accordance with (45.21) $\Sigma_R^{(2)}$ can be obtained from $\Sigma^{(2)}$ by subtracting the two leading terms of the expansion of $\Sigma^{(2)}$ into a series in powers of $p-p_0$, whence it follows that $\Sigma_R^{(2)}$ is proportional to the second derivative of $\Sigma^{(2)}(p)$ with respect to $p$ evaluated at some point intermediate between $p_0$ and $p$. Since the momentum $p$ appears in the integrand of $\Sigma^{(2)}(p)$ in the form of a sum involving the variable of integration, then the integrand in $\Sigma_R^{(2)}(p)$ will be a rational fraction of degree two units lower than the degree of the corresponding integrand for $\Sigma^{(2)}(p)$. Since in accordance with (46.1) the integral for $\Sigma^{(2)}$ contains a linear divergence, the integral defining $\Sigma_R^{(2)}$ will be absolutely convergent.

A similar situation exists in the case of the irreducible photon self-energy part $\Pi_R^{(2)}$. It follows from (45.22) that the expression for $\Pi_R^{(2)}(k)$ can be obtained from the expression for $\Pi^{(2)}(k)$ by subtracting the two leading terms in the expansion of $\Pi^{(2)}(k)$ into a power series in $k$. It can be easily shown by repeating the arguments referring to $\Sigma_R^{(2)}$ that this subtraction guarantees the absolute convergence of the integral defining $\Pi_R^{(2)}$.

However, here we encounter a peculiar contradiction. For the derivation in § 45 of the fundamental formulas defining $G_c^\gamma$ and $P_R$ we used the property of the polarization operator $P(0) = 0$ which was proved in § 44. For the derivation of this property we naturally made use of the fact that the quantity $P(k)$ is finite. And yet, as we have seen earlier, $\Pi^{(2)}(k)$ contains a quadratically divergent integral, and after the introduction of the cut-off momentum $\Pi(0) \sim L^2$. We thus see that the introduction of the cut-off momentum, which makes various quantities finite, violates the gauge invariance of the theory. Therefore, strictly speaking, the simple replacement of the integrals appearing in the theory by integrals taken over a finite region is not justified.

We can formally avoid this difficulty by means of the following device. We consider the quantity $P_1(k) = P(k) - P(0)$. In an exact gauge-invariant theory $P_1(k) = P(k)$. In our approximate treatment we will introduce the cut-off momentum not in the integrals defining $P(k)$, but in integrals defining $P_1(k)$. This will ensure both the finiteness and the gauge invariance of the resulting expressions.

We encounter a similar difficulty in the discussion of diagrams corresponding to photon-photon scattering (cf. Fig. 88 (*3*)). In accordance with (46.1) the expressions corresponding to such diagrams

$$M(k_1, k_2, k_3, k_4) = \int Q(k_1, k_2, k_3, k_4, q_1, \dots, q_n)\, d^4q_1 \dots d^4q_n$$

diverge logarithmically. The same divergence will also be contained in the quantity

$$M(0,0,0,0) = \int Q(0,0,0,0,q_1 \ldots, q_n)\, d^4q_1 \ldots d^4q_n.$$

On the other hand, the gauge invariance of the theory requires that the equation $M(0,0,0,0) = 0$ should hold, since physical quantities can depend only on fields, and not on potentials. We see that the divergence of the quantity $M(k_1, k_2, k_3, k_4)$ is associated with the violation of gauge invariance of the theory by the introduction of the cut-off momentum.

In an exact gauge invariant theory the following relation must hold

$$M_R(k_1, k_2, k_3, k_4) = M(k_1, k_2, k_3, k_4),$$

where

$$M_R(k_1, k_2, k_3, k_4) = M(k_1, k_2, k_3, k_4) - M(0,0,0,0). \quad (46.4)$$

Therefore, we replace the matrix element $M(k_1, k_2, k_3, k_4)$ by its regularized value $M_R(k_1, k_2, k_3, k_4)$. The quantity $M_R(k_1, k_2, k_3, k_4)$, which can be written in the form

$$M_R(k_1, k_2, k_3, k_4)$$
$$= \int \{Q(k_1, k_2, k_3, k_4, q_1, \ldots, q_n) - Q(0,0,0,0,q_1, \ldots, q_n)\}\, d^4q_1 \ldots d^4q_n,$$

will be finite, since $Q(k_1 \ldots q_1 \ldots) - Q(0 \ldots q_1 \ldots)$ is a fraction of degree two units lower than the degree of the fraction $Q(k_1 \ldots q_1 \ldots)$.

The problem of photon-photon scattering will be discussed in detail in § 54.

### 46.4. *Convergence of Regularized Quantities in the Case of Reducible Diagrams*

We now proceed to investigate reducible vertex parts. In order to obtain the regularized vertex part $\Lambda_R(Y)$ corresponding to a reducible diagram $Y$, we must replace it by an equivalent irreducible skeleton diagram.

We begin with the simplest example of a fifth order vertex part $\Lambda^{(5)}$ shown in Fig. 89. The irreducible skeleton diagram which is equivalent to it is shown in Fig. 90, where the point $A$ corresponds to a third order vertex part $\Lambda^{(3)}$. In virtue of the results of § 45 in order to obtain $\Lambda_R^{(5)}$ we must first replace $\Lambda^{(3)}$ by $\Lambda_R^{(3)}$ in accordance with formula (46.2′), and we must then regularize the quantity evaluated in accordance with the diagram of Fig. 90, which we denote by $\Lambda^{(5)\prime}$, by using the general formula (45.7). Since we are interested here only in the problem of the nature of the divergence of the quantity $\Lambda^{(5)\prime}$, then by omitting the spinor matrices we can schematically represent the expression for the integral in $\Lambda^{(5)\prime}$ in the following form:

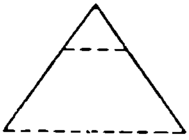
Fig. 89.

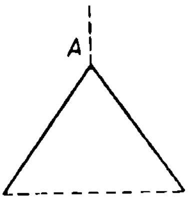

Fig. 90.

$$\Lambda^{(5)\prime}(p_1, p_2\,; k) \sim \int R(p_1, p_2, q)\Lambda_R^{(3)}(p-q, p_2-q\,; k)d^4q, \quad (46.5)$$

where $R$ is the same function (a rational fraction of degree $-4$ in $q$) which appears in the integral for $\Lambda^{(3)}$ (46.2) and whose asymptotic dependence on $q$ for $|q|^2 \gg |p_1^2|, |p_2|^2$ is of the following nature:

$$R(p_1, p_2, q)d^4q \sim d(\ln q^2).$$

On the other hand, as we shall see later (cf. 47.4), for large $q$ $\Lambda_R^{(3)}$ behaves like $\ln q^2$, and therefore the divergent part of the integral defining $\Lambda^{(5)\prime}$ is of the form

$$\Lambda^{(5)\prime} \sim \int \ln q^2\, d(\ln q^2) \sim (\ln L)^2,$$

i.e., $\Lambda^{(5)\prime}$ diverges logarithmically.

Since the integrand in the integral for $\Lambda^{(5)\prime}$ differs from the corresponding expression for $\Lambda^{(3)}$ only by a factor which depends on the momenta logarithmically, then by utilizing the same arguments which were given earlier in subsection 46.3 we can easily show that the regularized vertex part $\Lambda_R^{(5)}$ is convergent.

We now consider the fifth order vertex parts shown in Fig. 91. The irreducible skeleton diagrams equivalent to them are shown in Fig. 92 where the thick lines correspond to the propagators $G_c^e$ and $G_c^\gamma$ evaluated on the basis of the second order self-energy parts $\Sigma_R^{(2)}$ and $\Pi_R^{(2)}$.

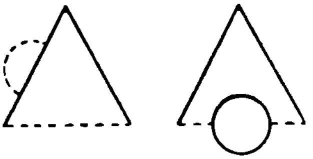

Fig. 91.

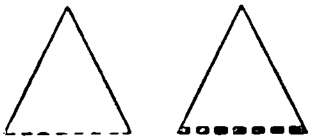

Fig. 92.

Since $\Sigma_R^{(2)} = \int\limits_{p_0}^{p} \Lambda_R^{(3)}(q)\,dq$, then asymptotically for large $p$, $\Sigma_R^{(2)} \sim p \ln p^2$, whence we obtain for the asymptotic behavior of $G_c^e$

$$G_c^e(p) \sim \frac{1}{p(1+C\ln p^2)}\,.$$

In order to determine the behavior of $G_c^\gamma(k)$ for large values of $k$ we utilize the properties of the three-photon vertex part $V^{(3)}$. Since it diverges logarithmically, it follows from dimensional considerations that for large $k$ it must be of the form, $V_\mu(k) \sim k_\mu \ln(L^2/k^2)$.

From this it follows that $V_{R\mu}^{(3)} \sim k_\mu \ln k^2$. Therefore

$$G_c^{(\gamma)} \sim \frac{1}{k^2(1+C\ln k^2)}$$

(cf. the corresponding expressions in § 47).

If with the aid of these functions we evaluate the vertex parts $\Lambda^{(5)\prime}$ corresponding to the diagrams of Fig. 89–91, they will contain only a logarithmic divergence. After regularization in accordance with formula (45.7) we obtain a finite quantity. These results can be easily generalized to the case of arbitrary reducible diagrams. On replacing such diagrams by skeleton irreducible diagrams in which the lines and the vertices denote respectively $G_c^e$, $G_c^\gamma$, $\Gamma_c$, and on reducing the latter to the simplest irreducible diagrams, it can be easily shown that asymptotically for large momenta $|p|^2 \gg m^2$, $|k^2| \gg m^2$

$$G_c^e(p) \sim S^c(p) f_s\left(\ln\frac{p^2}{m_c^2}\right),$$

$$G_c^\gamma(k) \sim D^c(k) f_D\left(\ln\frac{k^2}{m_c^2}\right),$$

$$\Gamma_c(q, q, 0) \sim f_\Gamma\left(\ln\frac{q^2}{m_c^2}\right),$$

where $f_s$, $f_D$ and $f_\Gamma$ are dimensionless polynomials in the corresponding logarithmic arguments. Thus, the diagram as a whole will correspond to an integral the integrand of which will contain, in addition to rational fractions of the type discussed in subsection 46.1, only logarithms of momenta which do not alter the character of the divergence of the integral.

In particular, these results also apply to expressions (45.23) and (45.24) for the regularized fourth order self-energy parts. It can be seen from formula (45.23) that after the removal of the divergence associated with the internal self-energy part $\Sigma^{(2)}$, the divergence in $\Sigma_I^{(4)\prime}$ can be eliminated by subtracting the two leading terms in the expansion of this quantity in powers of $(p-p_0)$. In the case of the self-energy part $\Sigma_{III}^{(4)}$ the divergences are associated with the presence of vertex parts at points $a$ and $b$. Formula (45.24) eliminates the so-called overlapping divergences associated with them.

## § 47. Evaluation of Self-Energy and Vertex Parts

### 47.1. *Evaluation of Integrals over Four-Dimensional Regions*

The integrals which we encounter in the evaluation of quantities corresponding to various diagrams are all of the form

$$I = \int \frac{F(p)d^4p}{a_1 a_2 \dots a_n}, \tag{47.1}$$

where $a_i$ are polynomials of second order and $F(p)$ is a polynomial of order $n$ with respect to $p_\mu$. Such integrals can be conveniently evaluated if we first utilize the identity

$$\frac{1}{a_1 a_2 \ldots a_n} = (n-1)! \int_0^1 d\xi_1 \int_0^1 d\xi_2 \ldots \int_0^1 d\xi_n \frac{\delta(\xi_1+\xi_2+\ldots+\xi-1)}{(a_1\xi_1+a_2\xi_2+\ldots+a_n\xi_n)^n}$$

$$= (n-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots$$

$$\int_0^{x_{n-2}} \frac{dx_{n-1}}{[a_1 x_{n-1}+a_2(x_{n-2}-x_{n-1})+\ldots+a_n(1-x_1)]^n} \tag{47.2}$$

or the equivalent identity

$$\frac{1}{a_1 a_2 \ldots a_n} = (n-1)! \int_0^1 u_1^{n-2}\, du_1 \int_0^1 u_2^{n-3}\, du_2 \ldots$$

$$\int_0^1 du_{n-1} \frac{1}{[a_1 u_1 \ldots u_{n-1}+a_2 u_1 \ldots u_{n-2}(1-u_{n-1})+\ldots+a_n(1-u_1)]^n}, \tag{47.3}$$

where $x_1 = u_1, x_2 = u_1 u_2, \ldots, x_{n-1} = u_1 u_2 \ldots u_{n-1}$ and $x_1, x_2, \ldots, x_{n-1}$ are scalar parameters. On substituting this expression into (47.1) we obtain an integral in $p$-space

$$J(x_1, \ldots, x_{n-1}) = \int \frac{F(p)\, d^4p}{[(p-a)^2+l]^n}, \tag{47.4}$$

where $a$ and $l$ depend on the parameters $x_1 \ldots x_{n-1}$. The desired integral $I$ is obtained from $J(x_1 \ldots x_{n-1})$ by integration over $x_1 \ldots x_{n-1}$:

$$I = (n-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-2}} dx_{n-1} J(x_1 \ldots x_{n-1}). \tag{47.4'}$$

We call this method of evaluating the integrals $I$, *the method of parametrization* (second of references (62)),

First of all we demonstrate the validity of identity (47.3). For $n = 2$ it is self-evident, since

$$\frac{1}{a_1 a_2} = \int_0^1 \frac{du}{[a_1 u + a_2(1-u)]^2}.$$

We now show that if it is valid for some $n$, then it is also valid for $n+1$. On noting that integrations over the different $u_i$ in (47.3) may be interchanged we obtain

$$\frac{1}{a_1 \dots a_{n+1}} = (n-1)! \int_0^1 u_1^{n-2}\, du_1 \dots \int_0^1 du_{n-1} \frac{1}{A^n a_{n+1}},$$

where $A^n$ is the denominator in the right hand side of formula (47.3). But

$$\frac{1}{nA^n a_{n+1}} = \int_0^1 \frac{u_0^{n-1}\, du_0}{[Au_0+a_{n+1}(1-u_0)]^{n+1}}.$$

Therefore

$$\frac{1}{a_1 a_2 \dots a_n a_{n+1}} = n! \int_0^1 u_0^{n-1}\, du_0 \dots \int_0^1 du_{n-1} \frac{1}{[Au_0+a_{n+1}(1-u_0)]^{n+1}},$$

which is what we set out to prove.

We now proceed to evaluate the integral (47.4). We first assume that the integral (47.4) is convergent. In this case we can shift the variable $p-a \to p$, as a result of which the integral assumes the form

$$K = \int \frac{f(p)}{(p^2+l)^n}\, d^4p. \tag{47.5}$$

Obviously, we can confine ourselves to the evaluation of only the scalar integral, since if $f(p) = p_\mu f_1(p^2)$, then $K = 0$; if $f(p) = p_\mu p_\nu f_2(p^2)$, then

$$\int \frac{p_\mu p_\nu f_2(p^2)}{(p^2+l)^n}\, d^4p = \frac{1}{4}\, \delta_{\mu\nu} \int \frac{p^2 f_2(p)}{(p^2+l)^n}\, d^4p$$

etc.

We now demonstrate the method of evaluating the integral (47.5). In this integral the integration over $p_0$ is performed in accordance with the rule for going around the poles (cf. § 17), i.e., along the contour $C$ shown in Fig. 93. But this contour can, evidently, be rotated by $\pi/2$, as shown in Fig. 94. As a result $p_0$ will be replaced by $ip_0'$ and $p^2 = \mathbf{p}^2 - p_0^2$ will go over into $\mathbf{p}'^2 + p_0'^2$ ($p_0'$ is a real quantity).

Thus, rotation of the contour of integration $C$ by $\pi/2$ corresponds to the transition in $p$-space to the usual Euclidean metric, in which the square of a length is equal to the sum of the squares of all four coordinates with the fourth coordinate being real.

Since

$$d^4p = i\,d^4p' = i\,dp'\,dp_0' = 2\pi^2 i q^3\,dq = i\pi^2 q^2\,dq^2, \tag{47.6}$$

where $d^4p'$ is an element of volume in $p$-space after the contour of integration over $p_0$ has been rotated by $\pi/2$, $p' \equiv p$, $q = \sqrt{p'^2 + p_0^2}$, we obtain the one-dimensional integral

$$\int f(p^2)\,d^4p = i\pi^2 \int_0^\infty f(z)\,z\,dz. \tag{47.6'}$$

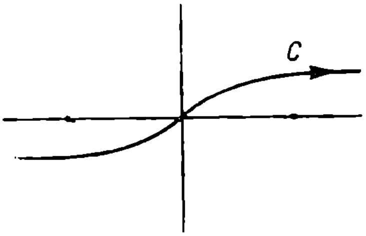

Fig. 93.

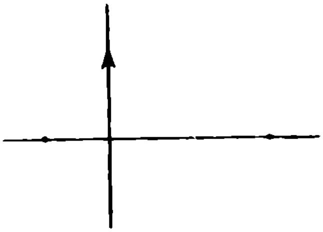

Fig. 94.

In particular,

$$\begin{aligned} &\int \frac{d^4p}{(p^2+l)^3} = \frac{\pi^2 i}{2l}, \\ &\int \frac{d^4p}{(p^2+l-2pk)^3} = \frac{\pi^2 i}{2(l-k^2)}, \\ &\int \frac{p_\sigma\,d^4p}{(p^2+l-2pk)^3} = \frac{\pi^2 i k_\sigma}{2(l-k^2)}. \end{aligned} \tag{47.7}$$

We now proceed to investigate divergent integrals of type (47.4). Since in quantum electrodynamics no divergences higher than quadratic occur, we shall be dealing with integrals of the type

$$\begin{aligned} J^{(2)}(k, l) &= \int \frac{d^4p}{(p^2-2pk+l)^2}, \\ J^{(2)}_\mu(k, l) &= \int \frac{p_\mu\,d^4p}{(p^2-2pk+l)^2}, \\ J^{(2)}_{\mu\nu}(k, l) &= \int \frac{p_\mu p_\nu\,d^4p}{(p^2-2pk+l)^2}, \\ J^{(3)}_{\mu\nu}(k, l) &= \int \frac{p_\mu p_\nu\,d^4p}{(p^2-2pk+l)^3}. \end{aligned}$$

In all these integrals and in other similar divergent integrals we shall carry out the integration over a certain finite and invariant region ($N$) characterized by a number $N$, such that $N \to \infty$ corresponds to the whole infinite four-dimensional $p$ space (such a finite invariant region may be defined, for example, by the inequalities $|p^2| \leqslant N^2$, $(ps)^2/s^2 \leqslant N^2$, where $s$ is an arbitrary time-like four-vector).

Since integration over $p_0$ with $p^2$ kept constant does not lead to a divergence, the contour of integration with respect to $p_0$ (cf. Figs. 93, 94) may be rotated, just as in the evaluation of the convergent four-dimensional integrals (47.4), through $\pi/2$, i.e., it may be replaced by a vertical straight line. After such a rotation we obtain a four-dimensional Euclidean $p$-space with a real fourth coordinate, whose element of volume $d^4p'$ is related to the element of volume of the initial $p$-space by formula (47.6). The region of integration in this case is a four-dimensional sphere, whose radius we denote by $L$ and which we identify with the cut-off momentum introduced in § 46.

In proceeding to the evaluation of the divergent integrals of interest to us we begin with the logarithmically divergent integral

$$J^{(2)}(0, l) = \int \frac{d^4p}{(p^2+l)^2}.$$

On rotating the path of integration with respect to $p_0$ by $\pi/2$ and on utilizing formula (47.6) we obtain

$$\int \frac{d^4p}{(p^2+l)^2} = i \int \frac{d^4p'}{(p'^2+l)^2} = 2\pi^2 i \int_0^L \frac{q^3 dq}{(q^2+l)^2}.$$

The last integral can be evaluated in an elementary fashion and is equal to

$$\frac{1}{2}\left[\ln\frac{L^2}{l} - 1 + O\left(\frac{1}{L^2}\right)\right].$$

Therefore

$$\int \frac{d^4p}{(p^2+l)^2} = \pi^2 i \left(\ln\frac{L^2}{l} - 1\right). \tag{47.8}$$

Further, we consider the integral

$$J^{(2)}(k, l) = \int \frac{d^4p}{(p^2-2pk+l)^2},$$

which diverges logarithmically, like the integral $J^{(2)}(0, l)$. On rewriting this integral in the form

$$\int \frac{d^4p}{(p^2-2pk+l)^2} = \int \frac{d^4p}{[(p-k)^2+l-k^2]^2},$$

we introduce new variables by means of $p-k \rightarrow p$. Then the integral assumes the form (47.8) but with an altered range of integration, which we denote by $N'$. In accordance with (47.8) the integral $J^{(2)}(k, l)$ is equal to $\pi^2 i(\ln(L'^2/l-k^2)-1)$. But $L'$ differs from $L$ by a finite quantity so that if $L'$ is sufficiently great, $\ln L'$ differs from $\ln L$ by a small quantity of order $1/L$, and we finally obtain

$$\int \frac{d^4p}{(p^2-2pk+l)^2} = \pi^2 i\left(\ln \frac{L^2}{l-k^2}-1\right). \tag{47.9}$$

We now evaluate the logarithmically divergent integral

$$J^{(3)}_{\sigma\tau}(k, l) = \int \frac{p_\sigma p_\tau d^4p}{(p^2-2pk+l)^3}.$$

Just as in evaluating the integral (47.9) we can shift the origin of coordinates by introducing a new variable of integration $p-k \rightarrow p$:

$$\int\limits_{(N)} \frac{p_\sigma p_\tau d^4p}{(p^2-2pk+l)^3} = \int\limits_{(N')} \frac{(p_\sigma+k_\sigma)(p_\tau+k_\tau)d^4p}{(p^2+l-k^2)^3}.$$

On noting that

$$\int \frac{p_\sigma p_\tau d^4p}{(p^2+l^3)} = \frac{1}{4}\,\delta_{\sigma\tau}\int \frac{p^2 d^4p}{(p^2+l)^3} = \pi^2 i\frac{\delta_{\sigma\tau}}{2}\int\limits_0^L \frac{q^5 dq}{(q^2+l)^3} = \delta_{\sigma\tau}\frac{\pi^2 i}{4}\left(\ln\frac{L^2}{l}-\frac{3}{2}\right),$$

$$\int \frac{p_\sigma d^4p}{(p^2+l)^3} = 0,$$

$$k_\sigma k_\tau \int \frac{d^4p}{(p^2+l)^3} = \frac{\pi^2 i}{2}\frac{k_\sigma k_\tau}{l},$$

we obtain finally

$$\int \frac{p_\sigma p_\tau d^4p}{(p^2-2pk+l)^3} = \frac{\pi^2 i}{4}\,\delta_{\sigma\tau}\left(\ln\frac{L^2}{l-k^2}-\frac{3}{2}\right)+\frac{\pi^2 i}{2}\frac{k_\sigma k_\tau}{l-k^2}. \tag{47.10}$$

We now proceed to evaluate the integral

$$J^{(2)}_{\sigma}(k, l) = \int \frac{p_\sigma d^4p}{(p^2-2pk+l)^2}$$

On noting that

$$\int \frac{p_\tau p_\sigma d^4 p}{(p^2-2pk+l)^3} = \frac{\partial U_\sigma}{\partial k_\tau}, \tag{47.11}$$

where

$$U_\sigma = \frac{1}{4}\int \frac{p_\sigma d^4 p}{(p^2-2kp+l)^2}$$

we integrate (47.11) over $k_\tau$ from the point $k_\tau = 0$ to the point $k_\tau$:

$$\sum_\tau \int \frac{\partial U_\sigma}{\partial k_\tau} dk_\tau = -\frac{\pi^2 i}{4}\sum_\tau \delta_{\sigma\tau}\int \left[\ln(l-k^2)+A_0+\frac{1}{2}\right] dk_\tau$$
$$+\frac{\pi^2 i}{2}\sum_\tau \int \frac{k_\sigma k_\tau}{l-k^2} dk_\tau,$$

where

$$A_0 = -\ln L^2 + 1.$$

The left hand side of this equation is equal to one quarter of the desired integral, while the right hand side is equal to

$$-\frac{\pi^2 i}{4} k_\sigma \left\{\ln(l-k^2)+A_0+\frac{1}{2}\right\}.$$

Therefore finally we obtain

$$\int \frac{p_\sigma d^4 p}{(p^2-2pk+l)^2} = \pi^2 i k_\sigma \left(\ln\frac{L^2}{l-k^2} - \frac{3}{2}\right). \tag{47.12}$$

We note that if we had evaluated this integral by introducing a new variable (retaining the same range of integration $N$) $p-k \to p$, we would have obtained $\pi^2 i k_\sigma(\ln(L^2/l-k^2)-1)$; thus as a result of the transformation $p \to p-k$, i.e., of the displacement of the origin of coordinates by $k$, the divergent integral (47.12) acquires an increment $\pi^2 i k_\sigma/2$.

Finally, we investigate the quadratically divergent integral

$$J^{(2)}_{\sigma\tau}(k, l) = \int \frac{p_\sigma p_\tau d^4 p}{(p^2-2pk+l)^2}.$$

On integrating formula (47.12) with respect to $l$ we obtain

$$\int \frac{p_\sigma d^4 p}{p^2-2pk+l} = \pi^2 i k_\sigma \{(l-k^2)\ln(l-k^2)-(l-k^2)$$
$$+(A_0+\tfrac{1}{2})l+\varepsilon(N, k^2)\}, \tag{47.13}$$

where $\varepsilon$ is a function of $k^2$ and $N$, containing a constant which diverges quadratically as $N \to \infty$. It can be easily shown that $\varepsilon$ is a linear function of $k^2$. Indeed, the left hand side of (47.13) acquires a factor $n^3$ as a result of the transformation $k \to nk$. $p \to np$, $l \to n^2 l$, where $n$ is an arbitrary number, which is only possible provided $\varepsilon$ is of the form

$$\varepsilon = \tfrac{1}{2}A_2 + k^2(aA_0 + b), \tag{47.13'}$$

where $A_2$ is a quadratically divergent constant, while $a$ and $b$ are certain numbers ($k^2$ is multiplied by the logarithmically divergent constant $A_0$, and not by $A_2$, since only under this condition will both terms in $\varepsilon$ have the same dimensions).

On differentiating (47.13) with respect to $k_\tau$, we obtain the desired integral

$$\int \frac{p_\tau p_\sigma d^4p}{(p^2-2pk+l)^2} = \frac{\pi^2 i}{2}\delta_{\sigma\tau}\{(l-k^2)\ln(l-k^2)-(l-k^2)+l(A_0+\tfrac{1}{2})$$
$$+k^2(aA_0+b)+\tfrac{1}{2}A_2\}-\pi^2 i k_\sigma k_\tau[\ln(l-k^2)-aA_0-b]. \tag{47.14}$$

In order to obtain $a$ and $b$ and to express $A_2$ in terms of $L$ we can proceed as follows. We write $J^{(2)}_{\sigma\tau}(k, l)$ in the form

$$J^{(2)}_{\sigma\tau}(k, l) = K_{\sigma\tau}(k, l) + \int \frac{p_\sigma p_\tau d^4p}{(p^2+l)^2}, \tag{47.15}$$

where

$$K_{\sigma\tau}(k, l) = \int \frac{p_\sigma p_\tau d^4p}{(p^2-2pk+l)^2} - \int \frac{p_\sigma p_\tau d^4p}{(p^2+l)^2}.$$

The second integral in (47.15) can be evaluated immediately:

$$\int \frac{p_\sigma p_\tau d^4p}{(p^2+l)^2} = \delta_{\sigma\tau}\frac{\pi^2 i}{4}\{L^2-4l\ln L+2l\ln l+l\}.$$

The integral $K_{\sigma\tau}(k, l)$ can with the aid of

$$\frac{1}{\alpha^n}-\frac{1}{\beta^n} = -\int_0^1 \frac{n(\alpha-\beta)}{[(\alpha-\beta)z+\beta]^{n+1}}\,dz \tag{47.16}$$

be brought into the form

$$K_{\sigma\tau}(k, l) = 4k_\mu \int_0^1 dz \int \frac{p_\sigma p_\tau p_\mu d^4p}{[p^2-2pkz+l]^3}.$$

Since

$$\int \frac{p_\sigma p_\tau p_\mu d^4p}{(p^2+l)^3} = 0,$$

then by applying once again formula (47.16) we obtain

$$K_{\sigma\tau}(k,l) = 24k_\mu k_\nu \int_0^1 z\,dz \int_0^1 dt \int \frac{p_\mu p_\nu p_\sigma p_\tau\, d^4p}{[p^2-2pkzt+l]^4}$$

The integral with respect to $p$ appearing in this expression diverges logarithmically, and can be evaluated in the same way as integrals (47.12) and (47.14). By proceeding in this fashion we obtain $a$, $b$ and $A_2$, where $a = -1$, $b = -\frac{5}{6}$ and $A_2 = L^2$.

In conclusion we give a summary of the integrals which we have evaluated:

$$\int \frac{d^4p}{(p^2-2pk+l)^2} = \pi^2 i\left(\ln\frac{L^2}{l-k^2}-1\right),$$

$$\int \frac{p_\sigma d^4p}{(p^2-2pk+l)^2} = \pi^2 ik_\sigma\left(\ln\frac{L^2}{l-k^2}-\frac{3}{2}\right),$$

$$\int \frac{p_\sigma p_\tau d^4p}{(p^2-2pk+l)^2} = -\frac{\pi^2 i}{2}\delta_{\sigma\tau}\left\{(l-k^2)\ln\frac{L^2}{l-k^2}+\frac{5k^2-3l}{6}-\frac{L^2}{2}\right\}$$
$$+\pi^2 ik_\sigma k_\tau\left(\ln\frac{L^2}{l-k^2}-\frac{11}{6}\right), \tag{47.17}$$

$$\int \frac{d^4p}{(p^2-2pk+l)^3} = \frac{\pi^2 i}{2}\frac{1}{l-k^2},$$

$$\int \frac{p_\sigma d^4p}{(p^2-2pk+l)^3} = \frac{\pi^2 i}{2}\frac{k_\sigma}{l-k^2},$$

$$\int \frac{p_\sigma p_\tau d^4p}{(p^2-2pk+l)^3} = \frac{\pi^2 i}{4}\delta_{\sigma\tau}\left(\ln\frac{L^2}{l-k^2}-\frac{3}{2}\right)+\frac{\pi^2 i}{2}\frac{k_\sigma k_\tau}{l-k^2}.$$

### 47.2. *Second Order Electron Self-Energy Part*

We now undertake the explicit evaluation of self-energy and vèrtex parts. We begin by investigating the second order electron self-energy part $\Sigma^{(2)}(p)$. This quantity corresponds to the diagram shown in Fig.

56 (1). In accordance with Feynman's rules (cf. subsection 25.5.) $\Sigma^{(2)}(p)$ has the form

$$\Sigma^{(2)}(p) = \frac{e^2}{(2\pi)^4}\int \gamma_\mu S^c(p-k)\gamma_\mu D^c(k)\, d^4k, \tag{47.18}$$

where

$$S^c(p) = i\frac{i\hat{p}-m}{p^2+m^2}, \qquad D^c(k) = -i\frac{1}{k^2+\lambda^2}.$$

The quantity $\lambda$ in the expression for $D^c(k)$ which represents the photon "mass" has been introduced in order to eliminate the infrared divergence in the electron self-energy part, which appears as a result of regularization (cf. § 30).

On substituting into (47.18) the expressions for $S^c(p)$ and $D^c(k)$ and on utilizing the formula

$$\frac{1}{(k^2+\lambda^2)\,[(p-k)^2+m^2]} = \int\limits_0^1 \frac{dx}{[(px-k)^2+a^2]^2},$$

where

$$a^2 = (p^2+m^2)\,x(1-x)+m^2x^2+\lambda^2(1-x),$$

we rewrite (47.18) in the form

$$\Sigma^{(2)}(p) = -\frac{2e^2}{(2\pi)^4}\int d^4k \int\limits_0^1 dx \frac{2m+i(\hat{p}-\hat{k})}{[(k-px)^2+a^2]^2}.$$

Further, on introducing the change of variable $k \to k+px$ and on taking into account the resultant increment of the linearly divergent integral (cf. formula (47.12)), we obtain

$$\Sigma^{(2)}(p) = -\frac{2e^2}{(2\pi)^4}\left\{\int d^4k \int\limits_0^1 dx \frac{2m+i\hat{p}(1-x)}{(k^2+a^2)^2} - \frac{\pi^2}{4}\hat{p}\right\}. \tag{47.19}$$

We write this expression in the form of an expansion in terms of $\hat{p}-im$

$$\Sigma^{(2)}(p) = \Sigma_1^{(2)}+(\hat{p}-im)\Sigma_0^{(2)}+\Sigma_R^{(2)}(p), \tag{47.20}$$

where $\Sigma_1^{(2)}$ and $\Sigma_0^{(2)}$ are constants. On comparing this expression with (43.48), (43.49) and (45.21), we see that $\Sigma_R^{(2)}$ is the regularized value of the electron self-energy part, $\Sigma_1^{(2)}$ defines (in this approximation) the electromagnetic mass and $\Sigma_0^{(2)}$ defines the constant $Z_1$, where

$$\Sigma_1^{(2)} = -i\delta m, \quad \text{and} \quad \Sigma_0^{(2)} = Z_1^{-1}-1.$$

It follows from (47.19) that

$$\Sigma_1^{(2)} = -\frac{2e^2}{(2\pi)^4}\left\{-\frac{\pi^2}{4}\,im+\int d^4k\int_0^1 \frac{m(1+x)}{[k^2+m^2x^2+\lambda^2(1-x)]^2}\,dx\right\},$$

$$(\hat{p}-im)\Sigma_0^{(2)}+\Sigma_R^{(2)}(p) = \frac{2\pi^2e^2}{(2\pi)^4}\left\{\frac{1}{4}(\hat{p}-im)-\frac{1}{\pi^2}\int d^4k\right.$$

$$\times\int_0^1 \left(2m+i\hat{p}(1-x)\right)\left[\frac{1}{(k^2+a^2)^2}-\frac{1}{[k^2+m^2x^2+\lambda^2(1-x)]^2}\right]dx$$

$$\left.+(\hat{p}-im)\frac{1}{\pi^2 i}\int d^4k\int_0^1\frac{1-x}{[k^2+m^2x^2+\lambda^2(1-x)]^2}\,dx\right\}. \qquad (47.21)$$

For the evaluation of $\Sigma_1^{(2)}$, $\Sigma_0^{(2)}$ and $\Sigma_R^{(2)}(p)$ we utilize

$$\frac{1}{\alpha^2}-\frac{1}{\beta^2} = -2\int_0^1\frac{(\alpha-\beta)}{[\beta+(\alpha-\beta)z]^3}\,dz$$

and set in it $\alpha = k^2+a^2$, and $\beta = k^2+m^2x^2+\lambda^2(1-x)$. We then obtain

$$-\frac{1}{\pi^2}\int d^4k\left[\frac{1}{(k^2+a^2)^2}-\frac{1}{[k^2+m^2x^2+\lambda^2(1-x)]^2}\right]$$

$$= \frac{2}{\pi^2}\int d^4k\int_0^1 dz\,\frac{x(1-x)(p^2+m^2)}{[k^2+m^2x^2+\lambda^2(1-x)+zx(1-x)(p^2+m^2)]^3}$$

$$= ix(1-x)(p^2+m^2)\int_0^1\frac{dz}{m^2x^2+\lambda^2(1-x)+zx(1-x)(p^2+m^2)}\,. \qquad (47.22)$$

On noting that

$$\frac{(\hat{p}+im)[2m+i\hat{p}(1-x)]}{m^2x^2+\lambda^2(1-x)+(1-x)z(p^2+m^2)} = 2m^2i\,\frac{1+x}{m^2x^2+\lambda^2(1-x)}$$

$$+(\hat{p}-im)\frac{m(1+x)+(ip-m)(1-x)\left[1-\dfrac{2m^2x(1+x)z}{m^2x^2+\lambda^2(1-x)}\right]}{m^2x^2+\lambda^2(1-x)+x(1-x)z(p^2+m^2)}$$

and

$$p^2+m^2 = (\hat{p}-im)(\hat{p}+im),$$

we obtain the expressions for $\Sigma_R^{(2)}(p)$, $\Sigma_0^{(2)}$, $\Sigma_1^{(2)}$

$$\Sigma_R^{(2)}(p) = -\frac{2\pi^2 ie^2}{(2\pi)^4}\int\limits_0^1 x(1-x)\,dx$$

$$\times\int\limits_0^1 \frac{m(1+x)+(i\hat{p}-m)(1-x)\left[1-\dfrac{2m^2x(1+x)z}{m^2x^2+\lambda^2(1-x)}\right]}{m^2x^2\lambda^2(1-x)+x(1-x)z(p^2+m^2)}\,dz(\hat{p}-im)^2, \tag{47.23}$$

$$\Sigma_0^{(2)} = \frac{2\pi^2e^2}{(2\pi)^4}\left\{\frac{1}{4}-\frac{1}{2}A_0-\int\limits_0^1(1-x)\ln\,[m^2x^2+\lambda^2(1-x)]dx\right.$$

$$\left.-2\int\limits_0^1\frac{x(1-x^2)}{x^2+\dfrac{\lambda^2}{m^2}(1-x)}\,dx\right\},$$

$$\Sigma_1^{(2)} = \frac{2\pi^2}{(2\pi)^4}\,ie^2m\left\{\frac{1}{4}+\int\limits_0^1(1+x)\left[A_0+\ln\left(m^2x^2+\lambda^2(1-x)\right)\right]dx\right\},$$

where

$$A_0 = -2\ln L+1.$$

On assuming $\lambda \leqslant m$ we finally obtain the expression for the regularized second order electron self-energy part

$$\Sigma_R^{(2)}(p) = -\frac{i\alpha}{2\pi}\frac{(\hat{p}-im)^2}{m}\left\{\frac{1}{2(1-\varrho)}\left(1-\frac{2-3\varrho}{1-\varrho}\ln\varrho\right)\right.$$

$$\left.+\frac{i\hat{p}-m}{m}\left[\frac{1}{2\varrho(1-\varrho)}\left(-2-\varrho+\frac{4-8\varrho+\varrho^2}{1-\varrho}\ln\varrho\right)-\frac{1}{\varrho}-\frac{1}{\varrho}\ln\frac{\lambda^2}{\mathrm{m}^2}\right]\right\}, \tag{47.24}$$

where

$$\varrho = \frac{p^2+m^2}{m^2}.$$

We see that the regularized value of $\Sigma_R^{(2)}(p)$ contains the photon "mass" $\lambda$. This quantity did not appear in the original expression for $\Sigma^{(2)}(p)$ and has arisen as a result of regularization.

In view of the "infrared divergence" this expression is inapplicable for $|p^2| \ll \lambda^2$. For $|\varrho| \gg 1$, i.e., $|p|^2 \gg m^2$ formula (47.24) yields

$$\Sigma_R^{(2)}(p) = \frac{\alpha}{4\pi} \hat{p} \ln \frac{p^2}{m^2}. \tag{47.25}$$

The divergent constants $\Sigma_0^{(2)}$ and $\Sigma_1^{(2)}$ have the form for $\lambda \to 0$

$$\Sigma_0^{(2)} = \frac{\alpha}{2\pi} \left\{ \frac{1}{2} \ln \frac{L^2}{m^2} + \ln \frac{\lambda^2}{m^2} + \frac{9}{4} \right\}, \quad \Sigma_1^{(2)} = -i \frac{3}{4\pi} \alpha m \left( \ln \frac{L^2}{m^2} + \frac{1}{2} \right). \tag{47.26}$$

From this we obtain the expression for the electromagnetic mass of the electron,

$$\delta m = \frac{3}{4\pi} \alpha m \left( \ln \frac{L^2}{m^2} + \frac{1}{2} \right). \tag{47.27}$$

Formula (47.24) has been derived on the assumption that the arbitrary function $d_l$ which appears in $D^c_{\mu\nu}(k)$ (cf. (17.21)) has been chosen equal to zero. It can be easily shown that for an arbitrary gauge for the potentials the quantity $\Sigma_R^{(2)}$ is defined by the formula

$$\Sigma_R^{(2)}(p) = \frac{\alpha}{4\pi} (\hat{p} - im) \left\{ \left( 3 + d_l^0 \frac{p^2}{im^2} \right) \frac{1}{im} (\hat{p} + im) \frac{m^2}{p^2} \ln \frac{p^2 + m^2}{m^2} \right.$$
$$\left. - (3 - d_l^0) \ln \frac{\lambda^2}{m^2} + d_l^0 \left( \frac{im\hat{p}}{p^2} - 1 \right) \right\} \tag{47.24'}$$

where $d_l^0 = d_l(0)$.

### 47.3. *Second Order Photon Self-Energy Part*

We now evaluate the second order photon self-energy part $\Pi_{\mu\nu}^{(2)}(k)$. This quantity corresponds to the diagram shown in Fig. 58(*1*). In accordance with Feynman's rules the tensor $\Pi_{\mu\nu}^{(2)}(k)$ is defined by the formula

$$\Pi_{\mu\nu}^{(2)}(k) = \frac{e^2}{(2\pi)^4} Sp \int \frac{\gamma_\mu (i\hat{p} - m) \gamma_\nu \left( i(\hat{p} - \hat{k}) - m \right)}{(p^2 + m^2) \left[ (p-k)^2 + m^2 \right]} d^4 p. \tag{47.28}$$

This expression can be written as

$$\Pi_{\mu\nu}^{(2)}(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi^{(2)}(k) + \delta_{\mu\nu} A(k), \tag{47.28'}$$

where

$$\Pi^{(2)}(k)=\frac{1}{3}\Pi^{(2)}_{\mu\mu}(k)-\frac{4}{3}\frac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k),$$

$$A(k)=\frac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k).$$

In future we shall be interested in the transverse part of $\Pi^{(2)}_{\mu\nu}$, which is determined by the first term in (47.28′) since only this term has a physical meaning.

First of all we evaluate $\frac{1}{3}\Pi^{(2)}_{\mu\mu}$:

$$\frac{1}{3}\Pi^{(2)}_{\mu\mu}(k)=\frac{1}{3}\frac{e^2}{(2\pi)^4}\operatorname{Sp}\int\gamma_\mu\frac{i\hat{p}-m}{p^2+m^2}\gamma_\nu\frac{i(\hat{p}-\hat{k})-m}{[(p-k)^2+m^2]}\,d^4p$$

$$=\frac{e^2}{3(2\pi)^4}\operatorname{Sp}\int\frac{4m^2+2\hat{p}\,(\hat{p}-\hat{k})}{(p^2+m^2)\,[(p-k)^2+m^2]}\,d^4p.$$

After having evaluated the trace of $4m^2+2\hat{p}(\hat{p}-\hat{k})$ we rewrite $\frac{1}{3}\Pi^{(2)}_{\mu\mu}$ in the form

$$\tfrac{1}{3}\Pi^{(2)}_{\mu\mu}(k)=\frac{8e^2}{3(2\pi)^4}\int\frac{(p-k)^2+m^2+pk+m^2-k^2}{(p^2+m^2)[(p-k)^2+m^2]}\,d^4p$$

$$=\frac{8e^2}{3(2\pi)^1}\int\frac{d^4p}{p^2+m^2}+\frac{8e^2}{3(2\pi)^4}k_\sigma\int\frac{p_\sigma}{(p^2+m^2)[(p-k)^2+m^2]}\,d^4p$$

$$+\frac{8e^2}{3(2\pi)^4}(m^2-k^2)\int\frac{1}{(p^2+m^2)\,[(p-k)^2+m^2]}\,d^4p.$$

The first of the integrals appearing in this expression diverges quadratically and can be written in the form

$$\int\frac{d^4p}{p^2+m^2}=-i\pi^2m^2\ln\frac{L^2}{m^2}+\pi^2iL^2.$$

For the evaluation of the second and the third integrals we use the formula (cf. (47.3))

$$\frac{1}{(p^2+m^2)[(p-k)^2+m^2]}=\int\limits_0^1\frac{dx}{[p^2-2pK_x+l_x]^2},$$

where $K_x = kx$ and $l_x = k^2x+m^2$. Moreover, on utilizing the values of the integrals (47.9) and (47.12) we obtain

$$\int_0^1 dx \int \frac{d^4p}{[p^2-2pK_x+l_x]^2} = -\pi^2 i \int_0^1 dx\left\{-\ln\frac{L^2}{m^2}+1+\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]\right\},$$

$$\int_0^1 dx \int \frac{p_\sigma d^4p}{[p^2-2pK_x+l_x]^2} = -\pi^2 i k_\sigma \int_0^1 x\,dx\left\{-\ln\frac{L^2}{m^2}+\frac{3}{2}+\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]\right\}.$$

Therefore

$$\tfrac{1}{3}\Pi^{(2)}_{\mu\mu}(k) = \frac{8e^2\pi^2 i}{3(2\pi)^4}\left\{L^2-m^2\ln\frac{L^2}{m^2} - (m^2-k^2)\int_0^1\left(-\ln\frac{L^2}{m^2}+1+\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]\right)dx - k^2\int_0^1 x\left(-\ln\frac{L^2}{m^2}+\frac{3}{2}+\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]\right)dx\right\}. \quad (47.29)$$

We now evaluate $\dfrac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k)$:

$$\frac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k) = \frac{e^2}{(2\pi)^4}\,\mathrm{Sp}\int\frac{\hat{k}}{k^2}\,\frac{i\hat{p}-m}{p^2+m^2}\,\hat{k}\,\frac{i(\hat{p}-\hat{k})-m}{(p-k)^2+m^2}\,d^4p.$$

On noting that

$$\mathrm{Sp}\{\hat{k}(i\hat{p}-m)\hat{k}\,[i(\hat{p}-\hat{k})-m] = 4\left(k^2m^2+k^2p^2+k^2(kp)-2(kp)^2\right),$$

we obtain

$$\frac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k) = \frac{4e^2}{(2\pi)^4}\left\{\int\frac{d^4p}{p^2+m^2}+\int\frac{-k^2+3kp-2(kp)^2/k^2}{(p^2+m^2)\,[(p-k)^2+m^2]}\,d^4p\right\}.$$

The integrals appearing in this expression are equal to

$$\int \frac{d^4p}{p^2+m^2} = i\pi^2\big((L^2-m^2\ln(L^2/m^2)\big),$$

$$\int \frac{d^4p}{(p^2+m^2)[(p-k)^2+m^2]} = -i\pi^2\int_0^1 dx\big[-\ln(L^2/m^2)$$
$$+1+\ln\big(1+(k^2/m^2)(1-x)x\big)\big],$$

$$\int \frac{p_\sigma d^4p}{(p^2+m^2)[(p-k)^2+m^2]} = -i\pi^2k_\sigma\int_0^1 x\,dx\,\big[-\ln(L^2/m^2)$$
$$+\tfrac{3}{2}+\ln\big(1+(k^2/m^2)(1-x)x\big)\big],$$

$$\int \frac{p_\mu p_\nu d^4p}{(p^2+m^2)[(p-k)^2+m^2]}$$
$$= -\frac{i\pi^2}{2}\delta_{\mu\nu}\int_0^1 dx\Big[\big(m^2+k^2x(1-x)\big)\ln\big(L^2/(m^2+k^2x(1-x))\big)$$
$$+\tfrac{1}{6}\big(5k^2x^2-3(m^2+k^2x)\big)-\tfrac{1}{2}L^2\Big]$$
$$+i\pi^2k_\mu k_\nu\int_0^1 x^2\,dx\,\Big[\ln\big[L^2/\big(m^2+k^2x(1-x)\big)\big]-\tfrac{11}{6}\Big].$$

Therefore

$$\frac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k) = \frac{4i\pi^2e^2}{(2\pi)^4}\Big[\tfrac{1}{2}(L^2-m^2)$$
$$-\int_0^1\big(m^2-k^2x+3k^2x(1-x)\big)\ln\big(1+(k^2/m^2)x(1-x)\big)dx\Big]. \qquad (47.29')$$

On utilizing (47.29) and (47.29′) we obtain the expression for $\Pi^{(2)}(k)$

$$\Pi^{(2)}(k) = \frac{1}{3}\Pi^{(2)}_{\mu\mu}(k) - \frac{4}{3}\frac{k_\mu k_\nu}{k^2}\Pi^{(2)}_{\mu\nu}(k)$$

$$= \frac{8\pi^2ie^2}{(2\pi)^4}\Big[k^2\int_0^1 x(1-x)\ln\big(1+(k^2/m^2)x(1-x)\big)dx$$
$$-\frac{k^2}{6}\Big(\ln(L^2/m^2)-\frac{5}{6}\Big)\Big].$$

This expression must be regularized. To achieve this we must subtract from $\Pi^{(2)}(k)$ the quantity $\Pi^{(2)}(0)+k^2(\partial\Pi^{(2)}(k)/\partial k^2)_{k^2=0}$ which is equal to

$$\Pi^{(2)}(0)+k^2\left(\frac{\partial\Pi^{(2)}(k)}{\partial k^2}\right)_{k^2=0} = -\frac{8\pi^2 ie^2}{(2\pi)^4}\frac{k^2}{6}\left(\ln\frac{L^2}{m^2}-\frac{5}{6}\right). \tag{47.30}$$

Therefore the regularized expression for $\Pi^{(2)}(k)$ has the form

$$\Pi_R^{(2)}(k) = \frac{8\pi^2 ie^2}{(2\pi)^4}k^2\int_0^1 x(1-x)\ln[1+(k^2/m^2)x(1-x)]\,dx. \tag{47.31}$$

On comparing expression (47.30) with (43.53) we obtain the second perturbation theory approximation for $Z^{-1}=(e^2/e_c^2)$

$$Z^{-1} = 1+\frac{\alpha}{3\pi}\left(\ln\frac{L^2}{m^2}-\frac{5}{6}\right),$$

whence

$$Z = 1-\frac{\alpha}{3\pi}\left(\ln\frac{L^2}{m^2}-\frac{5}{6}\right). \tag{47.30$'$}$$

On introducing into (47.31) the new variable $x=\frac{1}{2}(1+\eta)$ we rewrite $\Pi_R^{(2)}(k)$ in the form

$$\Pi_R^{(2)}(k) = \frac{i\alpha}{4\pi}k^2\int_{-1}^{1}(1-\eta^2)\ln\left[1+\frac{1}{4}\frac{k^2}{m^2}(1-\eta^2)\right]d\eta. \tag{47.31$'$}$$

But

$$\int_{-1}^{1}\ln\left[1+\frac{1}{4}\frac{k^2}{m^2}(1-\eta^2)\right]d\eta = -4(1-\theta\cot\theta),$$

$$\int_{-1}^{1}\eta^2\ln\left[1+\frac{1}{4}\frac{k^2}{m^2}(1-\eta^2)\right]d\eta = -\frac{4}{9}+\frac{4}{3}(1-\theta\cot\theta)\cot^2\theta,$$

where $\theta$ is related to $k^2/m^2$ by the expression

$$\sin^2\theta = -\frac{k^2}{4m^2}.$$

Therefore

$$\Pi_R^{(2)}(k) = \frac{i\alpha k^2}{\pi}\left\{\frac{1}{9}+(1-\theta\cot\theta)\frac{4m^2-2k^2}{3k^2}\right\}. \tag{47.32}$$

We give the expressions for $\Pi_R^{(2)}(k)$ in two limiting cases.
If $|k^2| \ll m^2$, then

$$\Pi_R^{(2)}(k) = i\frac{\alpha}{15\pi}k^2\frac{k^2}{m^2}. \tag{47.33}$$

If $|k^2| \gg m^2$, then

$$\Pi_R^{(2)}(k) = i\frac{\alpha}{3\pi}|k^2|\ln\frac{|k^2|}{m^2}. \tag{47.34}$$

If we have the expression for $\Pi_R^{(2)}(k)$ we can determine the second approximation to the photon Green's function and the increment $\delta A_\mu^{(e)}$ of the external potential of the electromagnetic field due to the polarization of the electron-positron vacuum (cf. subsection 50.3).

The Fourier component of the increment of the external potential $A_\mu^{(e)}(q)$ due to vacuum polarization is determined by

$$\delta A_\mu^{(e)}(q) = D^c(q)\Pi_{\mu\nu}^{(3)}(q)A_\nu^{(e)}(q). \tag{47.35}$$

On substituting into this expression instead of $\Pi_{\mu\nu}^{(2)}(q)$ the regularized value (47.32) we obtain

$$\delta A_\mu^{(e)}(q) = \frac{\alpha}{\pi}\frac{1}{q^2}(q_\mu q_\nu - \delta_{\mu\nu}q^2)\left[\frac{4m^2-2q^2}{-3q^2}(1-\theta\cot\theta)-\frac{1}{9}\right]A_\nu^{(e)}(q). \tag{47.36}$$

On applying to $\delta A_\mu^{(e)}(x)$ the operator $-\Box$, we obtain the increment in the external current $j_\mu^{(e)}(x)$ due to vacuum polarization. The Fourier component of this increment is equal to

$$\begin{aligned}\delta j_\mu^{(e)}(q) &= q^2\,\delta A_\mu^{(e)}(q)\\ &= \frac{\alpha}{\pi}(q_\mu q_\nu - \delta_{\mu\nu}q^2)\left[\frac{4m^2-2q^2}{-3q^2}(1-\theta\cot\theta)-\frac{1}{9}\right]A_\nu^{(e)}(q).\end{aligned} \tag{47.37}$$

It can be easily shown that $\delta j_\mu^{(e)}(q)$ satisfies the law of conservation of charge:

$$q_\mu\,\delta j_\mu^{(e)}(q) = 0. \tag{47.37'}$$

On noting that the Fourier component of the external current density is given by $j_\mu^{(e)}(q) = q^2 A_\mu^{(e)}(q)$ and that $q_\mu A_\mu^{(e)}(q) = 0$ we can rewrite the expression for $\delta j_\mu^{(e)}(q)$ in the form

$$\delta j_\mu^{(e)}(q) = -\frac{\alpha}{\pi}\left[\frac{4m^2-2q^2}{-3q^2}(1-\theta\cot\theta)-\frac{1}{9}\right]j_\mu^{(e)}(q). \tag{47.38}$$

On expanding the quantity which appears in the square brackets into a series in powers of $q_\mu$ and on confining ourselves to fourth order terms we obtain

$$\delta j_\mu^{(e)}(q) = \frac{\alpha}{15\pi m^2} q^2 j_\mu^{(e)}(q). \tag{47.39}$$

On going over from the Fourier components to the functions themselves in terms of the coordinates and the time, we obtain

$$\delta j_\mu^{(e)}(x) = -\frac{\alpha}{15\pi m^2} \Box j_\mu^{(e)}(x). \tag{47.40}$$

If in the expansion of (47.38) in powers of $q_\mu$ we retain terms of the sixth order, then in place of (47.40) we obtain (199)

$$\delta j_\mu^{(e)}(x) = -\frac{\alpha}{15\pi m^2} \Box j_\mu^{(e)}(x) - \frac{\alpha}{170\pi}\left(\frac{1}{m^2}\Box\right)^2 j_\mu^{(e)}(x). \tag{47.41}$$

### 47.4. *Third Order Vertex Part in the Case of External Electron Lines*

We proceed to evaluate the third order vertex part $\Lambda_\mu^{(3)}(p_1, p_2; q)$. This quantity corresponds to the diagram shown in Fig. 65 (1). In accordance with Feynman's rules $\Lambda_\mu^{(3)}(p_1, p_2; q)$ can be written in the form

$$\Lambda_\mu^{(3)}(p_1, p_2; q) = \frac{e^2}{(2\pi)^4}\int \gamma_\nu S^c(p_2-k)\gamma_\mu S^c(p_1-k)\gamma_\nu D^c(k)\, d^4k. \tag{47.42}$$

We shall see later that the vertex part $\Lambda_\mu^{(3)}(p_1, p_2; q)$, like $\Sigma_R^{(2)}(p)$, will contain after regularization an infrared divergence. (If $p_1$ and $p_2$ are momenta of a free electron, then $\Lambda_\mu^{(3)}(p_1, p_2; q)$ contains an infrared divergence before regularization.) Therefore, for the evaluation of $\Lambda_\mu^{(3)}(p_1, p_2; q)$ we shall use, as we did in subsection 47.3, the expression for $D^c(k)$

$$D^c(k) = \frac{1}{i}\frac{1}{k^2+\lambda^2},$$

where $\lambda$ is the photon "mass". On substituting the expressions for $D^c(k)$ and $S^c(p)$ into (47.42) we obtain

$$\Lambda_\mu^{(3)}(p_1, p_2; q) = \frac{ie^2}{(2\pi)^4}\{\gamma_\nu(i\hat{p}_2-m)\gamma_\mu(i\hat{p}_1-m)\gamma_\nu J$$
$$-[\gamma_\nu(i\hat{p}_2-m)\gamma_\mu\gamma_\sigma\gamma_\nu+\gamma_\nu\gamma_\sigma\gamma_\mu(i\hat{p}_1-m)\gamma_\nu]J_\sigma-\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\nu J_{\sigma\tau}\}, \tag{47.43}$$

where

$$\begin{aligned} J &= \int \frac{d^4k}{(k^2-2p_1k+p_1^2+m^2)(k^2-2p_2k+p_2^2+m^2)(k^2+\lambda^2)}, \\ J_\sigma &= \int \frac{k_\sigma\, d^4k}{(k^2-2p_1k+p_1^2+m^2)(k^2-2p_2k+p_2^2+m^2)(k^2+\lambda^2)}, \\ J_{\sigma\tau} &= \int \frac{k_\sigma k_\tau\, d^4k}{(k^2-2p_1k+p_1^2+m^2)(k^2-2p_2k+p_2^2+m^2)(k^2+\lambda^2)}. \end{aligned} \tag{47.43'}$$

Of these three integrals only the third diverges (logarithmically) for large $|k|$.

In the region of small $k$ (for $\lambda = 0$) only the first integral can diverge, and only in the case when $p_1$ and $p_2$ are momenta of a free electron. For this reason we can set $\lambda = 0$ in the evaluation of the integrals $J_\sigma$ and $J_{\sigma\tau}$.

We investigate first of all the case when $p_1$ and $p_2$ are momenta of a free electron. In this case the integrals $J$, $J_\sigma$, $J_{\sigma\tau}$ can be evaluated simply.

On taking into account the fact that $p_1^2 = p_2^2 = -m^2$, and on utilizing the formulas

$$\frac{1}{(k^2-2p_1k)(k^2-2p_2k)} = \int_0^1 \frac{dy}{(k^2-2p_yk)^2},$$

$$\frac{1}{(k^2-2p_yk)^2(k^2+\lambda^2)} = \int_0^1 \frac{2x\,dx}{(k^2-2p_xk+l_x)^3},$$

where

$$p_y = yp_1+(1-y)p_2, \quad p_x = xp_y, \quad l_x = (1-x)\lambda^2,$$

we write $J$, $J_\sigma$, $J_{\sigma\tau}$ in the form

$$\begin{aligned} J &= \int_0^1 dy \int_0^1 2x\,dx \int \frac{d^4k}{(k^2-2p_xk+l_x)^3}, \\ J_\sigma &= \int_0^1 dy \int_0^1 2x\,dx \int \frac{k_\sigma d^4k}{(k^2-2p_xk)^3}, \\ J_{\sigma\tau} &= \int_0^1 dy \int_0^1 2x\,dx \int \frac{k_\sigma k_\tau d^4k}{(k^2-2p_xk)^3}. \end{aligned} \tag{47.44}$$

The integrals over $k$ appearing in the foregoing expressions are, in accordance with (47.17), equal to

$$\int \frac{d^4k}{(k^2-2p_xk+l_x)^3} = \frac{\pi^2 i}{2}\cdot\frac{1}{l_x-p_x^2},$$

$$\int \frac{k_\sigma d^4k}{(k^2-2p_xk)^3} = -\frac{\pi^2 i}{2p_x^2}p_{x\sigma},$$

$$\int \frac{k_\sigma k_\tau d^4k}{(k^2-2p_xk)^3} = -\frac{\pi^2 i}{4}\sigma_{\sigma\tau}\left[-\ln\frac{L^2}{-p_x^2}+\frac{3}{2}\right]-\frac{\pi^2 i}{2p_x^2}p_{x\sigma}p_{x\tau}.$$

On carrying out the integration over $x$ we obtain

$$J = -\pi^2 i\int_0^1 \frac{dy}{p_y^2}(\ln\sqrt{-p_y^2}-\ln\lambda),$$

$$J_\sigma = -\pi^2 i\int_0^1 \frac{p_{y\sigma}dy}{p_y^2}, \tag{47.44'}$$

$$J_{\sigma\tau} = -\frac{\pi^2 i}{4}\int_0^1\left\{\delta_{\sigma\tau}\left[-\ln\frac{L^2}{-p_y^2}+\frac{1}{2}\right]+2\frac{p_{y\sigma}p_{y\tau}}{p_y^2}\right\}dy.$$

In order to evaluate these integrals we note that

$$\begin{aligned}-p_y^2 &= -[p_1+q(1-y)]^2 = m^2+q^2(1-y)y\\ &= m^2[1-4y(1-y)\sin^2\theta]; \quad q = p_2-p_1,\end{aligned} \tag{47.45}$$

where $\theta$ is related to $q$ by the expression

$$-q^2 = 4m^2\sin^2\theta. \tag{47.45'}$$

Further, in place of $y$ we introduce the new variable $\xi$

$$2y-1 = \frac{\tan\xi}{\tan\theta}.$$

Then

$$-p_y^2 = m^2\frac{\cos^2\theta}{\cos^2\xi}, \quad p_{y\sigma} = \frac{\tan\xi}{2\tan\theta}(p_{1\sigma}-p_{2\sigma})+\frac{1}{2}(p_{1\sigma}+p_{2\sigma}), \tag{47.46}$$

and the integral $J$ assumes the form

$$J = \frac{2\pi^2 i}{m^2\sin 2\theta}\left\{-\int_0^\theta \ln\frac{\cos\xi}{\cos\theta}d\xi-\int_0^\theta \ln\frac{\lambda}{m}\,d\xi\right\}.$$

Since

$$\int_0^\theta \ln\frac{\cos\xi}{\cos\theta}\,d\xi = \int_0^\theta \xi\tan\xi\,d\xi,$$

then we finally have

$$J = -\frac{2\pi^2 i}{m^2\sin 2\theta}\left\{\int_0^\theta \xi\tan\xi\,d\xi + \theta\ln\frac{\lambda}{m}\right\}. \tag{47.47}$$

The integrals $J_\sigma$ and $J_{\sigma\tau}$ can be evaluated immediately after (47.46) and (47.45′) have been substituted into them:

$$J_\sigma = \frac{\pi^2 i}{m^2}\frac{\theta}{\sin 2\theta}(p_{1\sigma}+p_{2\sigma}),$$

$$J_{\sigma\tau} = \frac{\pi^2 i}{2}\left\{\delta_{\sigma\tau}\left(\ln\frac{L}{m}-\frac{1}{4}\right)+\frac{\theta(p_{1\sigma}+p_{2\sigma})(p_{1\tau}+p_{2\tau})}{2m^2\sin 2\theta}\right.$$
$$\left.+\left(\delta_{\sigma\tau}+\frac{q_\sigma q_\tau}{-q^2}\right)(1-\theta\cot\theta)\right\}. \tag{47.48}$$

We now return to formula (47.43) defining $\Lambda_\mu^{(3)}(p_1, p_2; q)$ and simplify the coefficients appearing in front of the integrals $J$, $J_\sigma$, $J_{\sigma\tau}$.

We investigate first of all the expression which appears in front of $J$; obviously it can be rewritten in the form

$$\gamma_\nu(i\hat{p}_2-m)\gamma_\mu(i\hat{p}_1-m)\gamma_\nu = 2\hat{p}_1\gamma_\mu\hat{p}_2-2im\,[\gamma_\mu\hat{p}_2+\hat{p}_2\gamma_\mu]$$
$$-2im\,[\gamma_\mu\hat{p}_1+\hat{p}_1\gamma_\mu]-2m^2\gamma_\mu.$$

Since in the evaluation of the matrix elements the quantity $\Lambda_\mu^{(3)}(p_1, p_2;\ q)$ is always multiplied on the left by $\bar{u}_2$, and on the right by $u_1$, where $u_2$ and $u_1$ are spinor amplitudes for an electron of momenta $p_2$ and $p_1$, then in the last expression the matrix $i\hat{p}_1$ appearing on the right, and the matrix $i\hat{p}_2$ appearing on the left, can be replaced by $-m$ (if the matrix $i\hat{p}_1$ appears on the left instead of the matrix $i\hat{p}_2$, or the matrix $i\hat{p}_2$ appears on the right instead of $i\hat{p}_1$, then before carrying out this replacement we must make the substitution $\hat{p}_1 = \hat{p}_2-\hat{q}$, $\hat{p}_2 = \hat{p}_1+\hat{q}$). After such transformations the factor in front of $J$ assumes the form

$$\gamma_\nu(i\hat{p}_2-m)\gamma_\mu(i\hat{p}_1-m)\gamma_\nu = 4m^2\gamma_\mu-2\hat{q}\gamma_\mu\hat{q}.$$

The last term on the right hand side of this expression can be replaced by $2q^2\gamma_\mu$. Indeed, on noting that $\gamma_\mu\hat{q} = -\hat{q}\gamma_\mu+2q_\mu$, and on multiplying this equation on the left by $\hat{q}$, we obtain $\hat{q}\gamma_\mu\hat{q} = -q^2\gamma_\mu+2q_\mu\hat{q}$.

But the last term gives no contribution in the evaluation of the matrix elements since $\bar{u}_2\hat{q}u_1 = \bar{u}_2(\hat{p}_2 - \hat{p}_1)u_1 = 0$, and can therefore be neglected. Thus, finally, the coefficient appearing in front of $J$ in the expression for $\Lambda_\mu^{(3)}(p_1, p_2; q)$ can be written in the form

$$\gamma_\nu(i\hat{p}_2-m)\gamma_\mu(i\hat{p}_1-m)\gamma_\nu = 4m^2\gamma_\mu+2q^2\gamma_\mu.$$

The factors appearing in front of the integrals $J_\sigma$ and $J_{\sigma\tau}$ can be subjected to an analogous transformation:

$$\gamma_\nu(i\hat{p}_2-m)\gamma_\mu\gamma_\sigma\gamma_\nu+\gamma_\nu\gamma_\sigma\gamma_\mu(i\hat{p}_1-m)\gamma_\nu = -4m\delta_{\sigma\mu}+2i(\hat{q}\gamma_\mu\gamma_\sigma-\gamma_\sigma\gamma_\mu\hat{q}),$$
$$\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\nu = -2\gamma_\tau\gamma_\mu\gamma_\sigma.$$

On substituting these expressions and the values of the integrals (47.47), (47.48) into (47.43) we obtain the expression for $\Lambda_\mu^{(3)}(p_1, p_2; q)$

$$\Lambda_\mu^{(3)}(p_1, p_2; q) = \frac{ie^2}{(2\pi)^4}\Big\{(4m^2+2q^2)\gamma_\mu J+\Big[4im(p_1+p_2)_\mu-2(\hat{p}_1+\hat{p}_2)\gamma_\mu\hat{q}$$
$$+2\hat{q}\gamma_\mu(\hat{p}_1+\hat{p}_2)+\frac{1}{2}(\hat{p}_1+\hat{p}_2)\gamma_\mu(\hat{p}_1+\hat{p}_2)\Big]K_1+2\hat{q}\gamma_\mu\hat{q}K_2-4\gamma_\mu K_3\Big\},$$

where

$$K_1 = \frac{\pi^2 i}{m^2}\frac{\theta}{\sin 2\theta}, \qquad K_2 = -\frac{\pi^2 i}{2q^2}(1-\theta\cot\theta),$$
$$K_3 = \frac{\pi^2 i}{2}\left[1-\theta\cot\theta+\ln\frac{L}{m}-\frac{1}{4}\right].$$

In this expression we can replace, as we have done previously, the matrix $i\hat{p}_1$ appearing on the right and the matrix $i\hat{p}_2$ appearing on the left by $-m$. As a result of this we obtain

$$\Lambda_\mu^{(3)}(p_1, p_2; q) = -\frac{e^2\gamma_\mu}{(2\pi)^2}\left\{\frac{2\theta}{\tan 2\theta}\left(\ln\frac{m}{\lambda}-1\right)-\frac{2}{\tan 2\theta}\int\limits_0^\theta \xi\tan\xi\,d\xi\right.$$
$$\left.-\frac{\theta}{2}\tan\theta-\frac{1}{8}-\frac{1}{4}\ln\frac{L^2}{m^2}\right\}+\frac{ie^2}{32\pi^2 m}(\gamma_\mu\hat{q}-\hat{q}\gamma_\mu)\frac{2\theta}{\sin 2\theta}. \qquad (47.49)$$

It remains now to regularize $\Lambda_\mu^{(3)}(p_1, p_2; q)$. In order to do this we must subtract from $\Lambda_\mu^{(3)}(p_1, p_2; q)$ the quantity $\Lambda_\mu^{(3)}(p_0, p_0; 0)$, where $p_0$ is the momentum of a free electron. Since in the case under consideration $p_1$ and $p_2$ are momenta of a free electron, then in order to regularize $\Lambda_\mu^{(3)}(p_1, p_2; q)$ it is sufficient to subtract from it its value at $q = 0$. This value is obviously equal to

$$\Lambda_\mu^{(3)}(p_1, p_2; 0) = -\gamma_\mu \frac{e^2}{(2\pi)^2}\left\{-\frac{1}{4}\ln\frac{L^2}{m^2} - \frac{9}{8} + \ln\frac{m}{\lambda}\right\} \equiv \gamma_\mu L^{(3)}. \quad (47.50)$$

The quantity $L^{(3)}$ is related to $Z_1$ by $Z_1^{-1} = 1 + L^{(3)}$. It can be easily shown that

$$L^{(3)} = -\Sigma_0^2. \quad (47.51)$$

The regularized value of $\Lambda_{\mu R}^{(3)}(p_1, p_2; q)$ is of the form

$$\Lambda_{\mu R}^{(3)}(p_1, p_2; q) = -\frac{\alpha}{\pi}\gamma_\mu\left\{\left(\frac{2\theta}{\tan 2\theta} - 1\right)\left(\ln\frac{m}{\lambda} - 1\right) - \frac{2}{\tan 2\theta}\int_0^\theta \xi \tan\xi \, d\xi - \frac{\theta}{2}\tan\theta\right\} + \frac{i\alpha}{8\pi m}(\gamma_\mu \hat{q} - \hat{q}\gamma_\mu)\frac{2\theta}{\sin 2\theta}. \quad (47.52)$$

For $|p^2| \gg m^2$, $|q^2| \gg m^2$

$$\Lambda_{\mu R}^{(3)}(p_1, p_2; q) = -\frac{\alpha}{\pi}\gamma_\mu\left\{\ln\frac{|q^2|}{4m^2}\ln\frac{m}{\lambda} + \frac{1}{4}\ln^2\frac{|q^2|}{4m^2}\right\}. \quad (47.53)$$

### 47.5. *Third Order Vertex Part in the Case of One External Electron Line*

We now consider the third order vertex part in the case when only one electron line represents a free state, while the second line represents a state which may or may not be free. Insofar as the photon line is concerned, we assume it to be an external line. Under these assumptions the vertex part can be written in the form

$$\Lambda_\mu^{(3)}(p, p+k; k) = \frac{ie^2}{(2\pi)^4}\int \gamma_\nu \frac{i(\hat{p}+\hat{k}-\hat{q}) - m}{q^2 - 2qp - 2pk + \varkappa m^2} \times \gamma_\mu \frac{i(\hat{p}-\hat{q}) - m}{q^2 - 2qp}\gamma_\nu \frac{d^4 q}{q^2 + \lambda^2}, \quad (47.54)$$

where

$$k^2 = 0, \quad p^2 = -m^2, \quad (p+k)^2 + m^2 = 2pk = \varkappa m^2.$$

In evaluating $\Lambda_\mu^{(3)}(p, p+k; k)$ we must keep in mind that on the side of the free electron line, i.e., on the right hand side, the matrix $\Lambda_\mu^{(3)}(p, p+k; k)$ is multiplied by the bispinor $u$ which satisfies $(i\hat{p}+m)u = 0$.

This fact enables us to utilize, for example, a relation such as $\hat{p}\hat{a}\hat{p} = 2impa+m^2\hat{a}$, since

$$\hat{p}\hat{a}\hat{p}u = im\hat{p}\hat{a}u = im\,(2pa-\hat{a}\hat{p})\,u = im\,(2pa-im\hat{a})\,u.$$

Moreover, we must keep in mind that $\Lambda_\mu^{(3)}(p,p+k;k)$ always appears in the matrix element in the form of the product $\Lambda_\mu^{(3)}e_\mu$ where $e_\mu$ is the photon polarization vector which satisfies the condition $e_\mu k_\mu = 0$. This enables us in the course of further calculations to utilize in the expression for $\Lambda_\mu^{(3)}e_\mu$ the relations

$$\hat{k}\hat{e}\hat{k} = 0,$$

$$\hat{k}\hat{e}\hat{p}+\hat{p}\hat{e}\hat{k} = 2(ep)\hat{k}-\varkappa m^2\hat{e},$$

$$im\hat{e}\hat{k}+\hat{p}\hat{k}\hat{e} = \varkappa m^2\hat{e}-2(ep)\hat{k},$$

$$im\hat{e}\hat{p}-\hat{p}\hat{e}\hat{k}-m^2\hat{e} = -2m^2\hat{e}+\hat{p}\hat{k}\hat{e}.$$

On utilizing these relations we can write $\Lambda_\mu^{(3)}e_\mu$ in the form

$$\Lambda_\mu^{(3)}(p,\, p+k;\, k)e_\mu = \frac{ie^2}{(2\pi)^4}\{2\gamma_\sigma\hat{e}\gamma_\tau J_{\sigma\tau}+2(im\hat{e}\gamma_\sigma-\gamma_\sigma\hat{e}\hat{k}+\hat{p}\gamma_\sigma\hat{e})J_\sigma$$

$$-2[\hat{p}\hat{k}\hat{e}+2im(ep)]J_0\},$$

where

$$J_{0,\sigma,\sigma\tau} = \int\frac{(1\,,\, q_\sigma,\, q_\sigma q_\tau)d^4q}{(q^2-2qp)(q^2-2qp-2qk+\varkappa m^2)q^2}$$

and the symbol $(1,\, q_\sigma,\, q_\sigma q_\tau)$ denotes 1 in the case of $J_0$, $q_\sigma$ in the case of $J_\sigma$ and $q_\sigma q_\tau$ in the case of $J_{\sigma\tau}$.

All these integrals may be expressed in terms of the Spence function

$$F(x) = \int\limits_0^x \ln(1+u)\frac{du}{u}, \qquad F(-1) = -\frac{\pi^2}{6},$$

viz.:

$$\frac{1}{i\pi^2}J_0 = \frac{1}{\varkappa}[F(\varkappa-1)-F(-1)],$$

$$\frac{1}{i\pi^2}J_\sigma = \left((J_0-\frac{\ln|\varkappa|}{\varkappa-1}\right)p_\sigma+\left(2J_0+\frac{\varkappa-2}{\varkappa-1}\ln|\varkappa|-2\right)\frac{1}{\varkappa}k_\sigma,$$

$$\frac{1}{i\pi^2}J_{\sigma\tau} = -\frac{\delta_{\sigma\tau}}{4}\left\{A_0+\frac{1}{2}-3+\frac{\varkappa-2}{\varkappa-1}\ln|\varkappa|+2J_0\right\}$$

$$+k_\sigma k_\tau\frac{1}{2\varkappa^2}\left\{12J_0+\frac{(\varkappa-2)(\varkappa^2+6\varkappa-6)}{(\varkappa-1)^2}\ln|\varkappa|-\frac{2\varkappa^2+9\varkappa-12}{\varkappa-1}\right\}+$$

$$+(k_\sigma p_\tau+k_\tau p_\tau)\frac{1}{2\varkappa}\left\{6J_0+\frac{2\varkappa^2-9\varkappa+6}{(\varkappa-1)^2}\ln|\varkappa|+\frac{1}{\varkappa-1}-5\right\}$$

$$+p_\sigma p_\tau\frac{1}{2}\left\{2J_0-\frac{3\varkappa-2}{(\varkappa-1)^2}\ln|\varkappa|+\frac{1}{\varkappa-1}\right\},$$

where $A_0$ is a logarithmically divergent constant.

In accordance with the general theory the expression (47.54) diverges logarithmically for large $q$. In order to regularize the vertex part $\Lambda_\mu^{(3)}(p, p+k; k)$ we must subtract from it the value of $\Lambda_\mu^{(3)}(p, p; 0)$ corresponding to $k=0$:

$$\Lambda_\mu^{(3(}(p,p;0)=\gamma_\mu\frac{e^2}{(2\pi)^2}\left(-\frac{1}{4}A_0+\frac{11}{8}-\ln\frac{m}{\lambda}\right).$$

As a result of this we obtain the following expression for the regularized vertex part $\Lambda_{\mu R}^{(3)}(p, p+k; k)$:

$$\Lambda_{\mu R}^{(3)}(p,p+k;k)=\frac{\alpha}{2\pi}\left\{-2\gamma_\mu\left(1+\ln\frac{\lambda}{m}\right)+\frac{p_\mu}{m^2(\varkappa-1)}\left(-im+\hat{k}\frac{\varkappa-2}{\varkappa}\right)\right.$$

$$+\frac{\ln|\varkappa|}{m^2(\varkappa-1)}\left[\left(\frac{\hat{p}\hat{k}-\hat{k}\hat{p}}{2}-m^2\right)\gamma_\mu+p_\mu\frac{3\varkappa-2}{\varkappa(\varkappa-1)}(\hat{k}+i\varkappa m)\right]$$

$$\left.+\frac{1}{m^2\varkappa^2}(-2\hat{k}p_\mu+\varkappa m^2\gamma_\mu)\big(F(\varkappa-1)-F(1)\big)\right\}. \qquad (47.55)$$

On noting that

$$(\hat{k}\hat{p}-\hat{p}\hat{k})\gamma_\mu=-\varkappa m^2\gamma_\mu-2im\hat{k}\gamma_\mu+4\hat{k}p_\mu,$$

we can write $\Lambda_{\mu R}^{(3(}(p, p+k; k)$ in the form

$$\Lambda_{\mu R}^{(3)}(p,p+k;k)=\frac{\alpha}{2\pi}(m^2A\gamma_\mu+imB\hat{k}\gamma_\mu+imCp_\mu+D\hat{k}p_\mu), \qquad (47.56)$$

where

$$A=-\ln\frac{\lambda^2}{m^2}-2+\frac{(\varkappa-2)\ln|\varkappa|}{2(\varkappa-1)}+\frac{1}{\varkappa}[F(\varkappa-1)-F(-1)],$$

$$B=\frac{\ln|\varkappa|}{\varkappa-1},$$

$$C=-\frac{1}{\varkappa-1}+\frac{(3\varkappa-2)\ln|\varkappa|}{(\varkappa-1)},$$

$$D=-\frac{1}{\varkappa-1}+\frac{2}{\varkappa}-\frac{(\varkappa-2)(2\varkappa-1)}{\varkappa(\varkappa-1)^2}\ln|\varkappa|-\frac{2}{\varkappa^2}[F(\varkappa-1)-F(-1)].$$

## § 48. Functional Properties of Green's Functions. Limits of Applicability of Quantum Electrodynamics

### 48.1. *Expansion Parameters of Perturbation Theory*

We have seen that the perturbation theory series of quantum electrodynamics, which are formal expansions in terms of the parameter $e$, have meaning only when we introduce the cut-off momentum $L$. When $L \to \infty$ the use of perturbation theory has no sense, since even the second term in the expansion becomes infinite. It is therefore clear that the true perturbation theory expansion parameter must depend not only on $e$, but also on $L$. We now consider the problem of finding this parameter.

With this aim in mind we investigate the structure of the perturbation theory series which determines the vertex function $\Gamma(p_1 p_2; k)$. In § 46 we have shown that the terms of this series can be expressed in terms of logarithmically divergent integrals, i.e., that they contain $\ln L$. Since the vertex function is a dimensionless quantity it can depend only on the ratio of $L$ to some quantity having the dimensions of momentum. Therefore, we can write the series for $\Gamma(p_1, p_2; k)$ in the form

$$\Gamma(p_1, p_2; k) = \sum_{n=0}^{\infty} a_n l^n, \quad \text{where} \quad l = \ln \frac{L^2}{m^2},$$

and the coefficients $a_n$ are functions of $p_1^2/m^2, p_2^2/m^2, k^2/m^2$ and $e$. Let us determine the nature of the dependence of $a_n$ on $e$. First of all, we note that, as we have shown in § 46, irreducible diagrams of arbitrary order correspond to quantities proportional to $\ln L$. On summing the contributions of all the irreducible diagrams we obtain

$$a_1 = \sum_{r=0}^{\infty} a_{1r} (e^2)^{r+1}.$$

Reducible diagrams are equivalent to irreducible skeleton diagrams whose vertices represent certain lower order vertex parts which also depend on $l$. In the general case the vertex part of the $(2n+1)$th order which is proportional to $(e^2)^n$, is a polynomial of the $n$th degree in $l$. Thus,

$$a_n = \sum_{r=0}^{\infty} a_{nr} (e^2)^{n+r}$$

and

$$\Gamma(p_1, p_2; k) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} a_{nr}(e^2)^r (e^2 l)^n, \tag{48.1}$$

where the coefficients $a_{nr}$ depend on $p_1^2/m^2, p_2^2/m^2$ and $k^2/m^2$.

The series (48.1) contains two parameters $e^2$ and $e^2 l$, and the criterion for the applicability of perturbation theory is the smallness of both these parameters

$$e^2 \ll 1, \qquad e^2 l \equiv e^2 \ln \frac{L^2}{m^2} \ll 1. \tag{48.2}$$

The perturbation theory series for the propagators $G^e(k)$ and $G^\gamma(k)$ will obviously have a structure analogous to (48.1). If we write these functions in the form

$$\begin{aligned} G^e(k) &= s(k) S^c(k), \\ G^\gamma(k) &= d(k) D^c(k), \end{aligned} \tag{48.3}$$

where $s(p)$ and $d(k)$ are dimensionless factors, then

$$\begin{aligned} s(k) &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} s_{nr}(e^2)^r (e^2 l)^n, \\ d(k) &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} d_{nr}(e^2)^r (e^2 l)^n, \end{aligned} \tag{48.4}$$

where $s_{nr}$ and $d_{nr}$ depend only on $k^2/m^2$.

We note that asymptotically the functions $d(k)$ and $s(k)$ cannot depend on $m$. Therefore the cut-off momentum can appear in the expressions for $d$ and $s$ only in the form $l_k = \ln L^2/|k^2|$.

In this case the series for $s(k)$ and $d(k)$ will be of the form (48.4) where $l$ has been replaced by $l_k$, and $s_{nr}$ and $d_{nr}$ are constants

$$\begin{aligned} s(k) &= \sum_{n,\,r} s_{nr}^0 (e^2)^r \left(e^2 \ln \frac{L^2}{k^2}\right)^n, \\ d(k) &= \sum_{n,\,r} d_{nr}^0 (e^2)^r \left(e^2 \ln \frac{L^2}{k^2}\right)^n, \end{aligned} \tag{48.4'}$$

where $|k^2| \gg m^2$, while $d_{nr}^0$ and $s_{nr}^0$ are constants.

The expression for the vertex function in the asymptotic case can also be written in an analogous form.

### 48. 2. *Zero Order Approximation in the Expansion in Powers of $e^2$*

The foregoing results ensure good applicability of perturbation theory up to very high momenta satisfying only the condition $\ln(|p^2|/m^2) \ll 1/e^2$.

Indeed, we can introduce the cut-off momentum $L$ in such a way that $L^2 \gg |p^2|$ and $e^2 \ln(L^2/m^2) \ll 1$ will hold. The criterion for the applicability of perturbation theory (48.2) will then be satisfied, and if the fundamental assumption made in subsection 46.2 is valid, we can then utilize the method of renormalizations developed in §§ 45, 46 which leads to results independent of $L$.

Therefore, in electrodynamics problems do not arise in practice which require the applications of methods differing from perturbation theory. Physically of considerably greater importance than the inaccuracy of perturbation theory will be processes of interaction of electrons and photons with electrically charged fields, for example meson fields, which have additional strong nonelectromagnetic interactions. Nevertheless, from a fundamental point of view the problem of the existence of exact solutions of the system of equations of quantum electrodynamics is of considerable interest.

We consider the problem (115) of obtaining the zero order approximation for the propagators in the form of a series in terms of the one parameter $e$, i.e., we shall attempt to eliminate the second condition (48.2). The solution of this problem is equivalent to summing the series (48.1)–(48.4) over $n$ for $r = 0$. For large momenta $|p^2| \gg m^2$ this approximation can be easily found by utilizing the property of the renormalizability of the propagators.

In order to do this we consider the renormalized photon propagator $G_c^\gamma(k) = Z^{-1}G^\gamma(k)$. We write it in the form (48.3)

$$G_c^\gamma(k) = d_c(k)\, D^c(k). \tag{48.5}$$

Then

$$d_c(k) = Z^{-1}\, d(k). \tag{48.5'}$$

In accordance with the proof given in § 45 the function $d_c(k)$ does not depend on $L$ if it is regarded as a function of $k$ and of the renormalized charge $e_c^2 = Ze^2$. We call this property *renormalizability*.

The quantities $Z$ and $d(k)$ are functions of the cut-off momentum. We emphasize this by providing these quantities with an index $L$ and by denoting them as $d_L(k)$ and $Z_L$. Since $L$ may be chosen arbitrarily, then for a fixed true electron charge $e_c^2$ the following equation, which expresses the renormalizability property, must hold:

$$Z_{L_0}^{-1}\, d_{L_0}(k) = Z_L^{-1}\, d_L(k), \tag{48.6}$$

and we shall use this equation to solve the problem which we have formulated (76), (29), (192).

We make the preliminary remark that as a result of (48.4') we have

$$d_L(L) = 1. \tag{48.7}$$

We now set $k = L_0$. Then it follows from (48.6) and (48.7) that

$$Z_{L_0}^{-1} = Z_L^{-1}\, d_L(L_0). \tag{48.8}$$

If $L$ is close to $L_0$, so that $e^2(L^2/L_0^2) \ll 1$, then we can utilize for $d_L$ the first term of the perturbation theory series. In accordance with (47.30') we have in this case

$$d_L(L_0) = 1 - \frac{e_L^2}{12\pi^2}(l - l_0), \tag{48.9}$$

where

$$l = \ln\frac{L^2}{m^2}, \qquad l_0 = \ln\frac{L_0^2}{m^2}, \qquad e_L^2 = Z^{-1}e_c^2.$$

We emphasize that (48.9) is the first order approximation in terms of the parameter $e_L^2\, l$ and the zero order approximation in terms of the parameter $e_L^2$. On substituting (48.9) into (48.8) we obtain

$$Z_L^{-1} - Z_{L_0}^{-1} = Z_L^{-2}\frac{e_c^2}{12\pi^2}(l - l_0). \tag{48.10}$$

From this relation follows the differential equation for $Z_L$

$$\frac{dZ_L^{-1}}{dl} = \frac{e_c^2}{12\pi^2} Z_L^{-2}. \tag{48.10'}$$

On integrating this equation we obtain an expression for $Z_L$ which is valid for $e_L^2 \ll 1$, but for arbitrary $L$. This expression is the desired zero order approximation with respect to the parameter $e_L^2$.

On integrating (48.10') we obtain

$$Z_L = -\frac{e_c^2}{12\pi^2}\, l + c \qquad \text{or} \qquad Z_L = \frac{c}{1 + \dfrac{e_L^2}{12\pi^2}\, l}\,.$$

The constant $c$ may be obtained from the condition that when $e_L^2 L \to 0$ the zero order approximation with respect to $e_L^2$ is given by $Z_L = 1$. Thus, $c = 1$ and

$$Z_L = 1 - \frac{e_c^2}{12\pi^2} \ln \frac{L^2}{m^2}, \qquad Z_L^{-1} = 1 + \frac{e_L^2}{12\pi^2} \ln \frac{L^2}{m^2}. \tag{48.11}$$

On substituting (48.11) into (48.8) we obtain the expression[1] for $d_L(k)$

$$d_L(k) = \frac{1}{1 + \frac{e_L^2}{12\pi^2} \ln \frac{L^2}{k^2}}. \tag{48.12}$$

The expression for the renormalized function $d_c(k)$ can be easily obtained directly from formula (48.5′) on setting $k = L$. Since in accordance with (48.7) $d_k(k) = 1$, then

$$d_c(k) = Z_k^{-1}. \tag{48.13}$$

On substituting into this equation the expression for $Z$ (48.11), we obtain

$$d_c(k) = \frac{1}{1 - \frac{e_c^2}{12\pi^2} \ln \frac{k^2}{m^2}}. \tag{48.14}$$

We can also easily obtain the zero order approximation for the functions $G^e(p)$ and $\Gamma_\mu(p, p, 0)$ for $|p^2|/m^2 \gg 1$.

These functions have their simplest form in the case of the transverse gauge for the function $D_{\mu\nu}^c$:

$$D_{\mu\nu}^c = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D^c.$$

As we have noted in § 46, in this case the expression for the vertex part $\Lambda^{(3)}$ contains no divergences.

This means that the series (48.1) for the vertex function contains no terms proportional to $e^2 l$. By utilizing the method which we employed earlier for making the transition from the approximation $e^2 l \ll 1$ to the approximation $e^2 \ll 1$, it can be easily shown that $Z_1$ contains no terms proportional to $e^2 l$, i.e.,

$$Z_1 = 1, \tag{48.15}$$

---

[1] This result is due to Landau, Abrikosov and Khalatnikov (115).

Thus, the electron Green's function and the vertex part are given for $e^2 \ll 1$ and for arbitrary $L$ by the same expressions which correspond to free fields. (This conclusion is valid if we take $d_l = 0$.)

We can also obtain an expression for the electromagnetic mass of the electron if we take $e^2 \ll 1$ and $L$ arbitrary. In accordance with (47.27) for small $e^2 l$ we have

$$\delta m = \frac{9}{4} \frac{e_L^2}{12\pi^2} ml.$$

If the cut-off momentum $L$ is changed the mass changes by the amount

$$dm = \frac{9}{4} \frac{e_L^2}{12\pi^2} m\, dl$$

or, after comparison with (48.10′),

$$\frac{dm}{m} = \frac{9}{4} d(Z^{-1}).$$

On integrating this expression, and on denoting by $m(Z)$ the value of the mass corresponding to a given value of the ratio $e_c^2/e^2$, we obtain (115)

$$m(Z) = m(1) Z^{-\frac{9}{4}}, \tag{48.16}$$

where $m(1)$ is a constant which evidently has the meaning of the mass in the absence of interaction ($Z = 1$), i.e., of the "mechanical" mass.

The quantity $e(k) = Z_k^{-\frac{1}{2}} e_c$ may be regarded as the effective charge corresponding to the interaction in the domain of momenta smaller than $k$, since $e(k)$ can be chosen as the primary charge for the description of interactions in this domain.

Formula (48.11) which we can write in the form

$$e^2(k) = \frac{e_c^2}{1 - (e_c^2/12\pi^2) \ln(k^2/m^2)} \tag{48.17}$$

shows the increase in the effective charge with increasing $|k|$, i.e., with decreasing distances.

Since momenta of the order of $k$ correspond to space-time distances $\lambda \sim 1/k$, we obtain the following simple picture of the spatial distribution of the electron charge. The primary charge $e$ gives rise to polarization of the vacuum, i.e., to a certain spatial distribution around the electron of charges of opposite sign which leads to a partial screening of the electron charge. Therefore, for processes in which distances smaller than $\lambda$ play no role the effective charge falls off in accordance with formula (48.17) as $\lambda \sim 1/k$ increases. For $\lambda \sim 1/m$ the effective charge assumes its minimum value $e_c$, which we call the physical charge of the electron determined, for example, from experiments on the scattering by electrons of light of long wavelength (cf. § 45).

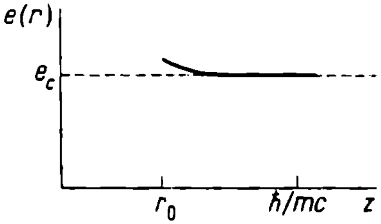

Fig. 95.

Schematically the dependence of the effective charge of the electron on the distance $z$ is shown in Fig. 95.

### 48.3. *Integral Equations for the Zero Order Approximation*

We have previously obtained asymptotic expressions for $G^\gamma$, $G^e$ and $\Gamma$ for $e^2 \ll 1$ and arbitrary $L$. In doing this we utilized the renormalizability of these expressions which was proved by methods of ordinary perturbation theory employing an expansion in powers of $e^2 \ln(L^2/m^2)$. It is therefore desirable to give a proof of these results without assuming the renormalizability property *a priori.*

In § 44 we have obtained two integral equations connecting the three functions $G^e$, $G^\gamma$ and $\Gamma$. As we have already noted, there exists no third exact equation which would make this system complete. However, we can obtain an equation which is valid in the zero order approximation in terms of $e^2$ (115). In order to obtain this equation it is sufficient to make use of the skeleton diagram of the vertex part shown in Fig. 83 (*1*).

Indeed, diagrams which are not taken into account by this equation, for example, the diagram of Fig. 83 (*2*), contain intersecting photon lines which in perturbation theory lead to terms proportional to $(e^2l)^n$ for $n > 1$, while the diagram of Fig. 83 (*1*) contains only terms $\sim e^2l$.

In order to obtain the desired integral equation it is sufficient to utilize the formula

$$\Gamma_\mu(p_1, p_2; k) = \gamma_\mu + \Lambda_\mu(p_1, p_2; k)$$

and to obtain the expression for $\Lambda_\mu$ on the basis of the skeleton diagram of Fig. 83 (*l*) by making each of its vertices correspond to a vertex function $\Gamma_\mu$, and each of its lines to the Green's functions $G^e$ and $G^\gamma$. We thus obtain the equation

$$\Gamma_\mu(p_1, p_2; k)$$
$$= \gamma_\mu + \frac{e^2}{(2\pi)^4}\int \Gamma_\alpha(p_2-q, p_2; -q) G^e(p_2-q)$$
$$\times \Gamma_\mu(p_1-q, p_2-q; k) G^e(p_1-q) \Gamma_\beta(p_1, p_1-q; q) G^\gamma_{\alpha\beta}(q) d^4q. \quad (48.18)$$

Together with the equations (44.4) we now have a complete system of zero order equations for the propagators $G^e$ and $G^\gamma$.

It can be easily shown that the expressions for $G^e$ and $G^\gamma$ obtained in subsection 48.1 are asymptotic solutions of these equations. This result is an independent proof of the renormalizability property of the Green's functions.

### 48.4. *The Renormalization Group*

In subsection 45.4 we have seen that the elements of the scattering matrix (cf. formula (45.19)) are not changed if we go over from the original values of the charge $e$ and the propagators $G^e(p)$, $G^\gamma(k)$, $\Gamma_\mu(p_1, p_2; k)$ to the renormalized values $e_c$, $G^e_c(p)$, $G^\gamma_c(k)$, $\Gamma_{\mu c}(p_1, p_2; k)$ related to the original values by

$$e_c = Z^{1/2}e, \quad G^e_c = Z_1 G^e, \quad G^\gamma_c = Z G^\gamma, \quad \Gamma_{\mu c} = Z_1^{-1}\Gamma_\mu. \quad (48.19)$$

We have utilized these transformations in order to remove in a unique and physically meaningful way those divergences which are encountered in quantum electrodynamics. However, the role played by equations (48.19) is not limited to this, since they are related to more general transformations admitted by the equations of quantum electrodynamics, the investigation of which enables us to obtain a number of general functional properties of the propagators[1].

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[1] A detailed investigation of the functional properties of the propagators is given in (30).

We start with the renormalized equations of quantum electrodynamics containing no divergences and consider the two systems of finite quantities $e_1, G_1^e(p), G_1^\gamma(k), \Gamma_1(p_1, p_2; k)$ and $e_2$, $G_2^e(p), G_2^\gamma(k), \Gamma_2(p_1, p_2; k)$. If these quantities are related by the expressions

$$\begin{aligned} \Gamma_2 &= z_1 \Gamma_1, \\ G_2^e &= z_2^{-1} G_1^e, \\ G_2^\gamma &= z_3^{-1} G_1^\gamma, \\ e_2^2 &= z_1^2 z_2^{-2} z_3^{-1} e_1^2, \end{aligned} \tag{48.19'}$$

where $z_1, z_2, z_3$ are arbitrary quantities, then on repeating the calculation carried out in subsection 45.4 we can easily show that the elements of the scattering matrix calculated with the aid of both systems of these quantities will be the same.

The transformations (48.19') obviously form a group which we call the *renormalization group*, or the group of *multiplicative renormalization* (191), (76), (29). We follow (29) in the following discussion.

On taking into account Ward's identity we can without loss of generality set $z_1 = z_2$ and rewrite the group (48.19') in the form

$$\begin{aligned} G_1^e &\to G_2^e = z_1 G_1^e, \\ G_1^\gamma &\to G_2^\gamma = z G_1^\gamma, \\ \Gamma_1 &\to \Gamma_2 = z_1^{-1} \Gamma_1, \\ e_1^2 &\to e_2^2 = z_1^{-1} e_1^2. \end{aligned} \tag{48.20}$$

We first of all establish the functional equations of the renormalization group. For the sake of simplicity we use the transverse gauge for the electromagnetic potentials in which $d_l = 0$, and we introduce in place of $G^\gamma(k)$ and $G^e(p)$ the quantities $d(k^2)$, $a(p^2)$, $b(p^2)$ which depend only on the square of the momentum:

$$\begin{aligned} G^\gamma(k) &= \frac{d(k^2)}{k^2}, \\ G^e(p) &= -\frac{a(p^2)\hat{p} + imb(p^2)}{p^2 + m^2}. \end{aligned} \tag{48.20'}$$

It is convenient to transfer the arbitrary factors which in virtue of (48.20) appear in the functions $d(k^2)$, $a(k^2)$, $b(k^2)$ into the arguments of the functions by assuming that for a certain value of $k^2 = \lambda^2$ these functions assume the values

$$d(\lambda^2) = 1, \quad a(\lambda^2) = a_0, \quad b(\lambda^2) = b_0.$$

We write $d(k^2)$, $a(k^2)$, $b(k^2)$ in the form $f(k^2/\lambda^2, m^2/\lambda^2, e^2)$. Therefore the normalization condition for $d(k^2)$ is of the form

$$d\left(1, \frac{m^2}{\lambda^2}, e^2\right) = 1.$$

It follows from (48.20) that

$$zd\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) = z'd\left(\frac{k^2}{\lambda'^2}, \frac{m^2}{\lambda'^2}, e'^2\right),$$

where $z$ and $z'$ are two values of the factor $z$ corresponding to the values of the charge $e$ and $e'$. On assuming here that $k^2 = \lambda^2$ we obtain

$$z = z'd\left(\frac{\lambda^2}{\lambda'^2}, \frac{m^2}{\lambda'^2}, e'^2\right).$$

Therefore

$$d\left(\frac{\lambda^2}{\lambda'^2}, \frac{m^2}{\lambda'^2}, e'^2\right) d\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) = d\left(\frac{k^2}{\lambda'^2}, \frac{m^2}{\lambda'^2}, e'^2\right) \tag{48.21}$$

and

$$e^2 d\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) = e'^2 d\left(\frac{k^2}{\lambda'^2}, \frac{m^2}{\lambda'^2}, e'^2\right). \tag{48.22}$$

Thus the quantity $e^2d$ does not depend on the choice of $\lambda$ and is, as is said, an invariant of the renormalization group. This quantity is called the *invariant charge.*

For the actual charge $e_c$ the function $d$ which we denote by $d_c(k^2/m^2, e_c^2)$ reduces to unity for $k^2 = 0$. Therefore, it follows from (48.22) that

$$e_\lambda^2 d\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e_\lambda^2\right) = e_c^2 d_c\left(\frac{k^2}{m^2}, e_c^2\right).$$

On setting $k^2 = \lambda^2$ in this expression we obtain

$$e_\lambda^2 = e_c^2 d_c\left(\frac{\lambda^2}{m^2}, e_c^2\right). \tag{48.23}$$

This relation defines the dependence of $e_\lambda$ on the magnitude of $\lambda$.

On introducing the notation $k^2/\lambda'^2 = x$, $m^2/\lambda'^2 = y$, $\lambda^2/\lambda'^2 = t$, we rewrite (48.21) in the form

$$e^2d(x, y, e^2) = e^2d(t, y, e^2)\, d\left(\frac{x}{t}, \frac{y}{t}, e^2d(t, y, e^2)\right). \quad (48.24)$$

This relation is the functional equation for the function $d$.

In a similar fashion we can obtain the functional equations for the functions $a$ and $b$, which determine $G^e(p)$ (cf. (48.20′)). On introducing for these functions the common notation $s(p^2/\lambda^2, m^2/\lambda^2, e^2)$ we obtain the functional equation

$$s(x, y, e^2)\, s\left(1, \frac{y}{t}, e^2d(t, y, e^2)\right) = s(t, y, e^2)\, s\left(\frac{x}{t}, \frac{y}{t}, e^2\, d(t, y, e^2)\right). \quad (48.25)$$

If we regard relations (48.24) and (48.25) as equations determining the unknown functions $d$, $s$, we can set ourselves the problem of finding their most general solution. Such a solution contains arbitrary functions of two arguments and therefore, generally speaking, is not of very great interest. However, in the particularly interesting domain of high momenta $|k^2|$, $|p^2| \gg m^2$ considerable simplifications occur, since we can omit the variable $y$ from the set of arguments of the unknown functions.[1] In this case, as we shall now show, the invariant charge is determined by only one function of one argument.

On setting $d(x, 0, e^2) \equiv d(x, e^2)$ and on introducing for the invariant charge the notation $g(x, \xi) = \xi d((x, \xi)$, we rewrite the functional equation (48.24) for the function $d$ in the asymptotic region of large momenta in the form

$$g(x, \xi) = g\left(\frac{x}{t}, g(t, \xi)\right). \quad (48.26)$$

It can be easily shown that the most general solution of this equation is

$$g(x, \xi) = f\left((xf^{-1}(\xi)\right), \quad (48.26')$$

where $f(x)$ is an arbitrary function and $xf^{-1}$ is the corresponding inverse function.

---

[1] The assumption that the variable $y$ can be omitted does not follow automatically from the fact that the functional equations (48.24) and (48.25) exist. A basis for this assumption can be found in perturbation theory.

Thus, for $|k^2| \gg m^2$ the invariant charge is given by the expression (76)

$$e^2 d\left(\frac{k^2}{m^2}, e^2\right) = f\left(\frac{k^2}{m^2} f^{-1}(e^2)\right), \qquad |k^2| \gg m^2. \tag{48.27}$$

The general solution of the functional equation (48.25) for the function $s$ in the region of large momenta has the form

$$s\left(\frac{p^2}{m^2}, e^2\right) = A(e^2) F\left(\frac{p^2}{m^2} f^{-1}(e^2)\right), \qquad |p^2| \gg m^2, \tag{48.28}$$

where $A$ and $F$ are arbitrary functions.

We can easily establish the physical meaning of the function $d$. In order to do this we note that the potential $\varphi(\boldsymbol{r})$ produced by the charge $e$ is related to the photon propagator by

$$\varphi(\boldsymbol{r}) = \frac{ie}{(2\pi)^3} \int G^{\gamma}(k, 0) e^{i\boldsymbol{kr}} d\boldsymbol{k}.$$

On taking the Laplacian of $\varphi(\boldsymbol{r})$ we obtain the charge density $\varrho(\boldsymbol{r})$ at a distance $\boldsymbol{r}$ from the charge

$$\begin{aligned} \varrho(\boldsymbol{r}) &= -\Delta\varphi = \frac{ie}{(2\pi)^3} \int k^2 G^{\gamma}(k, 0) e^{i\boldsymbol{kr}} d\boldsymbol{k} \\ &= \frac{ie}{(2\pi)^3} \int d(k, 0) e^{i\boldsymbol{kr}} d\boldsymbol{k}. \end{aligned} \tag{48.29}$$

The quantity $\varrho(\boldsymbol{r})$ obviously represents the charge density of the cloud of pairs surrounding the charge $e$. We see that this density is determined by the Fourier component of the function $d(k)$.

For large values of $|k^2|$ the function $d(k)$ is defined by formula (48.27). On utilizing this formula we obtain the following expression for the charge density $\varrho(\boldsymbol{r})$ at distances small compared to $\hbar/mc$:

$$\varrho(\boldsymbol{r}) = \frac{i}{(2\pi)^3 e} \int f\left(\frac{k^2}{m^2} f^{-1}(e^2)\right) e^{i\boldsymbol{kr}} d\boldsymbol{k}.$$

On introducing in place of $\boldsymbol{k}$ the new variable $(\boldsymbol{k}/m)\sqrt{f^{-1}(e^2)} = \boldsymbol{k}'$ we rewrite $\varrho(\boldsymbol{r})$ in the form

$$\varrho(\boldsymbol{r}) = C \int f(k'^2) e^{i\boldsymbol{k}'\boldsymbol{r}'} d\boldsymbol{k}', \tag{48.30}$$

where $C$ is a constant, $\boldsymbol{r}' = \boldsymbol{r}(m/\sqrt{f^{-1}(e^2)})$. This relation shows that the form of the charge distribution at distances small compared to $\hbar/mc$

does not depend on the magnitude of the charge $e$ which appears only in the scale factor.

Relation (48.27) also leads to important consequences with respect to the "bare" charge $e_0$.

On noting that in accordance with (48.23) $e_0^2 = e_c^2 d_c(\infty, e_c^2)$, and on taking into account that $e^2 d(k^2/m^2, e^2)$ is a positive increasing function of $k^2/m^2$ we can draw the following conclusions. If as $k^2/m^2 \to \infty$ the function $f\big((k^2/m^2) f^{-1}(e^2)\big) \to \infty$, then $e_0^2$ is infinite, and the singularity at the center of the charge distribution is stronger than a $\delta$-function corresponding to a finite point charge. However, if as $k^2/m^2 \to \infty$ the function $f\big((k^2/m^2) f^{-1}(e^2)\big)$ tends to a finite limit then $e_0^2$ is equal to this limit which does not depend on the value of $e_c^2$.

### 48.5. *Derivation of Asymptotic Expressions for the Green's Functions with the Aid of the Differential Equations of the Renormalization Group*

The arbitrariness arising in the solution of the functional equations (48.24) and (48.25) which manifests itself, in particular, in the fact that arbitrary functions appear in (48.27) and (48.28), can be removed if we seek solutions of equations (48.24) and (48.25) which in the limiting case of sufficiently small $e^2$ (for arbitrary values of the arguments $x, y$) reduce to the well known perturbation theory series. With this aim in view we shall first of all obtain the differential equations for the renormalization group which describe the infinitesimal transformations of the group (29).

On differentiating (48.24) and (48.25) with respect to $x$ and then on setting $t = x$ we obtain the desired equations

$$\begin{aligned} \frac{\partial}{\partial x} e^2 d(x, y, e^2) &= \frac{e^2 d(x, y, e^2)}{x} \varphi\left(\frac{y}{x}, e^2 d(x, y, e^2)\right), \\ \frac{\partial}{\partial x} \ln s(x, y, e^2) &= \frac{1}{x} \psi\left(\frac{y}{x}, e^2 d(x, y, e^2)\right), \end{aligned} \tag{48.31}$$

where

$$\varphi(y, e^2) = \left(\frac{\partial}{\partial \xi} d(\xi, y, e^2)\right)_{\xi=1},$$

$$\psi(y, e^2) = \left(\frac{\partial}{\partial \xi} \ln s(\xi, y, e^2)\right)_{\xi=1}.$$

Thus, in order to obtain the functions $d, s$ over the whole range of variation of their arguments we must know the functions $\varphi$ and $\psi$ which are defined by the values of the functions $d, s$ in the infinitesimal neighborhood of the point $x = 1$.

The functions $\varphi$ and $\psi$ can be obtained by using perturbation theory. Since the quantity $e^2$ appears in the definition of these functions in the form of the invariant charge $e^2 d$, the method of obtaining $d$ and $s$ with the aid of equations (48.31) requires that the condition $e^2 d \ll 1$ should hold.

We now show how the asymptotic behavior of the functions $d(k^2)$ and $s(k^2)$ can be obtained for $|k^2| \gg m^2$ with the aid of equations (48.31).

On omitting in (48.31) the variable $y$ we obtain

$$\begin{aligned} \frac{d}{dx} e^2 d(x, e^2) &= \frac{e^2 d(x, e^2)}{x} \varphi\big(e^2 d(x, e^2)\big), \\ \frac{d}{dx} \ln s(x, e^2) &= \frac{1}{x} \psi\big(e^2 d(x, e^2)\big), \end{aligned} \tag{48.32}$$

where

$$\begin{aligned} \varphi(z) &= \left(\frac{\partial}{\partial \xi} d(\xi, z)\right)_{\xi = 1}, \\ \psi(z) &= \left(\frac{\partial}{\partial \xi} \ln s(\xi, z)\right)_{\xi = 1}. \end{aligned} \tag{48.32'}$$

Integration of these equations yields

$$\begin{aligned} &\int\limits_{e^2}^{e^2 d(x, e^2)} \frac{dz}{z \varphi(z)} = \ln x, \\ &\ln \frac{s(x, e^2)}{s(x_0, e^2)} = \int\limits_{e^2 d(x_0, e^2)}^{e^2 d(x, e^2)} \frac{dz}{z \varphi(z)} \psi(z), \end{aligned} \tag{48.33}$$

where $e^2 d(x, e^2) = z(x)$.

We obtain first of all $d(x, e^2)$.

In order to find $\varphi(z)$ we use the perturbation theory expansion

$$d(x, e^2) = 1 + \frac{e^2}{12\pi^2} \ln x + \frac{e^4}{16\pi^2} \left(\frac{1}{4\pi^2} \ln x + \frac{1}{16\pi^2} \ln^2 x\right) + \cdots \tag{48.34}$$

From this in accordance with (48.32′) we obtain the expression for $\varphi(z)$,

$$\varphi(z)=\frac{z}{12\pi^2}\left(1+\frac{3z}{16\pi^2}+\ldots\right), \quad z\ll 1. \tag{48.34′}$$

Substitution of this expansion into the first relation (48.33) yields

$$\frac{e^2}{12\pi^2}\ln x = 1-d^{-1}-\frac{3e^2}{12\pi^2}\ln d+e^2O(e^2, e^2d),$$

whence

$$d^{-1}(x, e^2)= 1-\frac{e^2}{12\pi^2}\ln x+\frac{3e^2}{16\pi^2}\ln\left(1-\frac{e^2}{12\pi^2}\ln x\right)+\ldots \tag{48.35}$$

On confining ourselves to the first two terms of this expansion we obtain

$$e^2d(x, e^2)=\frac{e^2}{1-\dfrac{e^2}{12\pi^2}\ln x}, \quad \text{where} \quad x=\frac{k^2}{m^2} \tag{48.35′}$$

which coincides with the result (48.14) obtained previously (with $e_c \to e$).

We note that from the method used for the derivation of formula (48.35) it follows that $1-(e^2/12\pi^2)\ln x < 1$. Therefore, the logarithm appearing in the third term of this formula is negative and, consequently, the sign of the third term agrees with the sign of the second term. This circumstance is not accidental, and is a manifestation of the general situation which we shall discuss in greater detail in subsection 48.6.

As has been pointed out already, the method of integration of equations (48.32) is based on the assumption that $e^2d \ll 1$ since only under this condition is the expansion (48.34′) valid. Therefore, formula (48.35′) holds when the following inequality is satisfied:

$$1-\frac{e^2}{12\pi^2}\ln\frac{|k^2|}{m^2}\gg e^2,$$

whence it follows that $|k^2| \ll m^2e^{\frac{3\pi}{\alpha}}$. This inequality is at the same time the condition for the applicability of the series (48.35).

We see that in comparison with the usual perturbation theory series

$$f(x)=\sum_{n,m} f_{nm}e^{2(n+m)}\ln^n x, \quad x=\frac{k^2}{m^2},$$

the series (48.35) has been rearranged and has the structure

$$f(x)=\sum_n e^{2n}f_n(e^2\ln x),$$

where $f_n$ are functions of $e^2 \ln x$.

We can also obtain similar improved expansions of the functions $a(x, y, e^2)$, $b(x, y, e^2)$ which determine $G^e(x=p^2/\lambda^2,\ y=m^2/\lambda^2)$. In order to do this we must have the expansions of the functions $a$, $b$ by means of ordinary perturbation theory. They can be obtained if we substitute into the expression $G^e=S^c+S^c\Sigma S^c$ the expansion (47.24) or (47.24′) in place of $\Sigma$. For $d_l=0$ we obtain

$$\begin{aligned} a(x,y,e^2) &= 1+\frac{3e^2}{8\pi^2}\left\{\frac{y}{x}\ln\left|\frac{x+y}{y}\right|+c(y)\right\},\\ b(x,y,e^2) &= 1+\frac{3e^2}{8\pi^2}\left\{\frac{x+y}{2x}\ln\left|\frac{x+y}{y}\right|+c(y)\right\}, \end{aligned} \tag{48.36}$$

where $c(y)$ is a function of $y$. On utilizing these expressions, and also the relation (48.33) and the definition (48.32′) of the function $\psi(z)$, we can obtain the asymptotic formulas valid for large momenta ($x \gg y$),

$$\begin{aligned} \ln a(x,e^2) &= -\frac{3e^2}{8(4\pi)^2}\{d(x,e^2)-1\}+\dots,\\ \ln b(x,e^2) &= \frac{9}{4}\ln d^{-1}(x,e^2)+\dots, \end{aligned} \tag{48.37}$$

where $d(x, e^2)$ is defined by the expansion (48.35).

With the aid of equations (48.31) we can also obtain the behavior of the electron Green's function in the region $p^2 \approx -m^2$. In this region the function $G^2(p)$ has a singularity which is called the infrared singularity (the existence of this singularity can be seen, in particular, from formula (47.24′) for $\Sigma^{(2)}$ at $\varkappa \to 0$).

In order to investigate the infrared singularity of the function $s$ we use the second of equations (48.31) and set $y=-1$ in it. Integration of this equation yields

$$\ln\frac{s(x,-1,e^2)}{s(x_0,-1,e^2)}=\int_{x_0}^{x}\frac{dx'}{x'}\left\{\frac{\partial}{\partial\xi}\ln s\left(\xi,-\frac{1}{x'},e^2d(x',-1,e^2)\right)\right\}_{\xi=1}, \tag{48.38}$$

where $x=-(p^2/m^2)$.

We determine the quantity which appears in figure brackets in the integrand in the region $x \sim x_0 \sim 1$. By utilizing the usual perturbation theory expansion (48.36) we obtain in the case of the transverse gauge $(d_l = 0)$

$$\left\{\frac{\partial}{\partial \xi} \ln s\left(\xi, -\frac{1}{x'}, e^2 d(x', -1, e^2)\right)\right\}_{\xi=1} \approx -\frac{3e^2}{8\pi} \frac{x'}{x'-1}, \; x' \sim 1. \quad (48.38')$$

On substituting this expression into (48.38) we obtain the principal part of the integral in the region $x \sim 1$

$$\ln \frac{s(x, -1, e^2)}{s(x_0, -1, e^2)} \approx -\frac{3e^2}{8\pi} \ln \frac{x-1}{x_0-1},$$

whence

$$s(x, -1, e^2) \approx (x-1)^{-\frac{3e^2}{8\pi}}, \quad (48.39)$$

i.e.,

$$s\left(\frac{p^2}{m^2}, e^2\right) \approx (p^2+m^2)^{-\frac{3e^2}{8\pi}}, \quad p^2 \approx -m^2.$$

In the general case of an arbitrary gauge we would have obtained, for $\Sigma^{(2)}$ as can be easily seen by evaluating $\left\{\frac{\partial}{\partial \xi} \ln s\left(\xi, -\frac{1}{x}, e^2 d\right)\right\}_{\xi=1}$ with the aid of formula (47.24′), an expression which differs from (48.38′) by the replacement of 3 by $3\text{-}d_l^0$. Therefore, for $d_l^0 \neq 0$ the infrared singularity of $s$ has the form (1)

$$s\left(\frac{p^2}{m^2}, e^2\right) \approx (p^2+m^2)^{-(3-d_l^0)\frac{e^2}{8\pi}}, \quad \text{where} \quad p^2 \approx -m^2. \quad (48.40)$$

It can be shown (77) that the Green's function must have a singularity of this type not only for an electron, but for any charged particle of arbitrary spin.

### 48.6. *The Problem of Closure of Quantum Electrodynamics*

As we have repeatedly noted, the theory which contains a cut-off momentum $L$ is not a closed one. The possibility of its practical application was based on the assumption that we can choose $L$ in such a way that the region of momenta $|k| > L$ does not play an important role. But only the limit $L \to \infty$ has a rigorous meaning.

The perturbation theory method based on the smallness of the parameter $e^2 \ln (L^2/m^2)$ is obviously unsuitable in principle for the investigation of such a limiting transition. We have previously (cf. subsection 48.2) obtained expressions valid for small $e^2$ and arbitrary $L$.

For sufficiently large $L^2/k^2$ formula (48.17) assumes the form

$$\frac{e^2}{4\pi} d_L(k) = \frac{3\pi}{\ln (L^2/k^2)}. \tag{48.41}$$

In this case the renormalized charge $e_c = Z^{1/2} e$ turns out not to depend on the initial charge

$$\frac{e_c^2}{4\pi} = \frac{3\pi}{\ln (L^2/m^2)}. \tag{48.42}$$

Formula (48.42) is very instructive. The cut-off momentum $L$ which appears in the theory as an auxiliary parameter turns out for $\ln (L^2/m^2) \gg 1/e^2$ to be uniquely related to the physical charge $e_c$. If we take for $e_c^2$ its actual value $e_c^2/4\pi = 1/137$ then we obtain $L = m e^{(3\pi/2)137}$. Large values of $L$ lead to $e_c^2/4\pi < 1/137$, and in the limit $L \to \infty$ $e_c \to 0$.

Thus, although the solutions obtained previously provide a formal possibility for a limiting transition to the exact theory ($L \to \infty$), this transition cannot be actually carried out as it leads to a physically incorrect result — the vanishing of the electron charge.

We shall apply the method described in subsections 48.2 and 48.3 only in the case of small initial charge ($e^2 \ll 1$). Therefore, for the time being we can only conclude from the foregoing that in a consistent theory the primary charge $e$ cannot be small, i.e., that at very small distances a strong interaction must occur.

However, arguments can be given which show that the relation (48.42) must be valid for arbitrary values of $e^2$ (122).

In order to do this we consider the system of equations (44.4) which define the Green's functions. We note that in the first of equations (44.4) defining the electron Green's functions $G^e$ the charge $e^2$ and the photon Green's functions $G^\gamma$ appear only in the form of the product $e^2 G^\gamma$. The vertex function $\Gamma$ cannot be defined by means of an exact integral equation, but it can be characterized by the set of irreducible skeleton diagrams which correspond to expressions containing $e^2$ and $G^\gamma$ also in the form of the product $e^2 D^\gamma$ (cf. (45.20′)). Therefore, we can say that the quantity $e^2d(k) \equiv ie^2k^2G^\gamma(k)$ is defined by an equation of the type (43.43′)

$$\left(\frac{1}{e^2} + \mathrm{P}_1(k)\right) e^2d(k) = 1, \tag{48.43}$$

where $\mathrm{P}_1(k)$ is an operator which is independent of the parameter $e^2$.

Formula (48.41) gives a solution of this equation for large values of $L^2/k^2$ and small $e^2$. This solution corresponds to neglecting the term $1/e^2$ in (48.43) as a result of which the equation no longer contains $e^2$. Therefore, it is natural to suppose that for large $e^2$ we can neglect this term with even greater justification. In actual fact the situation is more complicated, since the operator $\mathrm{P}_1$ in (48.43), being an integral operator, includes the region of integration over $k$ close to $L$ where the approximation (48.41) is not valid. However, this difficulty is associated only with the sharpness of cut-off in momentum space. This could be avoided by introducing other types of cut-off. We could investigate, for example (154), a method of cut-off in which photon momenta are confined to a considerably smaller region of integration than electron momenta. In this case expression (48.41) remains valid over the whole range of variation of $k$ and is a solution of equation (48.43) valid for arbitrary values of $e^2$ (with the exception of very small ones).

A pictorial interpretation of the results (48.41) and (48.42) consists of saying that at small distances a strong polarization of the vacuum occurs. If the initial charge extends over distances of the order of $1/L$, then at large distances $1/k$ (for $\ln (L^2/k^2) \gg 1$) the effective charge $e(k)$ defined by formula (48.41),

$$\frac{e^2(k)}{4\pi} \equiv \frac{1}{4\pi} e^2d(k) = 3\pi/\ln (L^2/k^2),$$

will be small for arbitrarily large values of the primary charge $e$. However, if the initial charge is a point charge ($L \to \infty$), then it turns out to be completely screened at any arbitrary finite distance.

We, therefore, conclude that quantum electrodynamics is not a closed theory. It cannot in principle take into account interactions at distances smaller than $1/L$, where $L$ is in any case greater than the value given by formula (48.42). Therefore, we should expect that the value of the cut-off momentum must have a deep physical meaning, expressing a significant change in the properties of space-time or in the properties of interactions at small distances.

As we have mentioned previously, in the case of electrodynamics this question is not of great importance, since these distances are so small that in the case of phenomena associated with even much larger distances we must take into account phenomena which do not formally enter into the content of quantum electrodynamics (for example, meson processes).

The lack of closure of the formal structure of quantum electrodynamics is of greater importance for theories modelled on quantum electrodynamics for the investigation of strong (meson-nucleon) interactions. In such theories a relation of type (48.42) must also hold (155), and for a large value of the renormalized charge $e_c^2 \sim 1$ this yields $L \sim m$, where $m$ is a quantity of the order of magnitude of the masses of the particles participating in the interactions (mesons, nucleons). Such a value of $L$ corresponds to distances of the order of $10^{-14}$–$10^{-13}$ cm.

Of course, the method of investigation based on introducing a cut-off in momentum space cannot have a rigorous foundation, since it is, in fact, based on a nonlocal interaction of the type

$$\int \bar{\psi}(x)\hat{A}(y)\psi(z)\,\delta_1(x-y)\,\delta_2(x-z)\,dx\,dy\,dz,$$

where $\delta_1$ and $\delta_2$ are functions of effective width of the order of $1/L$ which go over into $\delta$-functions for $L\to\infty$.

But it is not possible to formulate a noncontradictory system of equations with such an interaction for finite $L$. Therefore, it cannot be proved that solution of these equations followed by a subsequent transition to the limit is equivalent to a solution of the equations involving a local interaction.

## § 49. Generalized Green's Functions

### 49.1. *Green's Functions in the Presence of External Fields*

So far we have been investigating Green's functions in the absence of external electromagnetic fields. But, as we know, in a number of problems it is convenient to utilize this auxiliary concept. Therefore, we shall now generalize the results obtained previously on Green's functions to the case when external fields are present. For this purpose it is convenient to consider the field due to external currents $J(x)$. The external currents appear in the Lagrangian in the form of an additional term $L_1(J) = -J_\mu(x) A_\mu(x)$, where $A(x)$ is the operator for the electromagnetic field. In the diagrams this term obviously corresponds to vertices from which only one photon line emerges.

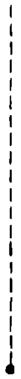


Fig. 96.

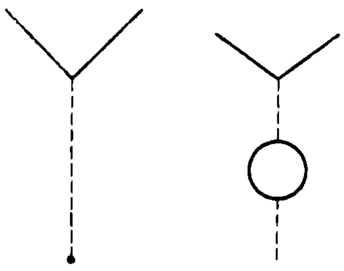


Fig. 97.

The simplest diagram of this type is shown in Fig. 96. It represents emission of a photon by external currents. The matrix element for this process has the form

$$S_{i\to f} = \frac{ie_\mu}{\sqrt{2\omega}} \int J_\mu(x) e^{-ikx}\, d^4x .$$

Figure 97 shows diagrams for pair creation by external currents. The matrix element for this process has in the first approximation the form

$$S_{i\to f} = -ie \int \bar{u}\gamma_\mu v e^{-i(p_1+p_2)x} J_\nu(y)\, D^c_{\mu\nu}(x-y)\, d^4x\, d^4y .$$

We note that this matrix element can be represented in the form

$$S_{i\to f} = -e \int A^{(e)}_\mu(x) \bar{u}\gamma_\mu v e^{-i(p_1+p_2)x}\, d^4x ,$$

where

$$A^{(e)}_\mu(x) = i \int J_\nu(y)\, D^c_{\mu\nu}(x-y)\, d^4y .$$

The quantity $A^{(e)}_\mu(x)$ may be regarded as the potential due to the external currents $J_\mu(x)$. Indeed, since $i D^c(x-y)$ is the Green's function for the wave equation the above expression for $A^{(e)}$ is a solution of the equation $\Box A^{(e)}_\mu = -J_\mu$.

In the general case the diagrams will contain parts joined to other parts by only one photon line (cf., for example, the second diagram of Fig. 97). Such a part is equivalent to a normal vertex which represents the effect of the external field $A^{(e)}$,

$$A^{(e)}_{\mu}(x) = i\int G^{\gamma}_{\mu\nu}(x-y)J_{\nu}(y)\,d^4y. \tag{49.1}$$

Thus, the addition of the interaction $L_1$ is equivalent to the addition of the interaction $L_1'$,

$$L_1' = -\mathrm{j}_{\mu}(x)A^{(e)}_{\mu}(x),$$

where $\mathrm{j}_{\mu}(x)$ is the current operator, and $A^{(e)}_{\mu}(x)$ is the external field.

All the quantities in which we are interested — the Green's function and the matrix elements—are obviously functionals of $J(x)$ or of $A^{(e)}(x)$. States which are stationary in the absence of external currents naturally cease being stationary when $J(x) \neq 0$. However, we can consider the set of states $\Phi_n(J)$ which become stationary when $J(x) = 0$. In this case the matrix element $(r|\mathrm{S}|r) = (\Phi_r(J), \mathrm{S}(J)\Phi_r(J))$ differs from unity when $J \neq 0$. In particular, this also applies to the "vacuum state" (we retain this name in the presence of external currents only for convenience).

In practice this is of no importance, since in actual problems we encounter only weak fields constant in time.

We define the electron propagator in the presence of external fields as

$$G^{e}_{\alpha\beta}(x, y) = \frac{\left(0\middle|T\left(\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)\,\mathrm{S}\right)\middle|0\right)}{(0|\mathrm{S}|0)}. \tag{49.2}$$

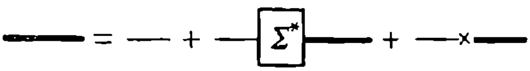


Fig. 98.

By introducing the denominator $(0|\mathrm{S}|0)$ which differs from unity in the presence of an external field we exclude consideration of vacuum loops. We can also use the former graphical definition of $G^e(x, y)$ as an effective electron line.

The equation for $G^e$ augmented by a term which takes the external field into account is shown graphically in Fig. 98, where the cross denotes the vertex involving the external field. The corresponding integral equation has the form

$$G^e(x, y) = S^c(x-y) - i \int S^c(x-x') M(x', y') G^e(y', y)\, d^4x'\, d^4y' \\ -e \int S^c(x-\xi) \hat{A}^e(\xi) G^e(\xi, y)\, d^4\xi. \qquad (49.3)$$

We define the photon propagator as

$$G^{\gamma}_{\mu\nu}(x, y) = \frac{\big(0\,|T\big(A_\mu(x) A_\nu(y) S\big)|\,0\big)}{(0|S|0)} - A^{(e)}_\mu(x) A^{(e)}_\nu(y). \qquad (49.4)$$

It satisfies the same equation (43.43) as in the absence of external currents. The difference consists only of the fact that $G^{\gamma}(x, y)$ and $P(x, y)$, like $G^e(x, y)$ and $M(x, y)$, are functions of the two variables $x, y$ and not of their difference $x-y$.

On applying to equation (49.3) the operator $i\hat{p}+m$, and to the equation (49.4) the operator $\Box$, where $p_\mu = -i\partial/\partial x_\mu$, we can write the equations for $G^e$ and $G^\gamma$ in the following form:

$$\begin{aligned} (i\hat{p}+m+M-ie\hat{A}^{(e)}) G^e(x, y) &= -i\,\delta(x-y), \\ (\Box + P)\, G^{\gamma}(x, y) &= -i\,\delta(x-y), \end{aligned} \qquad (49.5)$$

analogous to equations (43.42) and (43.43).

### 49.2. *Green's Function for Two Electrons. Equation for Bound States of the Electron-Positron System*

We have earlier discussed electron and photon Green's functions. We can also define more complicated Green's functions as vacuum expectation values of different products of Heisenberg field operators. Of the greatest importance is the Green's function for two electrons, since it enables us to obtain the approximate wave equation for two particles, and, in particular, the equation for the bound states of the electron-positron system.

We define the Green's function for two electrons as

$$G^{--}_{\alpha\beta\gamma\delta}(x_1 x_2;\, x_2 x_4) = \frac{\big(0|\,T\big(\psi_\alpha(x_1)\psi_\beta(x_2)\overline{\psi}_\gamma(x_3)\overline{\psi}_\delta(x_4) S\big)|\,0\big)}{(0|S|0)}. \qquad (49.6)$$

Graphically the function $G^{(--)}$ is represented by a set of diagrams terminating in four electron lines.

We divide diagrams defining $G^{(--)}$ into compact ones (Fig. 99) and noncompact ones (Fig. 100), calling those diagrams compact which cannot be divided into parts interconnected by only two electron lines.

It is also useful to introduce the self-energy part for two electrons as the part of the diagram connected to the other parts by four electron lines. We denote the set of such compact self-energy parts by $I(12;\, 34)$ where the indices denote both the coordinates and the spin variables.

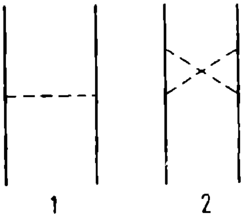


Fig. 99.

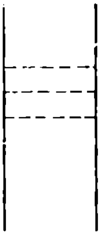


Fig. 100.

With the aid of this quantity we can construct the integral equation for the Green's function $G^{(--)}$. In order to do this we utilize the skeleton diagram of Fig. 101 which shows the relation between $G^{(--)}$ and $I$.

In order to be able to describe further the bound states of the electron-positron system we shall here use notation symmetric with respect to the electron and the positron, viz., we shall denote by $\psi(x)$ and $\bar{\psi}(x)$ the electron annihilation and creation operators, and by $\psi'(x)$ and $\bar{\psi}'(x)$ the positron annihilation and creation operators. These operators are interrelated by the equations

$$\psi'(x) = C\bar{\psi}(x), \qquad \bar{\psi}'(x) = C^{-1}\psi(x),$$

where C is the charge conjugation matrix.

On denoting the Green's function for the electron-positron system by $G^{-+}$ we have in accordance with (49.6)

$$C^{-+}_{\alpha\beta;\gamma\delta}(x_1 x_2;\, x_3 x_4) = \frac{1}{(0|S|0)}(0|\,T\{\psi_\alpha(x_1)\psi'_\beta(x_2)\bar{\psi}_\gamma(x_3)\bar{\psi}'_\delta(x_4)S\}\,|0)$$
$$-\frac{1}{(0|S|0)}(0|\,T\{\psi_\alpha(x_1)\psi'_\beta(x_2)S\}\,|0)\,\frac{1}{(0|S|0)}(0|\,T\{\bar{\psi}_\gamma(x_3)\bar{\psi}'_\delta(x_4)S\}\,|0). \tag{49.7}$$

The second term arises because of the absence of exchange diagrams for noninteracting electrons and positrons. On the basis of Fig. 101 we can write the integral equation (168)

$$G^{-+}(12;34) = G^{-}(13)G^{+}(24) + G^{-}(11')G^{+}(22')I(1'2';1''2'')G^{-+}(1''2'';34), \qquad (49.8)$$

where integration and summation over the variables corresponding to repeated indices are implied, and $G^-$ and $G^+$ denote Green's functions for the electron and the positron (in the general case in the presence of an external field).

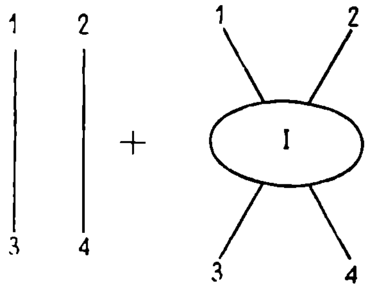
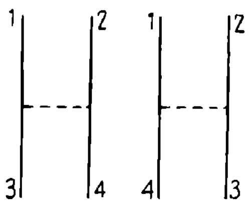


Fig. 101.



(a) (b)

Fig. 102.

We can easily obtain the form of the kernel $I(12;34)$ for an electron and a positron in the first approximation (proportional to $e^2$), which corresponds to the diagrams of Fig. 99 (*1*):

$$I^{(2)}_{\alpha\beta;\gamma\delta}(x_1 x_2; x_3 x_4) = -e^2(\gamma_\mu)_{\alpha\gamma}(C^{-1}\gamma_\mu C)_{\delta\beta} D^c(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_2-x_4) + e^2(\gamma_\mu C)_{\alpha\beta}(C^{-1}\gamma_\mu)_{\delta\gamma} D^c(x_1-x_3)\,\delta(x_3-x_4)\,\delta(x_1-x_2). \qquad (49.9)$$

The matrix $C$ appears in this expression for the reason that we are describing the positron not by the operators $\psi(x)$ and $\bar{\psi}(x)$, but by the operators $\psi'(x)$ and $\bar{\psi}'(x)$. The opposite signs of the terms in (49.9) correspond to the opposite signs in the terms of (37.4); finally, the $\delta$-functions take into account the fact that in the diagram of Fig. 102(*a*) the point $x_1$ appears to coincide with the point $x_3$, while $x_2$ appears to coincide with $x_4$, while in the diagram of Fig. 102(*b*) the point $x_1$ coincides with $x_4$, and $x_2$ with $x_3$.

On utilizing the properties of the matrix $C$ we can rewrite (49.8) in the form

$$
\begin{aligned}
&I_{\alpha\beta;\gamma\delta}(x_1, x_2;\, x_3 x_4)\\
&\quad = e^2(\gamma_\mu)_{\alpha\gamma}(\gamma_\mu)_{\beta\delta} D^c(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_2-x_4)\\
&\quad\quad + e^2(\gamma_\mu C)_{\alpha\beta}(C^{-1}\gamma_\mu)_{\delta\gamma} D^c(x_1-x_3)\,\delta(x_1-x_2)\,\delta(x_3-x_4). \qquad (49.10)
\end{aligned}
$$

On applying to equation (49.10) the operator

$$(i\hat{\mathrm{p}}^{(1)} - ie\hat{\mathrm{A}}(x_1) + \mathrm{M}^{-(1)} + m)\,(i\hat{\mathrm{p}}^{(2)} + ie\hat{\mathrm{A}}(x_2) + \mathrm{M}^{+(2)} + m),$$

$$\mathrm{p}^{(1)}_\mu = -i\frac{\partial}{\partial x_{1\mu}}, \qquad \mathrm{p}^{(2)}_\mu = -i\frac{\partial}{\partial x_2{}^{\mu}},$$

where M is the mass operator defined in § 43, and on taking into account the fact that the single particle Green's functions satisfy equation (49.5), we obtain

$$\mathrm{K}G^{-+}(x_1 x_2;\, x_3 x_4) = -\delta(x_1-x_3)\,\delta(x_2-x_4),$$

where

$$\mathrm{K} = (i\hat{\mathrm{p}}^{(1)} - ie\hat{\mathrm{A}}(x_1) + \mathrm{M}^{-(1)} + m)(i\hat{\mathrm{p}}^{(2)} + ie\hat{\mathrm{A}}(x_2) + \mathrm{M}^{+(2)} + m) + I \qquad (49.11)$$

and

$$IG^{-+}(x_1 x_2;\, x_3 x_4) = \int I(x_1 x_2;\, y_1 y_2)\, G^{-+}(y_1 y_2;\, x_3 x_4)\, d^4 y_1\, u^4 y_2.$$

We see that the function $G^{-+}$ is the Green's function for the equation $\mathrm{K}\varphi(x_1 x_2) = 0$, which is the equation for the wave function $\varphi(x_1 x_2)$ which describes the state of an interacting electron and positron.

The kernel $I$ in the absence of an external field can depend only on the difference of coordinates. On introducing the new variables $x = x_1 - x_2$, $x' = x_3 - x_4$, $X = \frac{1}{2}(x_1 + x_2)$ and $X' = \frac{1}{2}(x_3 + x_4)$, we can say that in this case $I$ will be a function of $x$, $x'$ and $X - X'$:

$$I = I(x, x';\, X - X').$$

In the absence of an external field we seek the wave function $\varphi(x_1 x_2)$ in the form $\varphi(x_1 x_2) = \varphi(x) e^{iKX}$. On going over to the momentum representation we obtain, in the approximation proportional to $e^2$,

$$\varphi(p) = \frac{1}{m + i\hat{p}_1}\,\frac{1}{m + i\hat{p}_2}\int I^{(2)}(p, p';\, K)\,\varphi(p')\, d^4 p', \qquad (49.12)$$

where

$$p_1 = p + \frac{1}{2}K, \qquad p_2 = -p + \frac{1}{2}K,$$

$$\varphi(p) = \frac{1}{(2\pi)^2}\int \varphi(x)\, e^{-ipx} d^4x,$$

$$I^{(2)}(p, p'; K) = \frac{1}{(2\pi)^4}\int I^{(2)}(x, x'; Y) e^{-i(px - p'x' + KY)} d^4x\, d^4x'\, d^4Y$$

$$= -\frac{ie^2}{(2\pi)^4}\left((\gamma_\mu)_{\alpha\gamma}(\gamma_\mu)_{\beta\delta}\frac{1}{(p-p')^2 - i0} + (\gamma_\mu C)_{\alpha\beta}(C^{-1}\gamma_\mu)_{\delta\gamma}\frac{1}{K^2}\right).$$

If we go over to the coordinate system in which the center of mass of the particles is at rest, i.e., if we set $K = (0,0,0, iK_0)$, we obtain an equation for the determination of the energy eigenvalues $K_0$ of the system of two interacting particles. In this case, if $K_0$ is smaller than $2m$, we shall obviously obtain a bound state of the electron-positron system (in any coordinate system this condition has the form $K^2 > (2m)^2$).

In the nonrelativistic approximation (up to terms of order $v^2/c^2$) we can obtain from this expression the Schrödinger equation for the electron-positron system which was used in Chapter VI.

### 49.3. *Equations for Green's Functions in Terms of Variational Derivatives*

The foregoing equations for the Green's functions $G^e$ and $G^\gamma$ define these quantities if the polarization and the mass operators $P = i\Pi^*$, $M = i\Sigma^*$ are known. Equations (44.2) and (44.3) express these operators in terms of the Green's functions $G^e$, $G^\gamma$ and the vertex function $\Gamma$.

As we have noted previously (cf. subsection 44.2), it is impossible to construct an integral equation which would in turn express $\Gamma$ in terms of $G^e$ and $G^\gamma$, and which together with the first two equations would give us a complete system of equations determining the Green's functions.

We shall now show that it is possible to find a relation connecting the vertex function $\Gamma$ with the electron Green's function $G^e$. However, this relation has the form not of an integral, but of a functional (variational) equation.

With this aim in view we investigate the change in the mass operator $M$ resulting from a variation of the external field $A^{(e)}$. Let an infinitesimal term $\delta A^{(e)}$ be added to the function $A^{(e)}$. The corresponding addition to the mass operator $M$ is shown in Fig. 103. We see that the diagram of Fig. 103 represents a certain vertex part $\Lambda$, and therefore

$$\delta\Sigma^*(x,y) = -e\int \Lambda_\mu(x,y;\xi)\,\delta A_\mu^{(e)}(\xi)\,d^4\xi,$$

i.e.,

$$-e\Lambda_\mu(x,y;\xi) = \frac{\delta\Sigma^*(x,y)}{\delta A_\mu^{(e)}(\xi)}. \tag{49.13}$$

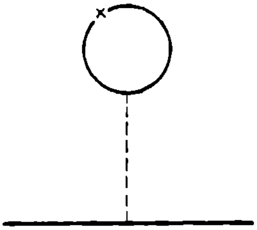


Fig. 103.

On utilizing (43.40) we obtain

$$\Gamma_\mu(x,y;\xi) = \frac{\delta\left(G^e(x,y)\right)^{-1}}{\delta A_\mu^{(e)}(\xi)}, \tag{49.13'}$$

where $\Gamma_\mu(x,y;\xi) = \Lambda(x,y;\xi)+\gamma_\mu\delta(x-y)\delta(y-\xi)$ and $\left(G^e(x,y)\right)^{-1}$ can be obtained from the relation

$$\int \left(G^e(x,y)\right)^{-1} G(y,z)\,d^4y = \delta(x-z).$$

On substituting this expression for $\Gamma$ into (44.2), (44.3), and on noting that in accordance with (49.1)

$$\delta A_\mu^{(e)}(x) = i\int G^\gamma_{\mu\nu}(x,y)\,\delta J_\nu(y)\,d^4y,$$

we obtain

$$\begin{aligned}\int \Sigma^*(x,z)\,G^e(z,y)\,d^4z &= ie\gamma_\mu\frac{\delta G^e(x,y)}{\delta J_\mu(x)},\\ i\int \Pi^*(x,z)\,G^\gamma(z,y)\,d^4z &= e\,\mathrm{Sp}\,\gamma_\mu\frac{\delta G^e(x,x)}{\delta J_\mu(y)}.\end{aligned} \tag{49.14}$$

Thus, equations (49.5) assume the form

$$\begin{aligned}\left[i\hat{p}+m-ieA^{(e)}(x)-e\gamma_\mu\frac{\delta}{\delta J_\mu(x)}\right]G^e(x,y) &= -i\delta(x-y),\\ \Box G^\gamma(x,y) = -i\delta(x-y)-e\,\mathrm{Sp}\,\gamma_\mu&\frac{\delta G^e(x,x)}{\delta J_\mu(y)}.\end{aligned} \tag{49.15}$$

This system of variational equations for the Green's function was obtained by Schwinger (177).

Equations (49.15) can also be obtained in a formal manner without resorting to graphical methods. In order to do this we evaluate the variation with respect to $J(x)$ of the expression for $G^e(x,y)$. On utilizing the definition (49.2) and the explicit expression for the scattering matrix S in the presence of external currents

$$\mathrm{S} = T\{e^{i\int j(x)+J(x))\mathrm{A}(x)d^4x}\},$$

we obtain

$$\begin{aligned}\frac{\delta G^e_{\alpha\beta}(x,y)}{\delta J_\mu(x)} &= \frac{i}{(0|\mathrm{S}|0)}\Big(0|\,T\big(\psi_\alpha(x)\overline{\psi}_\beta(y)\mathrm{A}_\mu(x)\big)\mathrm{S}\,|0\Big)\\ &-i\frac{\Big(0|\,T\big(\psi_\alpha(x)\overline{\psi}_\beta(y)\mathrm{S}\big)|0\Big)}{(0|\mathrm{S}|0)}\cdot\frac{\Big(0|T\big(\mathrm{A}_\mu(x)\mathrm{S}\big)|0\Big)}{(0|\mathrm{S}|0)}.\end{aligned} \tag{49.16}$$

On defining the external field

$$A^{(e)}(x) = \frac{\Big(0|\,T\big(\mathrm{A}(x)\mathrm{S}\big)|0\Big)}{(0|\mathrm{S}|0)},$$

we can write the second term in the right hand side of this equation in the form $iG^e(x,y)A^{(e)}_\mu(x)$. The first term, which is equal to

$$i\Big(0|\,T\big(\psi_\alpha(x)\overline{\psi}_\beta(y)\,\mathrm{A}_\mu(x)\big)|0\Big),$$

can be transformed by means of the relation

$$(i\hat{p}+m)_{\alpha\alpha'}T\big(\psi_{\alpha'}(x)\overline{\psi}_\beta(y)\big) = ieT\big(\hat{A}_{\alpha\alpha'}(x)\psi_{\alpha'}(x)\overline{\psi}_\beta(y)\big)-i\delta(x-y)\,\delta_{\alpha\beta}. \tag{49.17}$$

To prove this we utilize the fact that $T(\psi_\alpha(x)\overline{\psi}_\beta(y))$ represents a function which is continuous for $x_0 > y_0$ and for $x_0 < y_0$, but which has a discontinuity where it jumps from $\bar{\psi}_\beta(y)\,\psi_\alpha(x)$ at $x_0 = y_0-0$ to $\psi_\alpha(x)\,\bar{\psi}_\beta(y)$ at $x_0 = y_0+0$. The value of this discontinuity is equal to

$$\bar{\psi}_\beta(y)\psi_\alpha(x)+\psi_\alpha(x)\bar{\psi}_\beta(y) = \gamma_4\delta(x-y)\,\delta_{\alpha\beta}.$$

The derivative at the point of discontinuity is equal to the magnitude of the discontinuity multiplied by $\delta(x_0 - y_0)$. Thus, we have

$$i\hat{p}\Big(T\big(\psi_\alpha(x)\bar{\psi}_\beta(y)\big)\Big) = i\,T\big(\hat{p}\psi_\alpha(x)\bar{\psi}_\beta(y)\big)-i\delta(x-y)\,\delta_{\alpha\beta}.$$

Further, by utilizing the Dirac equation for the operator $\psi(x)$

$$(i\hat{p}+m-ie\hat{A}(x))\psi(x) = 0,$$

we obtain equation (49.17).

On substituting (49.17) into (49.16) we obtain equation (49.15).

In a similar manner we can easily obtain the variational equation for the photon Green's function. In order to do this we must utilize

$$\Box A_\mu(x) = -ie\left[\bar{\psi}(x),\gamma_\mu\psi(x)\right]-J_\mu(x).$$

from which it follows that $\Box A_\mu^{(e)} = -J_\mu(x)+ie\mathrm{Sp}\gamma_\mu G^e(x, x)$. By varying the last equation with respect to $J(x)$ we obtain second equation (49.15).

If in equations (49.15) we replace the variation with respect to $J(x)$ by variation with respect to $A^{(e)}(x)$ in accordance with (49.1), we obtain the system of equations

$$\big(i\hat{p}+m-ieA^{(e)}(x)\big)G^e(x, y) -ie\int G^\gamma(x, z)\gamma_\mu\frac{\delta}{\delta A_\mu^{(e)}(z)}G^e(z, y)\,d^4z = -i\delta(x-y), \qquad (49.17')$$

$$\Box G^\gamma(x, y) = -i\delta(x-y)-ie\,\mathrm{Sp}\int d^4zG^\gamma(z, y)\gamma_\mu\frac{\delta G^e(x, x)}{\delta A_\mu^{(e)}(z)}\,d^4z.$$

### 49.4. *Expressions for Green's Functions in Terms of Functional Integrals*

Green's functions can also be represented in the form of certain closed expressions containing functional (continuous-dimensional) integration (63), (74), (28), (54), (132). Although such a representation has not in practice led to any new results, nevertheless, its study presents a definite interest.

First of all we construct the expression for the vacuum expectation value of the operator S $= T(\sigma)$, where

$$\sigma = e^{i \int j_\mu(x) A_\mu(x) d^4} \tag{49.18}$$

and $T$ is the symbol for the chronological product.

We also write $T$ in the form $T = T_\psi T_A$, where the operators $T_\psi$ and $T_A$ correspond to time ordering of the operators of the electromagnetic field A and the electron-positron field $\psi$.

The chronological product of operators of the electromagnetic field can be expressed in terms of the normal product. In accordance with (24.17) we have $T_A(\sigma) = N_A(e^\Delta \sigma)$, where $N_A$ is the symbol for the normal product of the operators A and

$$\Delta = \frac{1}{2} \int d^4\xi d^4\eta D^c_{\mu\nu}(\xi - \eta) \frac{\delta^2}{\delta A_\mu(\xi) \delta A_\nu(\eta)}.$$

To simplify further calculations we formally replace integration over four-dimensional space by summation over a discrete number of points. On introducing the change of notation

$$\mathrm{j}(x)\sqrt{d^4x} = \mathrm{j}_x; \qquad \mathrm{A}(x)\sqrt{d^4x} = \mathrm{A}_x; \qquad D^c(\xi - \eta)\sqrt{d^4\xi d^4\eta} = D^c_{\xi\mu},$$

we obtain

$$\sigma = e^{i\mathrm{j}_x \mathrm{A}_x},$$

$$\Delta = \frac{1}{2} D^c_{\xi\eta} \frac{\partial^2}{\partial \mathrm{A}_\xi \partial \mathrm{A}_\eta}, \tag{49.19}$$

$$T_A(\sigma) = N_A\left(e^\Delta e^{i\mathrm{j}_x \mathrm{A}_x}\right)$$

where it is understood that a summation is taken over repeated indices. Now the operator $\sigma$, which in accordance with (49.19) is a functional of $\mathrm{A}(x)$ and $j(x)$, is written in the form of a function of an infinite number of variables $\mathrm{A}_x$ and $j_x$.

We evaluate the expectation value $\langle T_A(\sigma) \rangle$ of the operator $T_A(\sigma)$ for the vacuum state of the electromagnetic field. Since the vacuum expectation value of any normal product of operators is equal to zero, we have

$$\langle T_A(\sigma) \rangle = \left(e^{\frac{1}{2} D^c_{\xi\eta} \frac{\partial^2}{\partial A_\xi \partial A_\eta}} e^{\mathrm{j}_x A_x}\right)_{A_x = 0},$$

where $A_x$ may be regarded as $c$-numbers.

We evaluate the Fourier transform of the function $\sigma(\mathrm{j}_x, A_x)$ with respect to the variables $A_x$

$$\sigma(\Lambda) = \int \sigma(\mathrm{j}, A) e^{-i\Lambda_\xi A_\xi} \prod_x \frac{dA_x}{2\pi},$$

then we have

$$\langle T_A(\sigma) \rangle = \int \sigma(\mathrm{j}, A) \prod_y \frac{dA_y}{2\pi} \int e^{-\frac{1}{2} D^c_{\xi\eta} \Lambda_\xi \Lambda_\eta - i\Lambda_\xi A_\xi} \prod_\zeta d\Lambda_\zeta .$$

The quadratic form appearing in the exponent can be transformed to its principal axes. This changes the infinite-fold integral into a product of one-dimensional integrals, and allows us to carry out the integration. In this manner we obtain

$$\langle T_A(\sigma) \rangle = C \int \sigma(\mathrm{j}, A) e^{-\frac{1}{2} D^{c-1}_{\xi\eta} A_\xi A_\eta} \prod_x \frac{dA_x}{2\pi}, \tag{49.20}$$

where

$$C = \prod_\zeta \int e^{-\frac{1}{2} D^c_\zeta \Lambda^2_\zeta} d\Lambda_\zeta$$

and $D^c_\zeta \equiv D^c_{\zeta\zeta}$ is the matrix $D^c_{\zeta\eta}$ brought to diagonal form. This expression contains the operators $\psi$. Further, on averaging over the vacuum state of the electron-positron field, we obtain

$$(0|\mathrm{S}|0) = C \int S^0(A) e^{-\frac{1}{2} (D^c)^{-1}_{\xi\eta} A_\xi A_\eta} \prod_x \frac{dA_x}{2\pi}, \tag{49.21}$$

where

$$S^0(A) = \langle T_\psi(\sigma) \rangle$$

and the brackets denote averaging over the vacuum of the electron-positron field.

The expression for the Green's function can also be brought into a form analogous to (49.21). On carrying out the same transformations as before we obtain

$$(0|\mathrm{S}|0) G^e_{\alpha\beta}(x, y) = \int e^{-\frac{1}{2} (D^c)^{-1}_{\xi\eta} A_\xi A_\eta} \cdot \langle T_\psi \left( \psi_\alpha(x) \overline{\psi}_\beta(y) \sigma(A, \mathrm{j}) \right) \rangle \prod_z \frac{dA_z}{2\pi} . \tag{49.22}$$

We see that the problem reduces to the determination of the Green's function for an electron in the external field $A$

$$S^0(A)G_{\alpha\beta}(x,y;A) \equiv \langle T_\psi(\psi_\alpha(x)\bar{\psi}_\beta(y)\sigma(\mathrm{j},A))\rangle. \tag{49.23}$$

It follows from (49.23) that $G(x,y;A)$ satisfies the equation

$$\big(i\hat{\mathrm{p}}+m-ie\hat{A}(x)\big)G(x,y;A) = -i\delta(x-y).$$

If $G(x,y;A)$ and $S^0(A)$ are known, then $G^e(x,y)$ can be expressed in terms of these functions as

$$G^e(x,y) = \frac{\int G(x,y;A)e^{-\frac{1}{2}(D^c)^{-1}_{\xi\eta}A_\xi A_\eta}S^0(A)\prod\limits_x dA_x}{\int \exp\big(-\frac{1}{2}(D^c)^{-1}_{\xi\eta}A_\xi A_\eta\big)S^0(A)\prod\limits_x dA_x} \tag{49.24}$$

i.e., the Green's function $G^e(x,y)$ is obtained from the Green's function $G(x,y;A)$ in the given external field $A$ by averaging over all possible fields $A$ with a weighting function given by exp $(-\frac{1}{2}D^c_{\xi\eta}A_\xi A_\eta)S^0(A)$.

By utilizing a similar method the photon Green's function can be written in the form

$$G^\gamma_{\mu\nu}(x,y) = \frac{\int A_\mu(x)A_\nu(y)e^{-\frac{1}{2}(D^c)^{-1}_{\xi\eta}A_\xi A_\eta}S^0(A)\prod\limits_x dA_x}{\int \exp\big(-\frac{1}{2}(D^c)^{-1}_{\xi\eta}A_\xi A_\eta\big)S^0(A)\prod\limits_x dA_x}. \tag{49.25}$$

We now obtain the expression for $S^0(A)$. We introduce the notation

$$\chi(x) = e\bar{\psi}(x)\hat{A}(x),$$

$$\chi(x)\sqrt{d^4x} = \chi_x; \qquad \psi(x)\sqrt{d^4x} = \psi_x.$$

Then $S^0(A) = \langle T_\psi(e^{-\chi_x\psi_x})\rangle$. We expand this expression into a series in powers of $\chi_x\psi_x$, and in each term of the series express the $T$-product in terms of $N$-products in accordance with the formulas of § 24. On taking into account the fact that the expectation values of $N$-products vanish, we obtain

$$S^0(A) = \sum_n \frac{(-1)^n}{n!}D_n(K), \tag{49.26}$$

where $D_n(K)$ is a determinant of the $n$th order constructed of the quantities $K_{x_ix_j} = \psi_{x_i}\chi_{x_j}$.

The series (49.26) can be summed and yields

$$S^0(A) = D(K), \tag{49.27}$$

where

$$D(K) = \|\delta_{ij} - K_{x_i x_j}\|$$

is an infinite determinant (the Fredholm denominator).

Expression (49.27) can also be written in a different form. We assume that the matrix of the determinant $D(K)$ is brought to diagonal form in such a way that $K_{\alpha\beta} = \delta_{\alpha\beta} K_\alpha$. Then $S^0(A) = \prod\limits_\alpha (1-K_\alpha)$ and

$$\ln S^0(A) = \sum_\alpha \ln(1-K_\alpha) \equiv \operatorname{Sp} \ln(1-K). \tag{49.28}$$

Since the last expression contains only the trace of the matrix, it is invariant with respect to the choice of the representation.

The matrix $\ln(1-K)$ can be written in the form

$$\ln(1-K) = -K \int_0^1 \frac{d\lambda}{(1-\lambda K)}$$

or, by utilizing the definition of $K$,

$$\ln(1-K) = -e \int \frac{\hat{A}\, d\lambda}{\dfrac{1}{S^c} + e\lambda\hat{A}}. \tag{49.29}$$

We compare the integrand in (49.29) with the Green's function $G^e(x, y; A)$ for the Dirac equation for an electron in an external field $A$. In matrix form such an equation has the form

$$\left((S^c)^{-1} - e\hat{A}\right) G(A) = 1,$$

whence

$$\left((S^c)^{-1} - e\hat{A}\right)^{-1} = G(A). \tag{49.30}$$

We see that

$$\ln(1-K) = -e\hat{A} \int_0^1 G(\lambda A)\, d\lambda.$$

In the usual notation the last formula has the form

$$\operatorname{Sp} \ln(1-K) = -\operatorname{Sp} \int_0^1 d\lambda \int G(x, x; \lambda A) \hat{A}(x)\, d^4x, \tag{49.31}$$

where the symbol Sp refers only to the spinor variables.

Finally we obtain

$$S^0(A) = \exp\left(-e \int_0^1 d\lambda G(x, x; \lambda A) A(x) d^4x\right). \tag{49.32}$$

Thus, the foregoing method reduces the general problem of quantum electrodynamics to finding the Green's function for the Dirac equation in an arbitrary external field, and to subsequent averaging by means of functional integration.

On solving equation (49.30) by expansion into a series in powers of $eA$ and on substituting these solutions into (49.32) and corresponding integrals over $A$, we can develop a perturbation theory equivalent to the usual perturbation theory of quantum electrodynamics.

It is obvious that in the zero order approximation $G^e$ coincides with $S^c$ and $S^0 = 1$. In this case we can easily obtain from (49.25) that $G^\gamma = D^c$. In the general case the factor $S^0(A)$ in (49.25) takes into account the polarization of the vacuum, while the factor $\exp\left(-\frac{1}{2}(D^c)^{-1}_{\xi\eta} A_\xi A_\eta\right)$ corresponds to the free electromagnetic field. These two factors correspond to the two terms $P(k)$ and $k^2$ in the equation for $G^\gamma$.

We note that in accordance with (49.30) and (49.32) $S^0(A)$ is a function only of the product $eA$: $S^0(A) = F(eA)$. Therefore, on choosing for the variables of integration in (49.25) in place of $A_x$ the variables $B_x = eA_x$, we can write the expression for the function $G^\gamma$ in the form

$$e^2 G^\gamma_{\mu\nu}(x-y) = \frac{\int B_\mu(x) B_\nu(y) e^{-\frac{1}{2e^2}(D^c)^{-1}_{\xi\eta} B_\xi B_\eta} F(B) \prod\limits_z dB_z}{\int e^{-\frac{1}{2e^2}(D^c)^{-1}_{\xi\eta} B_\xi B_\eta} F(B) \prod\limits_z dB_z}. \tag{49.33}$$

We see that $e^2$ appears in the right-hand side of (49.33) only in the term corresponding to the free field, whose role becomes less important as $e^2$ increases. Thus, we obtain a result arrived at in a different manner in subsection 48.5: the function $e^2 G^\gamma$ does not depend on $e^2$ and therefore must be of the form (48.41).

# Radiation Corrections to Electromagnetic Processes

## § 50. Effective Potential Energy of the Electron. Radiation Corrections to the Electron Magnetic Moment and to Coulomb's Law

### 50.1. *Energy of Interaction of the Electron with the Electromagnetic Field Taking into Account Corrections of Order $\alpha$*

We now proceed to study specific physical effects associated with the higher order approximations of perturbation theory. Such effects have the general name of *radiation corrections.*

We start by determining the radiation corrections to the energy of interaction between an electron and an external electromagnetic field.

The interaction between an electron and an external electromagnetic field is determined in accordance with the Dirac equation by the quantity $U = e\gamma_\mu A_\mu^{(e)}(x)$, where $A_\mu^{(e)}(x)$ is the potential of the external field and $e$ is the physical charge of the electron (in this Chapter $e$ and $m$ will denote the physical charge and the physical mass of the electron).

The matrix element of this quantity determines the scattering of the electron by the field $A_\mu^{(e)}(x)$ in the first approximation of perturbation theory.

We wish to find the radiation corrections to the quantity $U$. For this it is obviously necessary to replace in the expression for $U$ the matrix $\gamma_\mu$ by $\Gamma_\mu$ and $A_\mu^{(e)}(x)$ by $A_\mu^{(e)}(x) + \delta A_\mu^{(e)}(x)$, where $\Gamma_\mu = \gamma_\mu + \Lambda_\mu$, $\Lambda_\mu$ is the vertex part and $\delta A_\mu^{(e)}(x)$ is the increment in the external potential due to the interaction between the electromagnetic field and the electron-positron field. The Fourier component of the first of these quantities is given in the second approximation of perturbation theory (in the case of external electron lines) by formula (47.52), while the Fourier component of the second quantity is given by formula (47.36). On neglecting the product $\Lambda_\mu \delta A_\mu^{(e)}(x)$, and on utilizing these formulas, we obtain the following expression for the Fourier component of the radiation correction to the quantity $U$ in the second approximation of perturbation theory:

$$\delta U(q) = e\big(\Lambda_\mu^{(3)}(q)\,A_\mu^{(e)}(q)+\gamma_\mu\delta A_\mu^{(e)}(q)\big) = F_\nu(q)\,A_\nu^{(e)}(q),$$

where $\Lambda_\mu^{(3)}(q)\equiv\Lambda_\mu^{(3)}(p_1, p_2;\, q)$ and

$$F_\nu(q) = -\frac{e^3}{(2\pi)^2}\Bigg\{\gamma_\nu\Bigg[\left(\frac{2\theta}{\tan 2\theta}-1\right)\left(\ln\frac{m}{\lambda}-1\right) - \frac{\theta}{2}\tan\theta - \frac{2}{\tan 2\theta}\int\limits_0^\theta \xi\tan\xi\, d\xi\Bigg] + \frac{i}{8m}(\hat{q}\gamma_\nu-\gamma_\nu\hat{q})\frac{2\theta}{\sin 2\theta} - \gamma_\mu\frac{1}{q^2}(q_\mu q_\nu - q^2\delta_{\mu\nu})\left[\frac{4m^2-2q^2}{-3q^2}(1-\theta\cot\theta)-\frac{1}{9}\right]\Bigg\}. \tag{50.1}$$

If we write $U(q)+\delta U(q)$ in the form

$$U(q)+\delta U(q) = e\gamma_\mu A_\mu^{(e)}(q)+ie\gamma_4\delta\varphi(q),$$

where

$$e\delta\varphi(q) = \frac{1}{i}\,\gamma_4\,\delta U(q), \tag{50.1'}$$

then the latter quantity can be regarded as the Fourier component of the increment of the potential energy of the electron due to the interaction between the electron and the vacuum. We call this increment *the effective potential energy* which determines the interaction of the electron with the vacuum.

If we know the Fourier component $e\delta\varphi(q)$ we can obtain the effective potential energy which determines the interaction of the electron with the vacuum as a function of the coordinates and the time:

$$e\delta\varphi(x) = \frac{1}{(2\pi)^4 i}\int \beta F_\nu(q) A_\nu^{(e)}(q)\, e^{iqx} d^4q,\ \beta = \gamma_4. \tag{50.2}$$

On substituting into this expression

$$A_\nu^{(e)}(q) = \int A_\nu^{(e)}(y) e^{-iqy} d^4y,$$

we obtain

$$e\delta\varphi(x) = \frac{1}{(2\pi)^4 i}\int \beta F_\nu(q)\, e^{iq(x-y)} A_\nu^{(e)}(y)\, d^4q\, d^4y$$

$$= \frac{1}{i}\int \beta F_\nu(x-y)\, A_\nu^{(e)}(y)\, dy,$$

where

$$F_\nu(x) = \frac{1}{(2\pi)^4}\int F_\nu(q)\, e^{iqx}\, d^4q.$$

We see that the effective potential energy of interaction of the electron with the vacuum is determined by an integral operator acting on the external field.

On noting that $q_\nu e^{iqx} = (1/i)(\partial/\partial x_\nu)e^{iqx}$, we can rewrite expression (50.2) in the form

$$e\delta\varphi(x) = \frac{1}{i}\beta F_\nu\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\frac{1}{(2\pi)^4}\int A_\nu^{(e)}(q)\, e^{iqx} d^4q = \frac{1}{i}\beta F_\nu\left(\frac{1}{i}\frac{\partial}{\partial x}\right)A_\nu^{(e)}(x). \tag{50.3}$$

We assume that the external field $A_\nu^{(e)}(x)$ is a slowly varying function of the coordinates and the time. In such a case the function $F_\nu(q)$ can be approximated by a power series in $q$; the operator $F_\nu(i^{-1}\partial/\partial x)$ will in this case be a differential operator acting on $A_\mu^{(e)}(x)$.

On confining ourselves to only the linear and the quadratic terms in the expansion for $F_\nu(q)$ we obtain in accordance with (50.1)

$$F_\nu(q) = -\frac{e^3}{(4\pi)^2}\left\{\frac{4}{3m^2}q^2\left(\ln\frac{m}{\lambda} - \frac{3}{8} - \frac{1}{5}\right)\gamma_\nu + \frac{i}{2m}(\hat{q}\gamma_\nu - \gamma_\nu\hat{q})\right\}. \tag{50.4}$$

Therefore, the effective potential energy of the electron will be given in the third approximation of perturbation theory by

$$e\delta\varphi(x) = \frac{e^3}{(4\pi)^2}\left\{\frac{4}{3m^2}\frac{1}{i}\left(\ln\frac{m}{\lambda} - \frac{3}{8} - \frac{1}{5}\right)\beta\gamma_\nu \Box A_\nu^{(e)}(x) - \frac{1}{2im}\beta\gamma_\mu\gamma_\nu F_{\mu\nu}(x)\right\}, \tag{50.5}$$

where

$$F_{\mu\nu}(x) = \frac{\partial}{\partial x_\mu}A_\nu^{(e)} - \frac{\partial}{\partial x_\nu}A_\mu^{(e)}.$$

On introducing the electron spin matrices

$$\Sigma_1 = \frac{1}{i}\gamma_2\gamma_3, \quad \Sigma_2 = \frac{1}{i}\gamma_3\gamma_1, \quad \Sigma_3 = \frac{1}{i}\gamma_1\gamma_2,$$

we can rewrite $e\delta\varphi(x)$ in the form

$$e\delta\varphi(x) = -\frac{e^3}{(4\pi)^2 m}(\beta\cdot\boldsymbol{\Sigma H}-i\beta\boldsymbol{\alpha E})$$

$$+\frac{e^3}{(4\pi)^2}\frac{4}{3m^2}\left(\ln\frac{m}{\lambda}-\frac{3}{8}-\frac{1}{5}\right)(\Box\varphi-\boldsymbol{\alpha}\,\Box \boldsymbol{A}), \qquad (50.6)$$

where $\boldsymbol{E}, \boldsymbol{H}$ are the electric and magnetic fields.

The expression for the effective potential energy $e\delta\varphi(x)$ contains the photon "mass" $\lambda$. This is associated with the fact that formula (50.2) was obtained by means of perturbation theory, which, as we have already pointed out in § 30, is, generally speaking, inapplicable to the description of the interaction of an electron with long wavelength photons. Therefore, formula (50.6) takes into account, strictly speaking, only the interaction of the electron with short wavelength photons, while if its interaction with long wavelength photons is required, it must be evaluated separately.

The individual terms in expression (50.6) admit a simple physical interpretation. Let us first of all consider the second term (207).

We assume for the sake of simplicity that the external field is an electrostatic field. Due to the existence of zero point oscillations of the field an electron experiences forces which result in its additional displacement. This displacement, which we denote by $\delta \boldsymbol{r}$, is associated with fluctuations in the potential energy of the electron, given by

$$e\varphi(\boldsymbol{r}+\delta\boldsymbol{r})-e\varphi(\boldsymbol{r}) = e\left(\delta\boldsymbol{r}\nabla+\tfrac{1}{2}(\delta\boldsymbol{r}\nabla)^2+\ldots\right)\varphi(\boldsymbol{r}).$$

On neglecting in this expression all powers of $\delta\boldsymbol{r}$ higher than the second, and on averaging over all possible values of $\delta\boldsymbol{r}$, we obtain

$$\langle e\varphi(\boldsymbol{r}+\delta\boldsymbol{r})-e\varphi(\boldsymbol{r})\rangle_0 = \tfrac{1}{6}e(\delta\boldsymbol{r})_0^2\Delta\varphi(\boldsymbol{r}),$$

where $(\delta\boldsymbol{r})_0^2$ is the average value of the square of the electron displacement due to the zero point oscillations of the field.

In order to make an estimate of the order of magnitude of $(\delta\boldsymbol{r})_0^2$ we start with the equation of motion for the electron, taking into account the effect of the zero-point oscillations of the field

$$\delta\ddot{\boldsymbol{r}} = \frac{e}{m}\boldsymbol{E}_0,$$

where $E_0$ is the electric field due to the zero-point oscillations. On expanding $\delta r$ and $E_0$ into monochromatic waves we obtain

$$|\delta r_\omega|^2 = \tfrac{1}{2}\frac{e^2}{m^2\omega^4}E_\omega^2,$$

where $\delta r_\omega$ and $E_\omega$ are the Fourier components of the quantities $\delta r$ and $E_0$. The component $E_\omega$ is determined by the relation $\frac{1}{2}E_\omega^2 = \omega/2$ (the normalization volume is assumed to be equal to unity), and therefore

$$|\delta r_\omega|^2 = \tfrac{1}{2}\frac{e^2}{m^2\omega^4}\omega .$$

On multiplying $|\delta r_\omega|^2$ by the number of modes of oscillation in the frequency interval $d\omega$, equal to $(2/(2\pi)^3)4\pi\omega^2 d\omega$, and on integrating over $\omega$, we obtain $(\delta r)_0^2$:

$$(\delta r)_0^2 = \frac{e^2}{2\pi^2 m^2}\int\frac{d\omega}{\omega}.$$

This expression diverges logarithmically both at high and at low frequencies. The divergence at high frequencies is associated with the general divergence in quantum electrodynamics in the region of large momenta of virtual particles; while the divergence at low frequencies is a manifestation of the previously mentioned infrared divergence, associated with the incorrect treatment of the interaction of the electron with long wavelength photons. We eliminate this divergence by ascribing to the photon a "mass" $\lambda$.

We take the upper limit in the "regularized" value of the integral to be equal in order of magnitude to $m$; thus, in order of magnitude $(\delta r)_0^2$ is equal to

$$(\delta r)_0^2 = \frac{e^2}{2\pi^2 m^2}\left(\ln\frac{m}{\lambda}+C\right), \tag{50.7}$$

where $C$ is a numerical constant of order of magnitude unity, which, naturally, cannot be determined in an elementary manner.

By utilizing this value of $(\delta r)_0^2$, we obtain the expression for the mean fluctuation of the electron potential energy

$$\langle e\varphi(r+\delta r)-e\varphi(r)\rangle_0 = \frac{e^3}{12\pi^2 m^2}\left(\ln\frac{m}{\lambda}+C\right)\Delta\varphi(r). \tag{50.7'}$$

On comparing this expression with formula (50.6) for $e\delta\varphi(x)$ we see that it coincides in order of magnitude with the second term in $e\delta\varphi(x)$ in the case of a purely electrostatic field.

We can easily obtain a relativistic generalization of formula (50.7′) for the case of an arbitrary electromagnetic field (which, however, varies sufficiently slowly). In order to do this it is, obviously, necessary to replace in the right hand side of (50.7′), firstly, the Laplacian by the d'Alembertian operator and, secondly, the scalar potential $\varphi$ by $\varphi-\boldsymbol{\alpha}\boldsymbol{A}$. We thus obtain the order of magnitude of the second term in expression (50.6) for the effective potential energy of the electron. Consequently, this term can be interpreted as the fluctuation of the potentials associated with the displacement of the electron under the action of the zero-point oscillations of the field.

### 50.2. *Radiation Corrections to the Electron Magnetic Moment*

We now investigate the first term in (50.6).

We assume that the electron is situated in a constant magnetic field. Then the effective potential energy of the electron, taking into account its interaction with the vacuum, will contain only the single term

$$e\delta\varphi(x) = -\frac{e^3}{(4\pi)^2 m}\beta\boldsymbol{\Sigma}\boldsymbol{H}.$$

On the other hand, the operator for the electron energy in a magnetic field is of the form

$$U_m = -\mu\beta\boldsymbol{\Sigma}\boldsymbol{H},$$

where $\mu$ is the magnetic moment of the electron (in the nonrelativistic approximation the matrix $\beta$ reduces to unity). A comparison of these formulas shows that as a result of its interaction with the vacuum the electron acquires an additional magnetic moment which in the third approximation of perturbation theory is equal to

$$\mu' = \frac{\alpha}{2\pi}\mu_0, \tag{50.8}$$

where $\mu_0 = e\hbar/2mc$ is the Bohr magnetron, and $\alpha = e^2/4\pi\hbar c$ is the fine structure constant. This additional term may be called the "anomalous" magnetic moment in contrast to the "normal" moment $\mu_0$.

Thus, up to terms of order $\alpha^2$ the magnetic moment of the electron is given by (this result is due to J. Schwinger (174))

$$\mu = \mu_0 \left(1 + \frac{\alpha}{2\pi}\right). \tag{50.8'}$$

In the presence of an electric field $\boldsymbol{E}$ the first term in (50.6) has the form

$$\mu'(\beta \boldsymbol{\Sigma H} - i\beta \boldsymbol{\alpha E}) = -\frac{\mu'}{2} \gamma_4 \gamma_{\mu\nu} F_{\mu\nu}, \tag{50.9}$$

where $\mu' = \mu_0 \frac{\alpha}{2\pi}$, $\gamma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$ and $F_{\mu\nu}$ is the electromagnetic field tensor. This expression agrees with the corresponding term in the phenomenological Dirac equation (15.12) for a particle with an anomalous magnetic moment $\mu'$. The Hamiltonian for the electron will in this case, up to terms of order $v^2/c^2$, contain terms proportional to $\mu'$ in (15.13).

The experimentally observed value of the "anomalous" magnetic moment of the electron differs from the theoretical value $(\alpha/2\pi)\mu_0$ given by formula (50.8). It is therefore of interest to evaluate the magnetic moment of the electron taking into account corrections of order $\alpha^2$, which correspond to fifth order perturbation theory effects. Such effects correspond to the diagrams shown in Fig. 104.

For the evaluation of the "anomalous" magnetic moment of the electron we can utilize the fact that the part of the effective potential energy (50.6) containing the "anomalous" magnetic moment $\mu'$, is of the form (50.9) A part of the effective potential energy of the electron will have the same structure also in higher order approximations of perturbation theory.

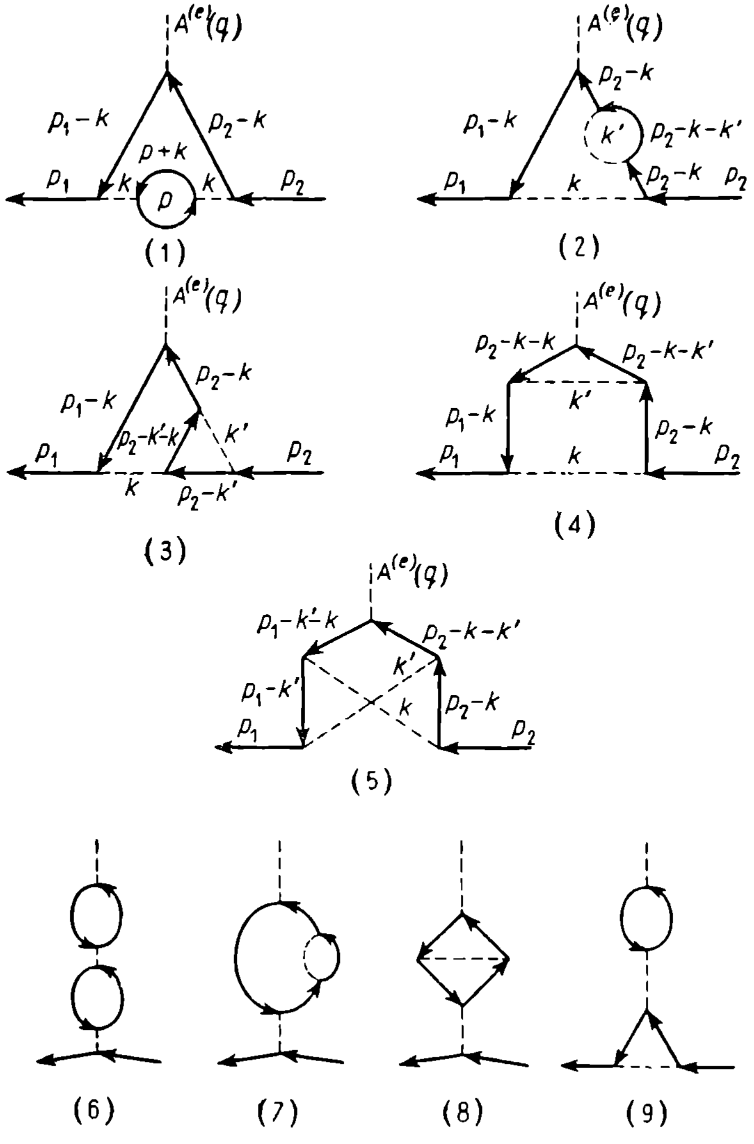
The energy (50.9) corresponds to the part of the element of the scattering matrix which has the form

$$M = -\mu' \bar{u}_2 A_\mu^{(e)}(q) \gamma_{\mu\nu} q_\nu u_1, \tag{50.9'}$$

where

$$A_\mu^{(e)}(q) = \int A_\mu^{(e)}(x) e^{-iqx} d^4x, \qquad q = p_2 - p_1$$

and $u_1$ and $u_2$ are the spinor amplitudes of the initial and final states of the electron of momenta $p_1$ and $p_2$. Therefore, for the determination of the "anomalous" magnetic moment of the electron in fifth order perturbation theory approximation it is sufficient to pick out from the matrix elements corresponding to the diagrams of Fig. 104 those parts which have the structure of (50.9').



(1) (2) (3) (4) (5) (6) (7) (8) (9)

Fig. 104.

It can be easily seen that contributions to the increment in the magnetic moment come only from diagrams 1, 2, 3, 4, 5, while diagrams 6, 7, 8, 9 determine corrections caused by polarization of the vacuum to the first and third order vertex parts, and make no contribution to the magnetic moment.

In accordance with the rules of subsection 25.5 the matrix elements corresponding to diagrams 1, 2, 3, 4, 5 are given by

$$\begin{aligned}
M_1 &= \frac{e^3}{(2\pi)^8}\bar{u}_2\int \gamma_\mu \frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m_2}\hat{A}^{(e)}(q)\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2} \\
&\qquad\qquad \times\gamma_\nu \frac{1}{k^2}\Pi^{(2)}_{\mu\nu}(k)\frac{d^4k}{k^2}u_1, \\
M_2 &= \frac{2e^3}{(2\pi)^8}\bar{u}_2\int \gamma_\mu \frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}\Sigma^{(2)}(p_2-k) \\
&\qquad \times\frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}\hat{A}^{(e)}(q)\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\gamma_\mu\frac{d^4k}{k^2}u_1, \\
M_3 &= \frac{2ie^2}{(2\pi)^4}\bar{u}_2\int \Lambda^{(3)}_\mu(p_2, p_2-k)\frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}\hat{A}^{(e)}(q) \\
&\qquad\qquad \times\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\gamma_\mu\frac{d^4k}{k^2}u_1, \\
M_4 &= \frac{ie^2}{(2\pi)^4}\bar{u}_2\int \gamma_\mu\frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}A^{(e)}_\nu(q)\Lambda^{(3)}_\nu(p_2-k, p_1-k) \\
&\qquad\qquad \times\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\gamma_\mu\frac{d^4k}{k^2+\lambda^2}u_1, \\
M_5 &= -\frac{e^5}{(2\pi)^8}\bar{u}_2\int \gamma_\lambda\frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}\gamma_\mu\frac{i(\hat{p}_2-\hat{k}-\hat{k}')-m}{(p_2-k-k')^2+m^2}\hat{A}^{(e)}(q) \\
&\quad \times\frac{i(\hat{p}_1-\hat{k}-\hat{k}')-m}{(p_1-k-k')^2+m^2}\gamma_\lambda\frac{i(\hat{p}_1-\hat{k}')-m}{(p_1-k')^2+m^2}\gamma_\mu\frac{d^4k}{k^2}\frac{d^4k'}{k'^2}u_1,
\end{aligned}\tag{50.10}$$

where the quantities $\Pi^{(2)}_{\mu\nu}(k)$, $\Sigma^{(2)}(p)$ and $\Lambda^{(3)}_\mu(p', p'')\,\Lambda(p', p''; p'-p'')$ are defined by formulas (47.31), (47.24), (47.52).

We do not reproduce here any detailed calculations, but give only the final expression for the correction to the magnetic moment of the electron of order $\alpha^2$ (188), (102):

$$\mu'' = -0.328\frac{\alpha^2}{\pi^2}\mu_0. \tag{50.11}$$

Thus, the magnetic moment of the electron taking into account radiation corrections of order $\alpha^2$ is equal to

$$\mu = \left(1+\frac{\alpha}{2\pi}-0.328\frac{\alpha^2}{\pi^2}\right)\mu_0 = 1.0011596\mu_0. \tag{50.11'}$$

### 50.3. *Radiation Corrections to Coulomb's Law*

We now obtain the radiation corrections to the electric field produced by a point charge $e$ at rest.

In accordance with (49.1) the general relation connecting the given current $J_\mu(x)$ with the field $A_\mu(x)$ produced by it, has the form

$$A_\mu(x) = i\int G^{\gamma}_{\mu\nu}(x-x')J_\nu(x')\,d^4x', \tag{50.12}$$

where $G^{\gamma}_{\mu\nu}(x)$ is the photon Green's function.

In the case of interest to us of a point charge at rest $J_\mu(x) = ie\delta_{\mu 4}\delta(\boldsymbol{r})$. Therefore, the scalar potential $\varphi(\boldsymbol{r})$ produced by the point charge $e$ situated at the point $\boldsymbol{r} = 0$ is given by equation

$$\varphi(\boldsymbol{r}) = ie\int\limits_{-\infty}^{\infty} G^{\gamma}(x)\,dt. \tag{50.13}$$

With the aid of the representation of the function $G^{\gamma}(x)$ in momentum space

$$G^{\gamma}(x) = \frac{1}{(2\pi)^4}\int G^{\gamma}(k)e^{ikx}d^4k \equiv \frac{1}{(2\pi)^4}\int G^{\gamma}(\boldsymbol{k}, k_0)e^{i(\boldsymbol{kr}-k_0t)}d^4k$$

we can rewrite (50.13) in the form

$$\varphi(\boldsymbol{r}) = \frac{ie}{(2\pi)^3}\int G^{\gamma}(\boldsymbol{k}, 0)e^{i\boldsymbol{kr}}d\boldsymbol{k}. \tag{50.13'}$$

On utilizing the general relation (43.37) defining the function $G^{\gamma}(k)$, and formula (47.31) we obtain the following expression for the photon Green's function taking into account radiation corrections of order $\alpha$:

$$G^{\gamma}(k) = D^c(k)\left\{1+\frac{e^2}{2\pi^2}\int\limits_0^1 x(1-x)\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]dx\right\},$$

where

$$D^c(k) = \frac{1}{i}\,\frac{1}{k^2-i0}.$$

On substituting this expression into (50.13′) we obtain the potential of a point charge taking into account corrections of order $\alpha$:

$$\varphi(\boldsymbol{r})=\frac{e}{(2\pi)^3}\int\frac{e^{i\boldsymbol{k}\boldsymbol{r}}}{k^2}\left\{1+\frac{e^2}{2\pi^2}\int\limits_0^1 x(1-x)\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]dx\right\}d\boldsymbol{k}. \tag{50.14}$$

On noting that

$$\int D^c(\boldsymbol{k},0)e^{i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{k}=-\frac{2\pi^2 i}{r}$$

and

$$\int\limits_0^1 x(1-x)\ln\left[1+\frac{k^2}{m^2}x(1-x)\right]dx=-\int\limits_0^1\frac{\frac{1}{2}x^2-\frac{1}{3}x^3}{1+\frac{k^2}{m^2}x(1-x)}\frac{k^2}{m^2}(1-2x)dx$$

$$=\frac{k^2}{8m^2}\int\limits_0^1\frac{v^2\left(1-\frac{1}{3}v^2\right)}{1+\frac{k^2}{4m^2}(1-v^2)}dv,$$

where $v=2x-1$, we rewrite $\varphi(r)$ in the form

$$\varphi(r)=\frac{e}{4\pi r}+\frac{e^3}{(2\pi)^5}\int\limits_0^1\frac{v^2\left(1-\frac{1}{3}v^2\right)}{1-v^2}dv\int\frac{e^{i\boldsymbol{k}\boldsymbol{r}}}{k^2+\frac{4m^2}{1-v^2}}d\boldsymbol{k}.$$

Further, on utilizing the relation

$$\int\frac{e^{i\boldsymbol{k}\boldsymbol{r}}}{k^2+(4m^2/1-v^2)}d\boldsymbol{k}=\frac{2\pi^2}{r}e^{-\frac{2mr}{\sqrt{1-v^2}}}$$

and on introducing instead of $v$ the new variable $\zeta=1/\sqrt{1-v^2}$ we obtain finally the following general formula for $\varphi(r)$ taking into account corrections of order $\alpha$ (173):

$$\varphi(r)=\frac{e}{4\pi r}\left\{1+\frac{e^2}{6\pi^2}\int\limits_1^\infty e^{-2mr\zeta}\left(1+\frac{1}{2\zeta^2}\right)\frac{(\zeta^2-1)^{1/2}}{\zeta^2}d\zeta\right\}. \tag{50.15}$$

The integral appearing in this expression can be evaluated in two limiting cases, when $mr \ll 1$ and when $mr \gg 1$:

$$\int_1^\infty e^{-2mr\zeta}\left(1+\frac{1}{2\zeta^2}\right)\frac{(\zeta^2-1)^{1/2}}{\zeta^2}\,d\zeta=\begin{cases}-\dfrac{5}{6}-\gamma-\ln mr, & mr \ll 1,\\ \dfrac{3\sqrt{\pi}}{8}\dfrac{e^{-2mr}}{(mr)^{3/2}}, & mr \gg 1\end{cases} \tag{50.16}$$

where $\gamma$ is Euler's constant. Therefore, the potential of the charge at small and at large distances (compared to $\hbar/mc$) is of the form

$$\varphi(r)=\begin{cases}\dfrac{e}{4\pi r}\left\{1+\dfrac{\alpha}{3\pi}\left(-\dfrac{5}{3}-2\gamma+\ln\dfrac{1}{(mr)^2}\right)\right\} & mr \ll 1,\\ \dfrac{e}{4\pi r}\left\{1+\dfrac{\alpha}{4\pi^{1/2}}\dfrac{e^{-2mr}}{(mr)^{3/2}}\right\}, & mr \gg 1.\end{cases} \tag{50.17}$$

We see that at large (compared to $\hbar/mc$) distances from the charge the potential differs from the Coulomb potential by an exponentially small term, while at small distances the deviation from Coulomb's law varies logarithmically with the distance.

Formula (50.17) may be interpreted in the following manner. If we rewrite it in the form

$$\varphi(r)=\frac{e(r)}{4\pi r}, \tag{50.17'}$$

then we can say that the potential $\varphi(r)$ is determined by Coulomb's law using an effective charge $e(r)$. This charge coincides with the physical charge $e$ at large distances $r \gg (\hbar/mc)$. But at small distances $r \ll (\hbar/mc)$ it is equal to

$$e(r)\approx e\left[1+\frac{\alpha}{3\pi}\ln\frac{1}{(mr)^2}\right]. \tag{50.18}$$

This expression agrees with the expression obtained in Chapter VII for the effective charge for $L \sim 1/r$.

## § 51. Radiation Corrections to Electron Scattering

### 51.1. *Electron Scattering by the Coulomb Field of a Nucleus in the Second Born Approximation*

We now proceed to determine the radiation corrections to different electron scattering processes. We begin with the radiation corrections to the scattering of an electron by the Coulomb field of a nucleus, which we regard as a perturbation.

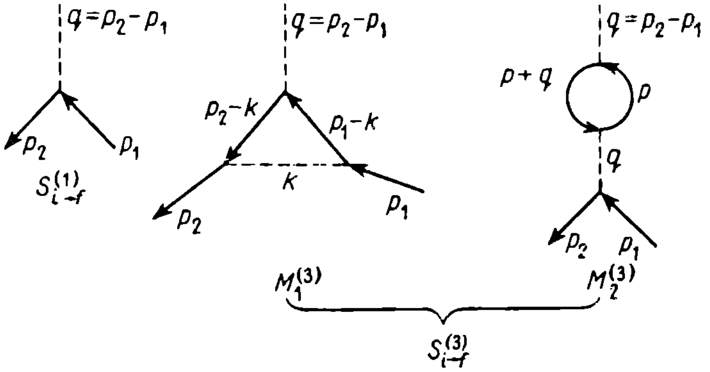


Fig. 105.

The diagrams representing the process of scattering by an external field in the first order and the radiation corrections to it in the third order of perturbation theory are shown in Fig. 105. The scattering matrix elements corresponding to these diagrams are given by

$$S^{(1)}_{i\to f} = e\bar{u}_2\hat{A}^{(e)}(q)u_1, \qquad S^{(3)}_{i\to f} = M_1^{(3)}+M_2^{(3)},$$

$$M_1^{(3)} = \frac{ie^3}{(2\pi)^4}\bar{u}_2\left(\int\gamma_\mu\frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}\hat{A}^{(e)}(q)\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\gamma_\nu\frac{d^4k}{k^2}\right)u_1, \tag{51.1}$$

$$M_2^{(3)} = -\frac{ie^3}{(2\pi)^4}\bar{u}_2\left(A_\mu^{(e)}(q)\frac{1}{q^2}\gamma_\nu \right.$$
$$\left.\times\int Sp\left\{\frac{i(\hat{p}+\hat{q})-m}{(p+q)^2+m^2}\gamma_\mu\frac{i\hat{p}-m}{p^2+m^2}\gamma_\nu\right\}d^4p\right)u_1,$$

where $u_1$ and $u_2$ are the spinor amplitudes of the initial and final states of the electron of momenta $p_1$ and $p_2$. $A^{(e)}(q)$ is the Fourier component of the external potential and $q=p_2-p_1$.

In the case of the Coulomb field of a nucleus of charge $-Ze$ (in Gaussian units)

$$A_0^{(e)}(q) = 2\pi\,\delta(\varepsilon_1-\varepsilon_2)\frac{Ze}{q^2} \equiv 2\pi A_0^{(e)}(q)\,\delta(\varepsilon_1-\varepsilon_2) \tag{51.1'}$$

and

$$\hat{A}^{(e)}(q) = i\gamma_4\frac{Ze}{q^2}\,2\pi\,\delta(\varepsilon_1-\varepsilon_2),$$

where $\varepsilon_1$ and $\varepsilon_2$ are the values of the electron energy before and after scattering. Therefore,

$$S_{i\to f}^{(1)} = \frac{Ze^2}{q^2}\,2\pi i(\bar{u}_2\gamma_4 u_1)\,\delta(\varepsilon_2-\varepsilon_1). \tag{51.2}$$

Further, the sum of $M_1^{(3)}$ and $M_2^{(3)}$ is obviously the matrix element (between the states $u_1$ and $u_2$) of the effective potential energy of the electron $e\delta\varphi(x)$ which takes into account its interaction with the vacuum. On utilizing formula (50.1′) for the Fourier component, and on substituting (51.1′) into it, we obtain the following expression for the sum of $M_1^{(3)}$ and $M_2^{(3)}$:

$$\begin{aligned}S_{i\to f}^{(3)} = -i\frac{e^2}{8\pi}\frac{Ze}{q^2}\bar{u}_2\gamma_4\Bigg\{4\Bigg[(1-2\Phi\coth 2\Phi)\left(1+\ln\frac{\lambda}{m}\right)\\ +\frac{\Phi}{2}\tanh\Phi+2\coth 2\Phi\int\limits_0^{\Phi}u\tanh u\,du\Bigg]\\ +4(1-\Phi\coth\Phi)\left(1-\frac{1}{3}\coth^2\Phi\right)\\ -\frac{4}{9}-\frac{i}{m}\hat{q}\frac{2\Phi}{\sinh 2\Phi}\Bigg\}u_1\,\delta(\varepsilon_1-\varepsilon_2),\end{aligned} \tag{51.3}$$

where $\Phi$ is related to the change in the momentum $q$ and to the scattering angle $\vartheta$ by the expressions

$$q^2 = |q|^2 = 4p^2\sin^2\frac{\vartheta}{2} = 4m^2\sinh^2\Phi, \tag{51.3'}$$

$$|p_1| = |p_2| = |p|.$$

In order to obtain the scattering cross section taking radiation corrections into account it is, however, insufficient to obtain the square of the absolute value of $S^{(1)}_{i\to f}+S^{(3)}_{i\to f}$ and to sum it in accordance with the rules of § 26 over the different initial and final states. Indeed, radiation corrections are third order perturbation theory effects. Therefore, along with them we must also take into account scattering in the second approximation of perturbation theory (the second Born approximation).

We note that it is not necessary to take scattering into account in the third Born approximation, since the matrix element corresponding to this process is proportional to $e^6$.

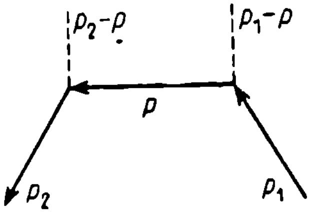


Fig. 106.

We investigate in greater detail scattering in the second Born approximation (42). The diagram representing this process is shown in Fig. 106 The matrix element which corresponds to it is equal to

$$S^{(2)}_{i\to f} = -\frac{ie^2}{(2\pi)^4}\bar{u}_2\left\{\int \hat{A}^{(e)}(p_2-p)\frac{i\hat{p}-m}{p^2+m^2}\hat{A}^{(e)}(p_1-p)\,d^4p\right\}u_1. \quad (51.4)$$

It can be easily shown that the integral appearing in this expression diverges for a purely Coulomb field, so that we carry out the calculations assuming the Coulomb field of the nucleus to be screened:

$$\varphi(r) = \frac{Ze}{4\pi r}e^{-\eta r}.$$

As we shall show later, we can make the limiting transition $\eta\to 0$ in the expression for the scattering cross section.

In the case of a screened Coulomb potential the quantity $\hat{A}^{(e)}(q)$ has the form

$$\hat{A}^{(e)}(q) = i\gamma_4\frac{Ze}{q^2+\eta^2}2\pi\,\delta(q_0). \quad (51.4')$$

On substituting this expression into (51.4) we obtain

$$S^{(2)}_{i\to f} = 4iZ^2\alpha^2\bar{u}_2\int\frac{\gamma_4}{\eta^2+(p_2-p)^2}\cdot\frac{i\hat{p}-m}{p^2+m^2}\,\frac{\gamma_4}{\eta^2+(p-p_1)^2}\,d^4p\,u_1\,\delta(\varepsilon_1-\varepsilon_2),$$

where $p_1 = i\varepsilon$ and $\varepsilon = \varepsilon_1 = \varepsilon_2$. On introducing the notation

$$\begin{aligned} \int \frac{ds}{[\eta^2+(p_2-s)^2]\,[\eta^2+(s-p_1)^2](p_1^2-s)} &= I_1, \\ \int \frac{s\,ds}{[\eta^2+(p_2-s)^2]\,[\eta^2+(s-p_1)^2](p_1^2-s^2)} &= \frac{p_1+p_2}{2} I_2 \end{aligned} \tag{51.5}$$

and on noting that

$$\begin{aligned} i\gamma p_1 u_1 &= (\gamma_4\varepsilon - m)u_1, \\ u_2 i\gamma p_2 &= \bar{u}_2(\gamma_4\varepsilon - m), \end{aligned}$$

we write $S^{(2)}$ in the form

$$S^{(2)}_{i\to f} = 4iZ^2\alpha^2\bar{u}_2\{m(I_1-I_2)+\gamma_4\varepsilon(I_1+I_2)\}\,u_1\,\delta(\varepsilon_1-\varepsilon_2). \tag{51.6}$$

This formula determines the correction to the scattering amplitude in the second Born approximation. The effective cross section and the polarization calculated taking this correction into account were given in § 14.

In order to evaluate the integrals $I_1$ and $I_2$ which appear in (51.6) we first of all evaluate the integral

$$\begin{aligned} L &= \int \frac{ds}{[(s-P)^2+\Lambda^2]\,(p^2-s^2+i0)} \\ &= 2\pi \int_0^\pi d\chi \int_0^\infty ds \frac{s^2 \sin\chi}{(s^2-2Ps\cos\chi+P^2+\Lambda^2)\,(p^2-s^2+i0)}. \end{aligned} \tag{51.7}$$

On introducing the variable $t = \cos\chi$, and on noting that the integrand is not altered by the substitution $s \to -s$, $t \to -t$ we rewrite $L$ in the form

$$L = \pi \int_{-1}^{1} dt \int_{-\infty}^{\infty} \frac{s^2\,ds}{(s^2-2Pst+P^2+\Lambda^2)\,(p^2-s^2+i0)}.$$

Further, on completing the contour of integration with respect to $s$ by a semicircle in the upper half-plane, and on applying the residue theorem we obtain

$$L = \frac{\pi^2 i}{P} \ln \frac{p-P+i\Lambda}{p+P+i\Lambda}. \tag{51.7'}$$

As we shall immediately see, the integrals $I_1$ and $I_2$ are related by simple expressions to the derivatives of $L$ with respect to $\varLambda$ and $P_r$

$$\int \frac{ds}{[(s-P)^2+\varLambda^2]^2\,(p^2-s^2+i0)} = -\frac{1}{2\varLambda}\frac{\partial L}{\partial \varLambda} = \frac{\pi^2}{\varLambda(p^2-P^2+\varLambda^2+2ip\varLambda)},$$

$$\int \frac{s_r\, ds}{[(s-P)^2+\varLambda^2]\,(p^2-s^2+i0)} = -\frac{1}{2}\frac{\partial L}{\partial P_r} - \frac{P_r}{2\varLambda}\frac{\partial L}{\partial \varLambda}$$

$$= -\pi^2 P_r\left[-\frac{1}{\varLambda(p+P+i\varLambda)\,(-P+i\varLambda)} + \frac{i}{2P^3}\ln\frac{p-P+i\varLambda}{p+P+i\varLambda} + \frac{i}{2P^2}\left(\frac{1}{p-P+i\varLambda}+\frac{1}{p+P+i\varLambda}\right)\right]. \qquad (51.8)$$

Indeed, we use the formula

$$\frac{1}{ab} = \frac{1}{2}\int_{-1}^{1}\frac{dz}{\left(a(1+z)/2+b(1-z)/2\right)^2}$$

and in it set $a = \eta^2+(s-p_2)^2$, and $b = \eta^2+(s-p_1)^2$. Since

$$\left(a\frac{1+z}{2}+b\frac{1-z}{2}\right)^2 = [(s-P)^2+\varLambda^2]^2,$$

where

$$P = \tfrac{1}{2}[(1+z)p_1+(1-z)p_2],$$

$$\varLambda^2 = \eta^2+p^2\sin^2\frac{\vartheta}{2}(1-z^2),$$

then we have

$$\frac{1}{[(s-p_2)^2+\eta^2]\,[(s-p_1)^2+\eta^2]} = \frac{1}{2}\int_{-1}^{1}\frac{dz}{[(s-P)^2+\varLambda^2]^2}.$$

On comparing this expression with the definitions of $I_1$ and $I_2$ and on utilizing formula (51.18) we obtain

$$I_1 = -\frac{1}{4}\int_{-1}^{1}\frac{1}{\varLambda}\frac{\partial L}{\partial \varLambda}\,dz,$$

$$\frac{p_{1r}+p_{2r}}{2}I_2 = -\frac{1}{4}\int_{-1}^{1}\left(\frac{\partial L}{\partial P_r}+\frac{P_r}{\varLambda}\frac{\partial L}{\partial \varLambda}\right)dz.$$

On carrying out the integration with respect to $z$ we finally obtain

$$I_1 = -\frac{\pi^2}{|p| \sin\frac{\vartheta}{2} \sqrt{\eta^4+4p^2\left(\eta^2+p^2 \sin^2\frac{\vartheta}{2}\right)}}$$

$$\times \left[ \operatorname{arctg} \frac{\eta |p| \sin\frac{\vartheta}{2}}{\sqrt{\eta^4+4p^2\left(\eta^2+p^2 \sin^2\frac{\vartheta}{2}\right)}} \right.$$

$$\left. +\frac{i}{2} \ln \frac{\sqrt{\eta^4+4p^2\left(\eta^2+p^2 \sin^2\frac{\vartheta}{2}\right)}+2p^2 \sin^2\frac{\vartheta}{2}}{\sqrt{\eta^4+4p^2\left(\eta^2+p^2 \sin^2\frac{\vartheta}{2}\right)}-2p^2 \sin^2\frac{\vartheta}{2}} \right],$$

$$I_2 = I_1 \frac{\eta^2+2p^2}{2p^2 \cos^2\frac{\vartheta}{2}} - \frac{i\pi^2}{2p^3 \cos^2\frac{\vartheta}{2}} \left[ \ln \frac{i\eta}{2p+i\eta} - i \frac{\operatorname{arctg} \dfrac{p \sin\frac{\vartheta}{2}}{\eta}}{\sin\frac{\vartheta}{2}} \right]. \qquad (51.9)$$

When $\eta \to 0$

$$I_1 = -\frac{i\pi^2}{2|p|^3 \sin^2\frac{\vartheta}{2}} \ln \frac{2|p| \sin\frac{\vartheta}{2}}{\eta},$$

$$I_2 = \frac{I_1}{\cos^2\frac{\vartheta}{2}} - \frac{i\pi^2}{2|p|^3 \cos^2\frac{\vartheta}{2}} \left( \ln \frac{i\eta}{2|p|} - \frac{i\pi}{2 \sin\frac{\vartheta}{2}} . \right) \qquad (51.10)$$

We note that for $\eta \to 0$

$$\operatorname{Re} I_1 = 0, \quad \operatorname{Re} I_2 = \frac{\pi^3}{4|p|^3 \cos^2\frac{\vartheta}{2}} \left( 1 - \frac{1}{\sin\frac{\vartheta}{2}} \right). \qquad (51.10')$$

### 51. 2. *Differential Cross Section for the Scattering of an Electron by the Coulomb Field of a Nucleus taking into Account Radiation Corrections of Order $\alpha$*

We now determine the differential cross section for purely elastic scattering of an electron averaged over the orientations of its spin in the initial and final states. In accordance with the general rules of § 26 this cross section is equal to

$$d\sigma = \frac{1}{2\pi J}\cdot\frac{1}{2}\sum_{\mu_1,\mu_2} |M|^2 \varrho_f\, do, \tag{51.11}$$

where $J$ is the electron flux density, which is equal to their velocity (the normalizing volume is assumed to be taken equal to unity); $do$ is the element of solid angle into which the electron is scattered; $\varrho_f$ is the density of the final states of the electron per unit energy and per unit solid angle

$$\varrho_f = \frac{|\boldsymbol{p}_2|^2\, d|\boldsymbol{p}_2|}{(2\pi)^3\, d\varepsilon_2} = \frac{|\boldsymbol{p}_2|\,\varepsilon_2}{(2\pi)^3}$$

and $M\,\delta(\varepsilon_1-\varepsilon_2)$ is the sum of the matrix elements for the basic process, the second Born approximation and the radiation corrections:

$$S^{(1)}_{\to f} + S^{(2)}_{i\to f} + S^{(3)}_{i\to f} \equiv M\delta(\varepsilon_1-\varepsilon_2).$$

On utilizing (51.1), (51.3), (51.6), we can write this quantity in the form

$$M\delta(\varepsilon_1-\varepsilon_2) = \bar{u}_2 \mathrm{Q} u_1 \delta(\varepsilon_1-\varepsilon_2), \tag{51.12}$$

where

$$\mathrm{Q} = (A_0+A_1)\gamma_4 + B\gamma_4\hat{q} + C, \qquad A_0 = 8\pi^2 i\frac{Z\alpha}{q^2},$$

$$A_1 = -8\pi i\frac{Z\alpha^2}{q^2}\left\{\left[(1-\Phi\coth\Phi)\left(1-\frac{1}{3}\coth^2\Phi\right)-\frac{1}{9}\right]\right.$$

$$\left.+\left[(1-2\Phi\coth 2\Phi)\left(1+\ln\frac{\lambda}{m}\right)+\frac{\Phi}{2}\tanh\Phi + 2\coth 2\Phi\int_0^{\Phi} u\tanh u\, du\right]\right\}$$

$$-4iZ^2\alpha^2\varepsilon(I_1+I_2),$$

$$B = -\frac{4\pi Z\alpha^2}{mq^2}\cdot\frac{\Phi}{\sinh 2\Phi},$$

$$C = -4iZ^2\alpha^2 m(I_1-I_2).$$

Summation over spin orientations can be carried out with the aid of the formula

$$\sum_{\mu_1,\mu_2} |M|^2 = \frac{1}{4\varepsilon_1\varepsilon_2} \mathrm{Sp}\{Q(i\hat{p}_1 - m)\bar{Q}(i\hat{p}_2 - m)\}$$

$$= \frac{1}{4\varepsilon_1\varepsilon_2} \mathrm{Sp}\{[(A_0 + A_1)\gamma_4 + B\gamma_4\hat{q} + C](i\hat{p}_1 - m)$$

$$\times [(A_0^* + A_1^*)\gamma_4 + B^*\gamma_4\hat{q} + C^*](i\hat{p}_2 - m)\}.$$

On neglecting the terms $|A_1|^2$, $|B|^2$, $B^*C$, $A_1^*C$, $A_1^*B$ and on utilizing the formulas of § 26 we obtain

$$\sum_{\mu_1,\mu_2} |M|^2 = \frac{1}{\varepsilon_1\varepsilon_2}\{|A_0 + A_1|^2(m^2 - p_1'p_2) + im(A_0^*B - A_0B^*)$$

$$+ 2m\varepsilon(A_0^*C + A_0C^*)\}, \quad p_1' = (-\mathbf{p}_1, \; i\varepsilon_1).$$

On substituting into this expression

$$|A_0 + A_1|^2 = |A_0|^2 + 2\mathrm{Re}(A_0^*A_1)$$

$$= 64\pi^4\frac{Z^2\alpha^2}{q^4} - \frac{128\pi^3Z^2\alpha^3}{q^4}\left\{\left[(1 - \Phi\coth\Phi)\left(1 - \frac{1}{3}\coth^2\Phi\right) - \frac{1}{9}\right]\right.$$

$$+ \left[(1 - 2\Phi\coth 2\Phi)\left(1 + \ln\frac{\lambda}{m}\right) + \frac{\Phi}{2}\tanh\Phi\right.$$

$$\left.\left. + 2\coth 2\Phi\int_0^{\Phi} u\tanh u\, du\right)\right\} - 64\pi^2\frac{Z^3\alpha^3}{q^2}\varepsilon\,\mathrm{Re}(I_1 + I_2),$$

$$A_0^*C + A_0C^* = 2\mathrm{Re}\,(A_0^*C) = -\frac{64\pi^2Z^3\alpha^3}{q^2}m\,\mathrm{Re}(I_1 - I_2),$$

$$A_0^*B - A_0B^* = 2i\,\mathrm{Im}\,(A_0^*B) = -\frac{64\pi^3Z^2\alpha^3}{q^4m}\frac{\Phi}{\sinh 2\Phi},$$

$$m^2 - p_1'p_2 = 2\varepsilon^2\left(1 - v^2\sin\frac{\vartheta}{2}\right),$$

we finally obtain

$$\sum_{\mu_1,\mu_2} |M|^2$$

$$= \frac{128\pi^4Z^2\alpha^2}{q^4}\left(1 - v^2\sin^2\frac{\vartheta}{2}\right)\left\{1 - \frac{\alpha}{\pi}\left[2(1 - \Phi\coth\Phi)\left(1 - \frac{1}{3}\coth^2\Phi\right)\right.\right.$$

$$-\frac{2}{9}+\Phi\tanh\Phi+2(1-2\Phi\coth 2\Phi)\left(1+\ln\frac{\lambda}{m}\right)$$

$$+4\coth 2\Phi\int_0^{\Phi}u\tanh u\,du+\frac{v^2\sin^2\frac{\vartheta}{2}}{1-v^2\sin^2\frac{\vartheta}{2}}\cdot\frac{2\Phi}{\sinh 2\Phi}\Bigg]$$

$$+\frac{\pi Z\alpha v\sin\frac{\vartheta}{2}\left(1-\sin\frac{\vartheta}{2}\right)}{1-v\sin^2\frac{\vartheta}{2}}\Bigg\}. \qquad (51.12')$$

Therefore the differential cross section for purely elastic scattering is given by

$$d\sigma_e=\left(\frac{Z\alpha}{2mv^2\sin^2\frac{\vartheta}{2}}\right)^2(1-v^2)\left(1-v^2\sin^2\frac{\vartheta}{2}\right)$$

$$\times\Bigg\{1-\frac{\alpha}{\pi}\Bigg[2(1-\Phi\coth\Phi)\left(1-\frac{1}{3}\coth^2\Phi\right)-\frac{2}{9}+\Phi\tanh\Phi$$

$$+2(1-2\Phi\coth 2\Phi)\left(1+\ln\frac{\lambda}{m}\right)$$

$$+4\coth 2\Phi\int_0^{\Phi}u\tanh u\,du+\frac{v^2\sin^2\frac{\vartheta}{2}}{1-v^2\sin^2\frac{\vartheta}{2}}\,\frac{2\Phi}{\sinh 2\Phi}\Bigg]$$

$$+\pi\alpha Zv\sin\frac{\vartheta}{2}\left(1-\sin\frac{\vartheta}{2}\right)\left(1-v^2\sin^2\frac{\vartheta}{2}\right)^{-1}\Bigg\}\,do. \qquad (51.13)$$

We see that the cross section $d\sigma_e$ contains the photon "mass" $\lambda$; insofar as the screening constant $\eta$ is concerned, it does not appear in $d\sigma_e$, as has been stated previously.

We note that the integral appearing in $d\sigma_e$ can be written in the form

$$\int_0^{\Phi} u\tanh u\,du = \frac{1}{4}\,\frac{1-v^2}{\sin\frac{\vartheta}{2}}\sinh 2\Phi \int_{\cos\frac{\vartheta}{2}}^{1} \frac{\zeta\ln(1-v^2\zeta^2)\,d\zeta}{(1-v^2\zeta^2)\sqrt{\zeta^2-\cos^2\frac{\vartheta}{2}}} + \frac{\Phi}{2}\ln\frac{1}{1-v^2}. \tag{51.14}$$

Indeed, we consider the integral

$$R = \int_{-1}^{1} \frac{\ln\sqrt{-P_z^2}}{-P_z^2}\,dz, \tag{51.15}$$

where

$$P_z = \tfrac{1}{2}(1+z)p_1 + \tfrac{1}{2}(1-z)p_2.$$

On introducing the new variable $u$

$$z = \frac{\tanh u}{\tanh\Phi},$$

we rewrite $R$ in the form

$$R = \frac{2}{m^2\cosh 2\Phi}\int_0^{\Phi} \ln\left(\frac{m\cosh\Phi}{\cosh u}\right)du.$$

After integration by parts we obtain

$$\int_0^{\Phi} u\tanh u\,du = \Phi\ln\cosh\Phi - \int_0^{\Phi}\ln\cosh u\,du,$$

and therefore

$$\int_0^{\Phi} u\tanh u\,du = \frac{m^2}{2}\sinh 2\Phi\cdot R - \ln m.$$

Finally, on utilizing the formula

$$\frac{1}{2}\int_{-1}^{1}\frac{dz}{-P_z} = \frac{2\Phi}{m^2\cosh^2\Phi}$$

and on introducing into (52.15) in place of $z$ the new variable $\zeta$

$$\zeta = \frac{|\boldsymbol{P}_z|}{v\varepsilon_z} = \frac{|\boldsymbol{P}_z|}{|\boldsymbol{p}_1|}$$

$$z = \frac{1}{\sin\frac{\vartheta}{2}}\sqrt{\zeta^2 - \cos^2\frac{\vartheta}{2}}$$

($\varepsilon_z$ is the fourth component of $P_z$), we obtain formula (51.14).

### 51. 3. *Elimination of the Photon "Mass" from the Scattering Cross Section*

We now show how the photon "mass" can be eliminated from the scattering cross section. In order to do this we must, as has been already explained in § 30, together with the purely elastic scattering also include in our investigation scattering accompanied by the emission of a soft photon whose energy is considerably lower than the electron energy.

The cross section for the emission of such a photon is determined by formula (30.25):

$$d\sigma' = \left(\frac{Z\alpha}{2mv^2\sin^2\frac{\vartheta}{2}}\right)^2 (1-v^2)\left((1-v^2\sin^2\frac{\vartheta}{2}\right)\frac{\alpha}{\pi}$$

$$\times\left\{2(2\Phi\coth 2\Phi-1)\ln\frac{2\Delta\varepsilon}{\lambda}+\frac{1}{v}\ln\frac{1+v}{1-v}-\frac{1-v^2}{v\sin\frac{\vartheta}{2}}\cosh 2\Phi\cdot G(v,\vartheta)\right\}do, \tag{51.16}$$

where

$$v(G,\vartheta) = \int\limits_{\cos\frac{\vartheta}{2}}^{1}\ln\frac{1+v\zeta}{1-v\zeta}\,\frac{d\zeta}{(1-v^2\zeta^2)\sqrt{\zeta^2-\cos^2\frac{\vartheta}{2}}}\,.$$

The photon mass appears in $d\sigma'$ in the form

$$\frac{2\alpha}{\pi}(2\Phi\coth 2\Phi-1)\ln\frac{2\Delta\varepsilon}{\lambda},$$

while in the elastic scattering cross section it appears in the form

$$\frac{2\alpha}{\pi}(2\Phi\coth 2\Phi-1)\ln\frac{\lambda}{m}.$$

Therefore, if we add the cross sections $d\sigma_e$ and $d\sigma'$ then the photon "mass" $\lambda$ will not appear in the total scattering cross section, which is the only one that has physical meaning.

We write this total cross section for scattering accompanied by an energy loss not exceeding $\Delta\varepsilon$ in the form

$$d\sigma_{\Delta\varepsilon} = \left(\frac{Z\alpha}{2mv^2\sin^2\frac{\vartheta}{2}}\right)^2(1-v^2)\left(1-v^2\sin^2\frac{\vartheta}{2}\right)(1+\delta_B-\delta_R)\,do, \quad (51.17)$$

where the quantities $\delta_R$ and $\delta_B$ take into account radiation corrections and scattering in the second Born approximation; in accordance with (51.13) they are equal to

$$\delta_R = \frac{\alpha}{\pi}\Bigg\{2(1-2\Phi\coth 2\Phi)\left(1+\ln\frac{2\Delta\varepsilon}{m}\right)+\Phi\tanh\Phi$$
$$+2(1-\Phi\coth\Phi)\left(1-\frac{1}{3}\coth^2\Phi\right)-\frac{2}{9}+\frac{1}{v}\ln\frac{1-v}{1+v}$$
$$+2\Phi\coth 2\Phi\ln\frac{1}{1-v^2}+\frac{v^2\sin^2\frac{\vartheta}{2}}{1-v^2\sin^2\frac{\vartheta}{2}}\cdot\frac{2\Phi}{\sinh 2\Phi}$$
$$+\frac{(1-v^2)\cosh\Phi}{v\sin\frac{\vartheta}{2}}\int_{\cos\frac{\vartheta}{2}}^{1}\left[\frac{\ln(1+v\zeta)}{1-v\zeta}-\frac{\ln(1-v\zeta)}{1+v\zeta}\right]\frac{d\zeta}{\sqrt{\zeta^2-\cos^2\frac{\vartheta}{2}}}\Bigg\}, \quad (51.18)$$

$$\delta_B = \pi\alpha vZ\sin\frac{\vartheta}{2}\left(1-\sin\frac{\vartheta}{2}\right)\left(1-v^2\sin^2\frac{\vartheta}{2}\right)^{-1}$$

(in the expression for $\delta_R$ we have used formula (51.14)).

We emphasize that these formulas can be used only for sufficiently small values of $\delta_R$ and $\delta_B$. In particular, the nucleus must be sufficiently light since the second Born approximation gives correct results only for $Z < 10$–$15$ (radiation corrections to scattering were first obtained by J. Schwinger (174); cf. also (55)).

Table 14 gives the values of $\delta_R$ and $\delta_B$ (in per cent) showing their dependence on $\varepsilon$, $\vartheta$, $\Delta\varepsilon$.

TABLE 14

|  | $\vartheta$ | 45° | 90° | 135° | 45° | 90° | 135° | 45° | 90° | 135° |
|---|---|---|---|---|---|---|---|---|---|---|
|  | $\varepsilon$ | 2.5 MeV |  |  | 4.0 MeV |  |  | 9.5 MeV |  |  |
|  | $\delta_B/Z$ | 0.61 | 0.89 | 0.86 | 0.62 | 0.92 | 0.99 | 0.63 | 0.94 | 1.07 |
| $\delta_R$ | $\Delta\varepsilon = 10$ keV | 4.8 | 7.4 | 8.7 | 6.9 | 9.9 | 11.3 | 12.4 | 15.9 | 17.5 |
|  | $\Delta\varepsilon = 25$ ,, | 3.9 | 6.0 | 7.1 | 5.7 | 8.1 | 9.3 | 10.5 | 13.5 | 14.8 |
|  | $\Delta\varepsilon = 50$ ,, | 3.2 | 5.0 | 5.9 | 4.7 | 6.8 | 7.9 | 9.0 | 11.7 | 12.8 |
|  | $\Delta\varepsilon = 100$ ,, | 2.5 | 3.9 | 4.7 | 3.8 | 5.5 | 6.4 | 6.7 | 9.9 | 10.8 |

Formula (51.18) for $\delta_R$ becomes inapplicable for $\Delta\varepsilon \to 0$ since as $\Delta\varepsilon \to 0$ the scattering cross section, strictly speaking, tends to zero.

We note that in the nonrelativistic domain the quantities $\delta_B$ and $\delta_R$ tend to zero as $v \to 0$.

### 51.4. *Removal of the Infrared Divergence for an Arbitrary Scattering Process*

We have seen earlier that the radiation corrections to the cross section for the scattering of an electron by an external field contain the photon "mass."

We shall now trace how the infrared divergence arises formally in the matrix elements. Prior to regularization neither the electron self-energy part, nor the vertex part (with internal electron lines) diverges in the domain of low frequencies of the virtual photons. But the vertex part and the electron self-energy part with external lines do contain an infrared divergence (cf. § 47).

In the course of regularizing the internal vertex part $\Lambda_\mu$ we subtract from it the value of $\Lambda_\mu$ for free electron lines, which contains an infrared divergence. Thus, the regularized value of $\Lambda_{\mu R}$ now begins to contain the photon "mass" $\lambda$. In a similar manner $\lambda$ appears in the expression for the electron self-energy part. For this reason the photon "mass" begins to appear in the radiation corrections to a great variety of scattering processes.

We shall now show, by generalizing the result obtained for the radiation corrections to the scattering of an electron in the nuclear Coulomb field, that if in the case of any arbitrarily complicated scattering process, which is an $n$th order perturbation theory effect, we take into account, along with the radiation corrections to this effect corresponding to

$(n+2)$th order perturbation theory, also the emission of a soft real photon, then the total scattering cross section, which takes into account both the radiation corrections and the additional emission of a soft photon, will not contain the photon "mass".

For the sake of simplicity we assume that only a single electron participates in the scattering process. The matrix element for such a process can be written in the form

$$M = \bar{u}(p_2)Q(p_1, p_2)u(p_1), \tag{51.19}$$

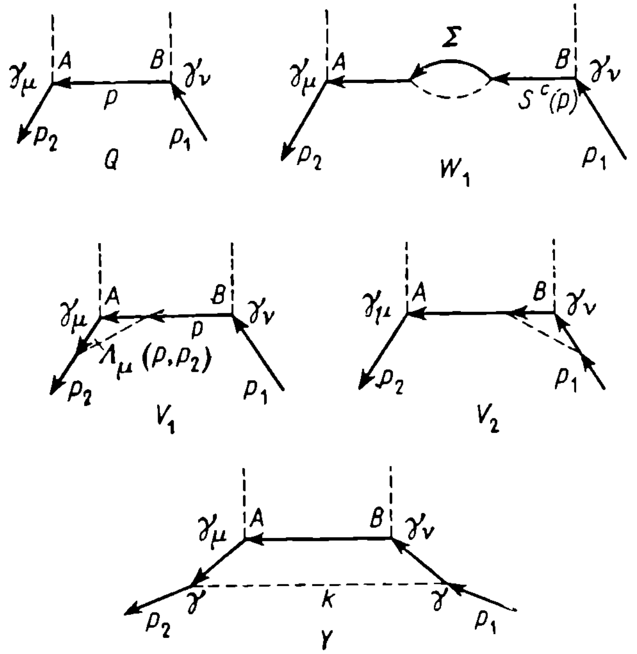


Fig. 107.

where $p_1$ and $p_2$ are the electron momenta before and after scattering, $u(p_1)$ and $u(p_2)$ are the corresponding spinor amplitudes, and $Q$ is a certain matrix.

We consider the lowest order radiation corrections to this scattering process associated with the emission and subsequent absorption of a virtual photon and leading to an infrared divergence. For example, let us take bremmstrahlung from an electron as the basic process (diagram $Q$ in Fig. 107). Then the radiation corrections to this process which lead to an infrared divergence will correspond to diagrams $W_1$, $V_1$, $V_2$, shown in Fig. 107.

We show first of all that in the contributions made by the radiation corrections the terms containing $\lambda$, and arising from an equal number of electron self-energy parts and vertex parts, mutually cancel.

We take, for example, the vertex $A$ (Fig. 107). The matrix $\gamma_\mu$ in the basic process $Q$ corresponds to the contribution from the diagrams for the radiation corrections $W_1$ and $V_1$

$$\gamma_\mu S^c(p)\Sigma(p)+\Lambda_\mu(p,p_2)=\gamma_\mu S^c(p)\Sigma_1+\gamma_\mu(\hat{p}-im)^{-1}\Sigma_0(\hat{p}-im)$$
$$+\gamma_\mu S^c(p)\Sigma_R(p)+\gamma_\mu L+\Lambda_\mu R(p,p_2), \quad (51.20)$$

where the notations of subsections 47.2 and 47.4 have been used. As has been explained previously, the left hand side of this equation does not contain an infrared divergence which appears only in the constants $\Sigma_0$ and $L$. But these constants are related by the identity (47.51) $L=-\Sigma_0$ and therefore cancel in (51.20). For this reason the regularized value of the left hand side of (51.20) will not contain any infrared divergence, as has been stated previously.

If the fundamental diagram $Q$ contains $n$ internal electron lines, then in the diagrams for the radiation corrections $W_i$ and $V_i$ there will be $n$ internal electron self-energy parts and $n-1$ internal vertex parts. From all these parts there will remain only one uncompensated photon "mass" $\lambda$ coming from one electron self-energy part. On taking into account the existence of two free electron lines with which two vertex parts are associated, we finally obtain from all the diagrams $W_i$ and $V_i$ one "uncompensated" $\lambda$ from the vertex part. In accordance with (47.52) $\lambda$ appears in this part in the form of ln $(\lambda/m)$. On taking $\lambda\to 0$ we can, therefore, write the expression for the principal part with respect to $\lambda$ of the matrix element corresponding to diagrams $W_i$ and $V_i$

$$M^\lambda_{W+V}=-\frac{\alpha}{\pi}M\ln\frac{\lambda}{m}. \quad (51.21)$$

We now consider the last diagram for the radiation corrections $Y$ and determine the matrix element $M_Y$ corresponding to it. In accordance with the general rules of subsection 25.5 we have

$$M_Y=\frac{i\alpha}{4\pi^3}\int\bar{u}(p_2)\gamma_\sigma\frac{i(\hat{p}_2-\hat{k})-m}{(p_2-k)^2+m^2}Q(p_1-k,p_2-k)$$
$$\times\frac{i(\hat{p}_1-\hat{k})-m}{(p_1-k)^2+m^2}\gamma_\sigma u(p_1)\frac{d^4k}{k^2+\lambda^2},$$

where because of the infrared divergence we have introduced into the function $D^c(k)$ the photon "mass" $\lambda$. We are interested in the contribution made to $M_Y$ by small values of $k$, and, therefore, wherever possible, we neglect quantities of order $k$.

On noting that

$$\bar{u}(p_2)\gamma_\sigma[i(\hat{p}_2-\hat{k})-m]$$
$$= \bar{u}(p_2)\{-i(\hat{p}_2+m)-2ik_\sigma+2ip_{2\sigma}+\hat{k}\gamma_\sigma\} \approx 2ip_{2\sigma}\bar{u}(p_2),$$

we obtain

$$M_Y \approx -\frac{i\alpha}{\pi^3}(p_1\, p_2)\bar{u}(p_2)$$
$$\times Q(p_1, p_2)\, u(p_1) \int \frac{d^4k}{k^2+\lambda^2}\, \frac{1}{[(p_2-k)^2+m^2]\,[(p_1-k)^2+m^2]}$$

or

$$M_Y = Ma_Y,$$

where

$$a_Y = -\frac{i\alpha}{\pi^3}(p_1 p_2) \int \frac{d^4k}{k^2+\lambda^2} \cdot \frac{1}{[(p_2-k)^2+m^2]\,[p_1-k)^2+m^2]}\,.$$

The integral

$$J \equiv \int \frac{d^4k}{k^2+\lambda^2}\, \frac{1}{[(p_2-k)^2+m^2]\,[(p_1-k)^2+m^2]}$$

appearing in the previous expression is, in accordance with (47.16′), equal to

$$J = -\frac{2\pi^2 i}{m^2 \sin 2\theta}\left\{\int_0^\theta \ln\frac{\cos\xi}{\cos\theta}\, d\xi + \int_0^\theta \ln\frac{\lambda}{m}\, d\xi\right\}.$$

Since we are interested only in the dependence of $J$ on $\lambda$, then, without loss of generality, we can assume that the electron momentum $p_1$ before scattering is equal to zero. In this case

$$p_1 p_2 = -m\varepsilon_2, \qquad \sin^2\theta = \frac{-\varepsilon^2+m}{2m},$$

$$\sin 2\theta = i\,\frac{p^2}{m}, \qquad \theta = \frac{i}{2}\operatorname{arccot} v_2,$$

where $v_2$ is the electron velocity after scattering. Therefore $\lambda$ appears in $a_Y$ in the form of the term

$$a_Y^\lambda = \frac{\alpha}{\pi} \frac{\operatorname{arctanh} v_2}{v_2} \ln \frac{\lambda}{m},$$

and in $M_Y$ in the form of the term

$$M_Y^\lambda = M \frac{\alpha}{\pi} \frac{\operatorname{arc\,tanh} v_2}{v_2} \ln \frac{\lambda}{m}. \tag{51.21'}$$

The total matrix element for the radiation corrections will contain the photon "mass" in the form $M_R^\lambda = M_{W+V}^\lambda + M_Y^\lambda$, while the scattering cross section, taking radiation corrections into account, will contain it in the form

$$|M + M_R^\lambda|^2 \approx M^2 \left\{ 1 + \frac{2\alpha}{\pi} \left( \frac{\operatorname{arctanh} v_2}{v_2} - 1 \right) \ln \frac{\lambda}{m} \right\}.$$

Therefore, the scattering cross section $d\sigma_R$, taking radiation corrections into account, will contain $\lambda$ in the form of the added term

$$d\sigma_R^\lambda = b_R^\lambda \, d\sigma_0, \tag{51.22}$$

where

$$b_R^\lambda = \frac{2\alpha}{\pi} \left( \frac{\operatorname{arctanh} v_2}{v_2} - 1 \right) \ln \frac{\lambda}{m} \tag{51.22'}$$

and $d\sigma_0$ is the cross section for the basic process.

We now consider together with the radiation corrections also the emission of a soft real photon of momentum $k$ and polarization $e$.

The diagrams representing the emission of the photon $k$ are shown in Fig. 108. The matrix element corresponding to these diagrams is, in accordance with subsection 25.5, equal to

$$M_k = \frac{ie}{(2\pi)^{3/2}} \bar{u}(p_2) \left\{ \frac{\hat{e}}{\sqrt{2\omega}} \frac{i(\hat{p}_2 + \hat{k}) - m}{(p_2 + k)^2 + m^2} Q(p_2 + k, p_1) \right.$$

$$\left. + Q(p_2, p_1 + k) \frac{i(\hat{p}_1 + \hat{k}) - m}{(p_1 + k)^2 + m^2} \frac{\hat{e}}{\sqrt{2\omega}} \right\} u(p_1).$$

On assuming $\omega \ll \varepsilon_1$ and on noting that

$$(i\hat{p}_1 + m) u(p_1) = 0,$$

we write $M_k$ in the form

$$M_k = \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left( \frac{p_1 e}{p_1 k} - \frac{p_2 e}{p_2 k} \right) M. \tag{51.23}$$

The cross section for scattering accompanied by the emission of a soft photon of energy not exceeding $\Delta\varepsilon$ is evidently equal to

$$d\sigma_k = b_k d\sigma_0, \tag{51.24}$$

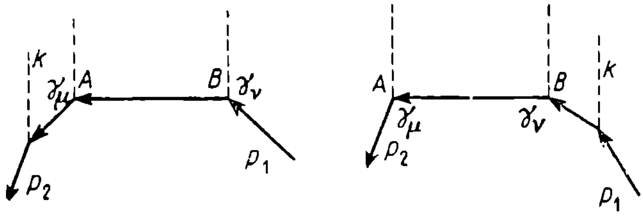


Fig. 108.

where

$$b_k = \frac{e^2}{(2\pi)^3} \sum_e \int_{\omega_{\min}}^{\Delta\varepsilon} \frac{d^3k}{2\omega} \left| \frac{p_1 e}{p_1 k} - \frac{p_2 e}{p_2 k} \right|^2, \quad \Delta\varepsilon \ll m.$$

On assuming, as we have done previously, that $p_1 = 0$, we obtain in accordance with subsection 30.3

$$b_k = \frac{2\alpha}{\pi} \left( \frac{\operatorname{arctanh} v_2}{v_2} - 1 \right) \ln \frac{\Delta\varepsilon}{\omega_{\min}} = \frac{2\alpha}{\pi} \left( \frac{\operatorname{arctanh} v_2}{v_2} - 1 \right) \left( \ln \frac{2\Delta\varepsilon}{\lambda} - \frac{5}{6} \right). \tag{51.24'}$$

Finally, we obtain the total scattering cross section which takes into account both the radiation corrections and the emission of a soft photon $d\sigma = d\sigma_R + d\sigma_k$. The part $d\sigma$ which can contain $\lambda$ has the form $d\sigma^\lambda = b^\lambda d\sigma_0$, where $b^\lambda = b_R^\lambda + b_k$. But in accordance with (51.22′) and (51.24′) this quantity does not contain the photon mass; therefore, as has been asserted earlier, $\lambda$ does not appear in the total scattering cross section (96).

## 51.5. *Scattering of High Energy Electrons by an External Field*

As the magnitude of the transferred momentum $q$ increases, the term $(\alpha/\pi) \ln^2(q^2/m^2)$ becomes the principal term in the expression for the quantity $\delta_R$ which determines the radiation corrections to the scattering of an electron by an external field. This quantity attains values of the order of magnitude unity when the transferred momenta attain values $q \sim m \exp(\frac{1}{4}\sqrt{\pi/\alpha})$. In such a case formula (51.17) no longer has meaning since the quantity $\delta_R$ is no longer small. In order to obtain the correct expression for the cross section for the scattering of an electron by an external field taking radiation corrections into account in this energy region we cannot confine ourselves to the approximation just considered, but must sum the whole perturbation theory series. As will beeome apparent from subsequent discussion, in each perturbation theory approximation the largest term in the domain of large values of transferred momenta will be an expression of the form $[(\alpha/\pi \ln^2(q^2/m^2)]^n$ where $n$ is the order of the approximation.

The remaining terms of the type $\alpha^n[\ln^2(q^2/m^2)]^s$, where $s < n$, can be neglected since

$$\alpha^n\left(\ln^2\frac{q^2}{m^2}\right)^s = \alpha^{n-s}\left(\frac{\alpha}{\pi}\ln^2\frac{q^2}{m^2}\right)^s \ll \left(\frac{\alpha}{\pi}\ln^2\frac{q^2}{m^2}\right)^s.$$

We shall refer to the largest terms as doubly logarithmic terms. Our problem consists of picking out the doubly logarithmic term from each successive perturbation theory approximation, and then to sum all such terms. Such a summation can be carried out in the domain of transferred momenta determined by $(\alpha/\pi)\ln^2(q^2/m^2) < 1$, and the expression so obtained can then be continued into the domain of transferred momenta defined by $(\alpha/\pi)\ln^2(q^2/m^2) > 1$.

In order to explain the manner in which the doubly logarithmic terms are picked out we consider the matrix element $M_1^{(3)}$ defined by formula (51.1). In the case of interest to us we are dealing with the calculation of the correction to the vertex part for which $q^2 = (p_1-p_2)^2 \gg p_1^2, p_2^2$. Such a relation between the vectors $q$, $p_1$, $p_2$ can evidently hold only for non-Euclidean vectors $p_1$, $p_2$, $q$ (Euclidean vectors satisfy the triangle rule).

The integral over $k$ in the expression for $M_1^{(3)}$ does not differ in the region $k_\alpha \gg q_\alpha$ from the similar integral for the vertex part in the case of Euclidean vectors $p_1$, $p_2$, $q$, and does not lead to the appearance of doubly logarithmic terms (cf. subsection 47.4). Therefore, there remains only the region of integration $k_\alpha \ll q_\alpha$, which, as we shall now see, leads to doubly logarithmic terms.

In place of $k$ we introduce new variables of integration $u, v$, $x$ related to $k$ by the expressions (first of references (192) and of (1))

$$k = p_1\left(u-\frac{2m^2}{q^2}v\right)+p_2\left(v-\frac{2m^2}{q^2}u\right)+k_\perp \qquad (51.25)$$

$$k_\perp^2 = x,$$

where the vector $k_\perp$ is perpendicular to the plane defined by the vectors $p_1$, $p_2$. Under the assumptions made with respect to $p_1$ and $p_2$ the vector $k_\perp$ is space-like, i.e., $x > 0$. In terms of the new variables the element of volume integrated over the angles in the plane of the vector $k_\perp$ has the form for $q^2 \gg p_1^2, p_2^2$

$$d^4k = \frac{\pi}{2} q^2\, du\, dv\, dx$$

The quantities $k^2$, $(p_1-k)^2+m^2$ and $(p_2-k)^2+m^2$ are related to $u$, $v$, $x$ by the expressions

$$\begin{aligned} k^2 &= -q^2uv+m^2(u^2+v^2)+x, \\ (p_1-k)^2+m^2 &= k^2+q^2v, \qquad (51.25') \\ (p_2-k)^2+m^2 &= k^2+q^2u. \end{aligned}$$

It can be easily shown that in integration over $x$ a real contribution comes only from going around the pole $k^2 = 0$. Since $x > 0$, $u$ and $v$ must vary in such a way that the point $k^2 = 0$ would correspond to positive $x$. From this it follows that the quantities $u$ and $v$ must have the same sign.

On carrying out the integration over $x$ we obtain in accordance with (51.25′) logarithmic integrals with respect to $u$ and $v$.

We now establish the limits of integration over $u$ and $v$. Since the region of integration over $x$ is restricted by the conditions $k_\alpha \ll q_\alpha$, it follows from (51.25) that $|u| \ll 1$, $|v| \ll 1$. Further, on taking into account the fact that the factor $k^2$ in the denominator of (51.1) must in fact be replaced by $k^2+\lambda^2$, where $\lambda$ is the photon "mass", which is equal with logarithmic accuracy to the frequency of the accompanying soft photon, we obtain the restrictions $|u| \gg \lambda^2/q^2$, $|v| \gg \lambda^2/q^2$. Finally, we must take into account the requirement $x > 0$ at the point $k^2 = 0$. It leads to the inequality

$$q^2uv-m^2(u^2+v^2) > 0.$$

On carrying out in $M_1^{(3)}$ the integration over $u$ and $v$ taking the above restrictions into account, we obtain the following expression for the additional term to the vertex part in the third approximation of perturbation theory:

$$F^{(3)} = -\frac{\alpha}{2\pi}\hat{A}_q \ln\frac{q^2}{m^2}\left(\ln\frac{q^2}{m^2}+2\ln\frac{m^2}{\lambda^2}\right).$$

This expression agrees, as it should, with formula (47.53).

We now consider the corrections to the vertex part in higher order approximations. In diagrams of the $n$th order for the vertex part there are $n$-1 virtual photon lines intersecting in an arbitrary manner. In this case the numerators of the fractions in the matrix elements corresponding to these diagrams coincide for $q^2 \gg m^2$, and only the denominators of the fractions are different.

We first consider diagrams with two virtual photon lines. There are two such diagrams—one with intersecting, and the other with parallel lines.

On introducing in place of $k_1$ and $k_2$ the new variables $u_1$, $v_1$, $x_1$ and $u_2$, $v_2$, $x_2$ in accordance with (51.25), and on carrying out the integration over $x_1$ and $x_2$, we obtain for these diagrams expressions proportional to

$$\{(p_1k_1)[(p_1k_1)+(p_1k_2)][(p_2k_1)+(p_2k_2)](p_2k_1)\}^{-1}$$

$$\{(p_1k_1)[(p_1k_1)+(p_2k_2)][p_2k_1)-(p_2k_2)](p_2k_1)\}^{-1}.$$

The integral with the highest power of the logarithm is obtained from the first expression when the conditions $(p_1k_2) \gg (p_1k_1)$; $(p_2k_1) \gg (p_2k_2)$ are satisfied, and from the second expression when the conditions $(p_1k_2) \gg (p_1k_1)$; $(p_2k_2) \gg (p_2k_1)$ are satisfied. In this case the expressions corresponding to the two diagrams simply coincide. Addition of these two expressions leads to the integrations over $u_1$ and $u_2$ becoming independent. On interchanging the variables $k_1 \rightleftarrows k_2$, and on taking half the sum of the resultant and the initial expressions we can easily show that the integrations over $v_1$ and $v_2$ also become independent. As a result, we obtain the following simple expression for the additional term to the vertex part proportional to $\alpha^2$

$$F^{(5)} = \hat{A}_q\frac{1}{2}\left[-\frac{\alpha}{2\pi}\ln\frac{q^2}{m^2}\left(\ln\frac{q^2}{m^2}+2\ln\frac{m^2}{\lambda^2}\right)\right]^2.$$

This formula can be easily generalized to the case of the $n$th approximation

$$F^{(2n+1)}=\hat{A}_q\frac{1}{n!}\left[-\frac{a}{2\pi}\ln\frac{q^2}{m^2}\left(\ln\frac{q^2}{m^2}+2\ln\frac{m^2}{\lambda^2}\right)\right]^n.$$

On summing all these expressions we obtain

$$F=\hat{A}_q\exp\left[-\frac{a}{2\pi}\ln\frac{q^2}{m^2}\left(\ln\frac{q^2}{m^2}+2\ln\frac{m^2}{\lambda^2}\right)\right]. \tag{51.26}$$

The continuation of this formula into the region $(a/\pi)\ln^2(q^2/m^2)>1$ obviously does not alter the form of $F$.

It follows from (51.26) that the cross section for the scattering of an electron by an external field for large $q^2$ taking into account radiation corrections up to terms of order $a$ differs from the cross section without the corrections by the square of the exponential factor appearing in (51.26) (first of references (1))

$$d\sigma=d\sigma_0\exp\left[-\frac{a}{\pi}\ln\frac{q^2}{m^2}\left(\ln\frac{q^2}{m^2}+2\ln\frac{m^2}{\lambda^2}\right)\right]. \tag{51.27}$$

In order to elucidate the physical meaning of the quantity $\lambda$ it is necessary to consider processes of scattering accompanied by emission of an arbitrary number of soft photons of total energy not exceeding a certain limiting value $\Delta\varepsilon$. (It is assumed that $\Delta\varepsilon \ll m$.) In this case it turns out that with logarithmic accuracy $\lambda$ should simply be replaced by $\Delta\varepsilon$.

### 51.6. *Radiation Corrections to Electron-Electron and Electron-Positron Scattering*

We now proceed to investigate radiation corrections to electron-electron and electron-positron scattering (5), (163), (153).

Since electron-electron scattering cannot be regarded as the scattering of an electron by an external field, then in order to determine the radiation corrections to this process we cannot directly use the foregoing results which refer to the scattering of an electron by the field due to a nucleus.

The principal types of diagrams which determine electron-electron scattering and radiation corrections to this process are shown in Fig. 109. In order to obtain all such diagrams we must in addition to the diagrams given in the figure also consider diagrams obtained from those given there by means of the substitutions 1) $p_1 \rightleftarrows p_2$, $p_1' \rightleftarrows p_2'$; 2) $p_1' \rightleftarrows p_2'$; 3) $p_1 \rightleftarrows p_2$.

We denote by $M_i$ the matrix element corresponding to the $i$th diagram of Fig. 109 (without the $\delta$-function). We respectively denote by $M_i'$, $\tilde{M}_i$, $\tilde{M}_i'$ the matrix elements obtained from $M_i$ by performing the substitutions 1) $p_1 \rightleftarrows p_2$, $p_1' \rightleftarrows p_2'$; 2) $p_1' \rightleftarrows p_2'$; 3) $p_1 \rightleftarrows p_2$. The differential cross section for purely elastic electron-electron scattering taking radiation corrections of order $\alpha$ into account can then be written in the form

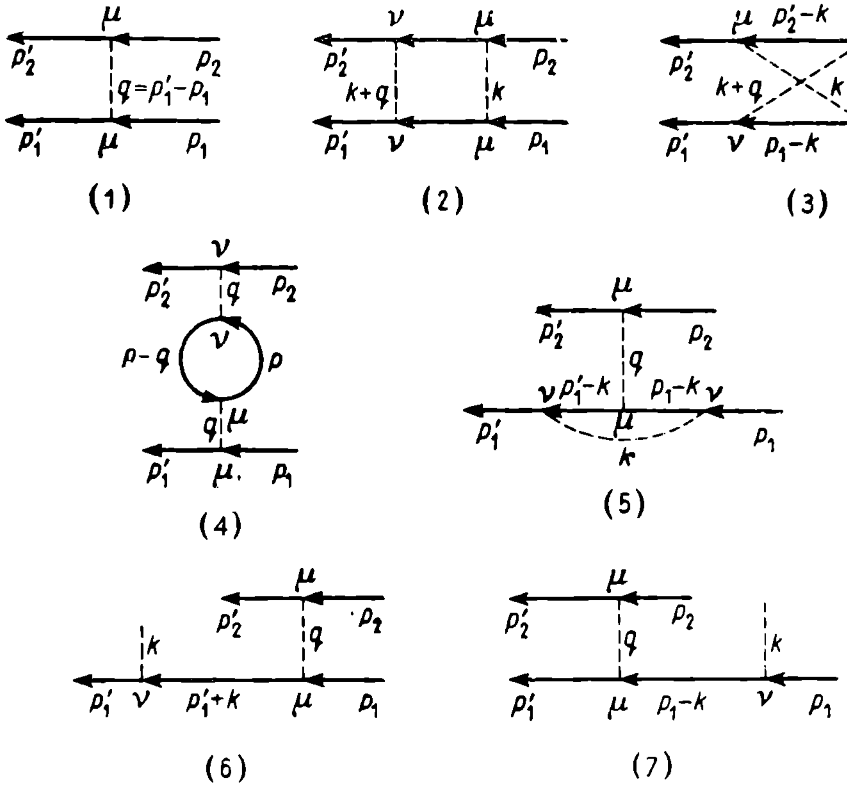


Fig. 109.

$$d\sigma_e = \frac{1}{2v}\,\frac{1}{4}\sum_{\mu_i,\mu_f}\{|M_1+M_1'+\tilde{M}_1+\tilde{M}_1'|^2$$

$$+\,2\mathrm{Re}\sum_{i=1}^{5}(M_1+M_1'+\tilde{M}_1+\tilde{M}_1')^*\,(M_i+M_i'+\tilde{M}_i+\tilde{M}_i')\}\,do,$$

where $v$ is the electron velocity in the center of mass system, and the summation is taken over the orientations of the electron spins in the initial and final states; *do* is the element of solid angle into which the electron is scattered.

In order to obtain the cross section for electron-positron scattering $d\sigma'_e$ we must in this expression set $p_1 = p_-, p'_1 = p'_-, p_2 = -p'_+, p'_2 = -p_+$, where $p_-$ and $p_+$ are the four-momenta of the electron and the positron.

The photon "mass" $\lambda$ appears in the scattering cross sections $d\sigma_e$ and $d\sigma'_e$ (it is contained in the matrix elements corresponding to diagrams 2, 3, 5 of Fig. 109).

In order to eliminate $\lambda$ it is necessary, as we have done in finding the radiation corrections to the scattering of an electron by an external field, to investigate the inelastic electron-electron and electron-positron scattering accompanied by emission of a soft photon. The diagrams corresponding to this process are shown in Fig. 109 (6, 7) (together with these diagrams we must also consider those diagrams which are obtained from the ones shown in the figure by means of the substitutions 1) $p'_1 \rightleftarrows p'_2$, 2) $p_1 \rightleftarrows p_2$, 3) $p_1 \rightleftarrows p_2$, $p'_1 \rightleftarrows p'_2$).

We shall reproduce here only the expression for the total scattering cross section $d\sigma = d\sigma_e + d\sigma_u$ which does not contain any infrared divergences in the limiting case of high energies.

If $p \gg m$, $\sin\vartheta \sim 1$, where $p$ is the electron momentum and $\vartheta$ is the scattering angle in the center of mass system, then the differential cross section for electron-electron scattering in the laboratory system has the form

$$\begin{aligned} d\sigma = \frac{\alpha^2(1-v^2)}{4\chi^2 m^2(1-\chi)} \Big\{ & \frac{1}{2}(2-3\chi+3\chi^2-\chi^3) \\ & + \frac{\alpha}{\pi}\Big[ 2(1-2\Phi+2\Phi_a-2\Phi_b)\ln\frac{m}{2\Delta\varepsilon}(2-3\chi+3\chi^2-\chi^3) \\ & + \frac{\Phi}{3}(28-48\chi+48\chi^2-17\chi^3) + \Phi_a(2-2\chi+\chi^2) \\ & + \Phi_b(2-2\chi+3\chi^2-\chi^3) - \Phi_a^2(2-\chi) \\ & - \Phi_b^2(2-\chi+2\chi^2-\chi^3) - \Phi^2(6-5\chi+5\chi^2-2\chi^3) \\ & + 2\Phi\Phi_a(6-7\chi+6\chi^2-2\chi^3) - 2\Phi\Phi_b(10-17\chi+16\chi^2-5\chi^3) \\ & + 2\Phi_a\Phi_b(2-3\chi+3\chi^2-\chi^3) \\ & - \frac{37}{18}(2-3\chi+3\chi^2-\chi^3) - \frac{\pi^2}{4}\chi(2-\chi) \Big]\Big\} do + \text{terms} \\ & \text{obtained by replacing } \chi \to 1-\chi,\ \Phi \rightleftarrows \Phi_b, \end{aligned} \tag{51.28}$$

where $\chi = \sin^2 \vartheta/2$ and

$$\begin{aligned}(p_1'-p_1)^2 &= q^2 = 4m^2 \sinh^2 \Phi,\\ (p_1-p_2)^2 &= q_a^2 = 4m^2 \sinh^2 \Phi_a,\\ (p_1'-p_2)^2 &= q_b^2 = 4m^2 \sinh^2 \Phi_b.\end{aligned}$$

In the same limiting case the electron-positron scattering cross section is given by

$$\begin{aligned}d\sigma' = \frac{a^2(1-v^2)}{4\chi^2 m^2}\Bigg\{&(1-\chi+\chi^2)^2\\ &+\frac{a}{\pi}\Bigg[4(1-2\Phi+2\Phi_b-2\Phi_a)\ln\frac{m}{2\Delta\varepsilon}(1-\chi+\chi^2)^2\\ &-\Phi_a^2(2-9\chi+19\chi^2-15\chi^3+6\chi^4)-\Phi^2(6-15\chi+19\chi^2-9\chi^3+2\chi^4)\\ &-2\Phi_b^2(1-3\chi+4\chi^2-3\chi^3+\chi^4)-2\Phi\Phi_a(10-17\chi+24\chi^2-17\chi^3+10\chi^4)\\ &+2\Phi\Phi_b(6-12\chi+13\chi^2-6\chi^3+2\chi^4)+2\Phi_a\Phi_b(2-6\chi+13\chi^2\\ &-12\chi^3+6\chi^4)+\frac{\Phi}{3}(28-42\chi+51\chi^2-23\chi^3+6\chi^4)+\frac{\Phi_a}{3}(6-23\chi\\ &+51\chi^2-42\chi^3+28\chi^4)+\Phi_b(2-5\chi+6\chi^2-5\chi^3+2\chi^4)\\ &-\frac{37}{9}(1-\chi+\chi^2)^2+\frac{\pi^2}{4}\chi^2(5-6\chi+4\chi^2)\Bigg]\Bigg\}\,do.\end{aligned} \tag{51.29}$$

In the limiting case of high energies and small scattering angles ($p \gg m$, $p\vartheta \sim m$) the electron-electron scattering cross section assumes the form

$$d\sigma = \frac{a^2(1-v^2)}{4m^2 \sin^4 \dfrac{\vartheta}{2}}(1-\delta_R)\,do, \tag{51.30}$$

where

$$\begin{aligned}\delta_R = \frac{a}{\pi}\Bigg[&2\left(1-\frac{1}{3}\coth^2\Phi\right)(1-\Phi\coth^2\Phi)-\frac{2}{9}\\ &+4(2\Phi\coth 2\Phi-1)\left(\ln\frac{m}{2\Delta\varepsilon}-1\right)+4\Phi_a(2\Phi\coth 2\Phi-1)\\ &-1-\frac{4\Phi}{\sinh 4\Phi}+\frac{4}{\tanh 2\Phi}\int_0^{\Phi} x\tanh x\,dx+\frac{2}{\tanh 2\Phi}\int_0^{2\Phi} x\coth x\,dx\Bigg].\end{aligned}$$

In the case $p \gg m$, $p\vartheta \sim m$ the electron-positron scattering cross section is given by a similar formula.

In the limiting case of ultrarelativistic energies, when $\ln(p/m) \gg 1$, $\sin\vartheta \sim 1$, the electron-electron and electron-positron elastic scattering cross section is determined by

$$d\sigma_e = d\sigma_0(1-\delta_R), \tag{51.31}$$

where $d\sigma_0$ is the cross section for the basic scattering process and

$$\delta_R = \delta_2+\delta_3+\delta_4+\delta_5+\delta_u,$$

$$-\delta_2 = \delta_3 = \delta_5 = \frac{4\alpha}{\pi}\left(2\Phi_a \ln\frac{m}{\lambda}+\Phi_a^2\right),$$

$$\delta_4 = -\frac{4\alpha}{\pi}\frac{\Phi_a}{3},$$

$$\delta_u = \frac{4\alpha}{\pi}\left(2\Phi_a \ln\frac{\lambda}{2\Delta\varepsilon}+\frac{5}{2}\Phi_a^2\right)$$

(the quantity $\delta_i$ corresponds to the *i*th diagram of Fig. 109; $\delta_u$ takes into account the emission of a soft photon).

Formulas (51.28) and (51.29) are no longer applicable if $q^2$, $q_a^2$ or $q_b^2$ become much larger than $m^2$, since in such a case the radiation corrections become of order of magnitude unity. In such a case we cannot confine ourselves to the approximation just considered, but must sum the whole perturbation theory series. For $q^2, q_a^2, q_b^2 \gg m^2$ such a summation in the case of electron-electron scattering does not differ essentially from the summation carried out in subsection 51.5 in discussing the problem of the scattering of a high energy electron by an external field. It leads to the replacement of formula (51.31) by

$$d\sigma_e = d\sigma_0 e^{-\delta'_R}, \tag{51.32}$$

where $\delta'_R$ is the principal term in $\delta_R$ containing squares of logarithms (third of references (1)).

A similar relation also holds for electron-positron scattering (third of references (1).

## § 52. Radiation Corrections to Photon-Electron Scattering, to Pair Creation and Annihilation, and to Bremsstrahlung

### 52.1. *Radiation Corrections to the Compton Effect*

We now proceed to determine the radiation corrections to scattering processes in which photons as well as electrons take part. We begin by considering the radiation corrections to the Compton effect (36).

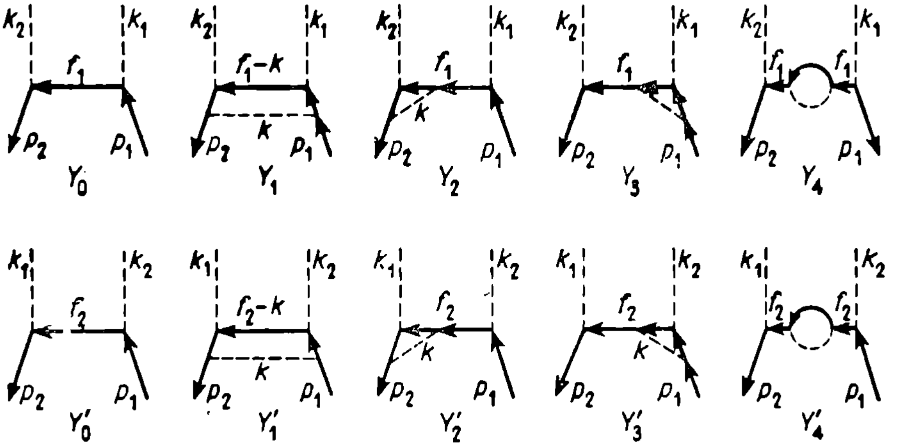


Fig. 110.

Figure 110 gives diagrams showing the basic Compton effect (diagrams $Y_0$ and $Y_0'$) and the radiation corrections to this effect of order $\alpha$ (diagrams $Y_1$, $Y_2$, $Y_3$, $Y_4$ and $Y_1'$, $Y_2'$, $Y_3'$, $Y_4'$).

The matrix elements which determine the basic effect and the radiation corrections are given in accordance with the rules of subsection 25.5 by

$$\begin{aligned} S^{(2)}_{i\to f} &= ie^2(2\pi)^4(\bar{u}_2 Q_0 u_1)\frac{\delta(p_1+k_1-p_2-k_2)}{2\sqrt{\omega_1\omega_2}}, \\ S^{(4)}_{i\to f} &= -e^4(\bar{u}_2 Q\, u_1)\frac{\delta(p_1+k_1-p_2-k_2)}{2\sqrt{\omega_1\omega_2}}, \end{aligned} \tag{52.1}$$

where

$$Q_0 = \hat{e}_2\frac{i\hat{f}_1-m}{m^2\varkappa_1}\hat{e}_1+\hat{e}_1\frac{i\hat{f}_2-m}{m^2\varkappa_2}\hat{e}_2,$$

$$f_1 = p_1+k_1 = p_2+k_2,$$

$$f_2 = p_1 - k_2 = p_2 - k_1,$$

$$m^2\varkappa_1 = f_1^2 + m^2,$$

$$m^2\varkappa_2 = f_2^2 + m^2,$$

$$Q = \sum_{i=1}^{4} (Y_i + Y_i'),$$

$$Y_1 = \int \gamma_\mu \frac{i(\hat{p}_2 - \hat{k}) - m}{(p_2 - k)^2 + m^2} \hat{e}_2 \frac{i(\hat{f}_1 - \hat{k}) - m}{(f_1 - k)^2 + m^2} \hat{e}_1 \frac{i(\hat{p}_1 - \hat{k}) - m}{(p_1 - k)^2 + m^2} \gamma_\mu \frac{d^4k}{k^2 + \lambda^2},$$

$$Y_2 = \int \frac{d^4k}{k^2} \gamma_\mu \frac{i(\hat{p}_1 - \hat{k}) - m}{(p_1 - k)^2 + m^2} \hat{e}_2 \frac{i(\hat{f}_1 - \hat{k}) - m}{(f_1 - k)^2 + m^2} \gamma_\mu \frac{i\hat{f}_1 - m}{m^2\varkappa_1} \hat{e}_1,$$

$$Y_3 = \hat{e}_2 \frac{i\hat{f}_1 - m}{m^2\varkappa_1} \int \gamma_\mu \frac{i(\hat{f}_1 - \hat{k}) - m}{(f_1 - k)^2 + m^2} \hat{e}_1 \frac{i(\hat{p}_1 - \hat{k}) - m}{(p_1 - k)^2 + m^2} \gamma_\mu \frac{d^4k}{k^2},$$

$$Y_4 = \hat{e}_2 \frac{i\hat{f}_1 - m}{m^2\varkappa_1} \int \gamma_\mu \frac{i(\hat{f}_1 - \hat{k}) - m}{(f_1 - k)^2 + m^2} \gamma_\mu \frac{d^4k}{k^2 + \lambda^2} \cdot \frac{i\hat{f}_1 - m}{m^2\varkappa_1} \hat{e}_1.$$

(we have adopted here the same notation as in § 28).

The matrices $Y_i'$ differ from the matrices $Y_i$ by an interchange of the photon momenta $k_1$ and $k_2$.

As we already know, the cross section for the basic process $d\sigma_0$ is equal to

$$d\sigma_0 = \frac{r_0^2}{4} \cdot \frac{\omega_2^2}{m^2\varkappa_2^2} 8U_0\, do,$$

where

$$U_0 = \frac{1}{8} \mathrm{Sp}\{Q_0(i\hat{p}_1 - m)Q_0(i\hat{p}_2 - m)\}$$

$$= 4\left(\frac{1}{\varkappa_1} + \frac{1}{\varkappa_2}\right)^2 - 4\left(\frac{1}{\varkappa_1} + \frac{1}{\varkappa_2}\right) - \left(\frac{\varkappa_1}{\varkappa_2} + \frac{\varkappa_2}{\varkappa_1}\right).$$

The cross section for Compton scattering taking radiation corrections of order $\alpha$ into account can be written in the form

$$d\sigma = d\sigma_0 + d\sigma_1$$

$$= \frac{r_0^2}{4} \frac{\omega_2^2\, do}{m^2\varkappa_1^2} \mathrm{Sp}\left\{\left[Q_0 - \frac{e^2}{i(2\pi)^4} Q\right](i\hat{p}_1 - m)\left[\bar{Q}_0 + \frac{e^2}{i(2\pi)^4} \bar{Q}\right](i\hat{p}_2 - m)\right\}$$

$$= d\sigma_0 + \frac{r_0^2}{4} \frac{\omega_2^2\, do}{m^2 \varkappa_1^2} \frac{e^2}{(2\pi)^4} \operatorname{Sp}\left\{ \frac{1}{i} Q_0 (i\hat{p}_1 - m) \overline{Q} (i\hat{p}^2 - m) \right.$$

$$\left. - \frac{1}{i} Q (i\hat{p}_1 - m) \overline{Q}_0 (i\hat{p}_2 - m) \right\}.$$

On noting that

$$\operatorname{Sp}\{Q_0(i\hat{p}_1 - m)\overline{Q}(i\hat{p}_2 - m)\} = \left(\operatorname{Sp}\{Q(i\hat{p}_1 - m)\, Q_0(i\hat{p}_2 - m)\}\right)^*,$$

we rewrite $d\sigma$ in the form

$$d\sigma = d\sigma_0 \left( 1 - \frac{a}{\pi} \frac{U_1}{U_0} \right), \tag{52.2}$$

where

$$U_1 = \frac{1}{16\pi^2} \operatorname{Re} \operatorname{Sp} \frac{1}{i} \{ [Q_1 + Q_1'] (i\hat{p}_1 - m) \overline{Q}_0 (i\hat{p}_2 - m) \},$$

$$Q_1 = \sum_{i=1}^{4} Y_i ,\, Q_1' = \sum_{i=1}^{4} Y_i' . \tag{52.3}$$

Further, we introduce the notation

$$\begin{aligned} P_1(\varkappa_1, \varkappa_2) &= \frac{1}{16\pi^2} \operatorname{Re} \operatorname{Sp} \frac{1}{i} \{ Q_1 (i\hat{p}_1 - m)\, \overline{Q}_0 (i\hat{p}_2 - m) \}, \\ P_2(\varkappa_1, \varkappa_2) &= \frac{1}{16\pi^2} \operatorname{Re} \operatorname{Sp} \frac{1}{i} \{ Q_1' (i\hat{p}_1 - m)\, \overline{Q}_0 (i\hat{p}_2 - m) \}. \end{aligned} \tag{52.4}$$

It can be easily shown in analogy with subsection 28.4 that

$$P_2(\varkappa_1, \varkappa_2) = P_1(\varkappa_2, \varkappa_1). \tag{52.4'}$$

Therefore $U_1$ can be rewritten in the form

$$U_1 = P_1(\varkappa_1, \varkappa_2) + P_1(\varkappa_2, \varkappa_1). \tag{52.5}$$

Thus, it suffices to evaluate $P_1(\varkappa_1, \varkappa_2)$. This quantity can be written in the form

$$P_1(\varkappa_1, \varkappa_2) = \sum_{n=1}^{4} P^{(n)}, \tag{52.6}$$

where

$$P^{(n)} = \frac{1}{16\pi^2} \operatorname{Re} \operatorname{Sp} \frac{1}{i} \{ Y_n (i\hat{p}_1 - m) \overline{Q}_0 (i\hat{p}_2 - m) \}. \tag{52.6'}$$

We first of all evaluate $P^{(1)}$. It is convenient to begin this calculation by evaluating the traces of the matrices, after which the integration variables $k_\sigma$ will appear only in the form of scalar products $k^2$, $kp_1$, $kp_2$, and this considerably simplifies the evaluation of the integrals appearing in $P^{(1)}$. The latter have the following structure:

$$J_{(1;\,\sigma\tau;\,\sigma\tau\varrho)} = \int \frac{(1\,;k_\sigma;k_\sigma k_\tau;k_\sigma k_\tau k_\varrho)}{(1)\,(2)\,(3)\,(0)}\,d^4k,$$

where

$$\begin{aligned}(1) &\equiv (p_1-k)^2+m^2 = k^2-2kp_1,\\ (2) &\equiv (p_2-k)^2+m^2 = k^2-2kp_2,\\ (3) &\equiv (f_1-k)^2+m^2 = k^2-2kf_1+m^2\varkappa_1,\\ (0) &\equiv k^2+\lambda^2,\end{aligned}$$

and the symbol $(1\,;k_\sigma;k_\sigma k_\tau;k_\sigma k_\tau k_\varrho)$ denotes 1 in the case of the integral $J$, $k_\sigma$ in the case of the integral $J_\sigma$, etc.

The simplification in the evaluation of the integrals due to the evaluation of the traces can be illustrated in the example of the quantity $2p_{1\varrho}\cdot J_{\sigma\tau\varrho}$. This quantity can be written in the form

$$2p_{1\varrho}\cdot J_{\sigma\tau\varrho} = \int \frac{2(kp_1)k_\sigma k_\tau d^4k}{(1)\,(2)\,(3)\,(0)} = \int \frac{[k^2-(k^2-2kp_1)]k_\sigma k_\tau}{(1)\,(2)\,(3)\,(0)}\,d^4k = J^{(0)}_{\sigma\tau}-J^{(1)}_{\sigma\tau},$$

where

$$\begin{aligned}J^{(0)}_{\sigma\tau} &= \int \frac{k_\sigma k_\tau d^4k}{(1)\,(2)\,(3)},\\ J^{(1)}_{\sigma\tau} &= \int \frac{k_\sigma k_\tau d^4k}{(2)\,(3)\,(0)}\end{aligned} \qquad (52.7)$$

(we also use analogous notation for the other integrals, by introducing a superscript to denote the factor of the type (1), (2), (3), (0) lacking in the denominator).

In this way we are able to reduce the quantities containing the integrals $J_{\sigma\tau\varrho}$ to the evaluation of simpler integrals. Analogous simplifications also arise in the evaluation of other integrals.

We list the values of the integrals which appear in the end result:

$$\frac{1}{\pi^2 i}\,J^{(0)} = -\frac{2y^2}{\varkappa_1+\varkappa_2},$$

$$\frac{1}{\pi^2 i}J^{(1)} = J^{(2)} = \frac{1}{\varkappa_1}\,[F(\varkappa_1-1)-F(-1)],$$

$$\frac{1}{\pi^2 i} J^{(3)} = \frac{2y}{\sinh 2y}\,[h(2y)-h(y)-\ln\lambda],$$

$$\frac{1}{\pi^2 i} J = \frac{2y}{\sinh 2y}\cdot\frac{1}{\varkappa_1}\left[h(2y)-2h(y)+\ln\frac{\varkappa_1}{\lambda}\right],$$

$$\frac{1}{\pi^2 i} J_\sigma^{(0)} = J^{(0)}\frac{p_{1\sigma}+p_{2\sigma}}{2} + \left(J^{(0)}-\frac{4b}{\varkappa_1+\varkappa_2}\right)\frac{q_{1\sigma}+q_{2\sigma}}{2},$$

$$\frac{1}{\pi^2 i} J_\sigma^{(1)} = \left(J^{(1)}-\frac{\ln\varkappa_1}{\varkappa_1-1}\right)p_{2\sigma} + \left(2J^{(1)}+\frac{\varkappa_1-2}{\varkappa_1-1}\ln\varkappa_1-2\right)\frac{1}{\varkappa_1}\cdot q_{2\sigma},$$

$$\frac{1}{\pi^2 i} J_\sigma^{(2)} = \left(J^{(1)}-\frac{\ln\varkappa_1}{\varkappa_1-1}\right)p_{1\sigma} + \left(2J^{(1)}+\frac{\varkappa_1-2}{\varkappa_1-1}\ln\varkappa_1-2\right)\frac{1}{\varkappa_1}\cdot q_{1\sigma},$$

$$\frac{1}{\pi^2 i} J_\sigma^{(3)} = \frac{y}{\sinh 2y}(p_{1\sigma}+p_{2\sigma}),$$

$$\begin{aligned}\frac{1}{\pi^2 i} J_{\sigma\tau}^{(0)} = {} & -\frac{\delta_{\sigma\tau}}{4}\left(A_0+\frac{1}{2}-1-2b+2J^{(0)}\right)\\ & -\frac{1}{\varkappa_1+\varkappa_2}\Big\{2y^2 f_{1\sigma}f_{1\tau}+b[2f_{1\sigma}(q_{1\tau}+q_{2\tau})+2f_{1\tau}(q_{1\sigma}+q_{2\sigma})\\ & -q_{1\sigma}q_{1\tau}-q_{2\sigma}q_{2\tau}]-\left(J^{(0)}-\frac{1}{2}\right)(q_{1\sigma}q_{2\tau}+q_{1\tau}\cdot q_{2\sigma})\Big\},\end{aligned} \tag{52.8}$$

$$\begin{aligned}\frac{1}{\pi^2 i} J_{\sigma\tau}^{(1)} = {} & -\frac{\delta_{\sigma\tau}}{4}\left(A_0+\frac{1}{2}-3+\frac{\varkappa_1-2}{\varkappa_1+1}\ln\varkappa_1+2J^{(1)}\right)\\ & +\frac{1}{2}p_{2\sigma}p_{2\tau}\left[2J^{(1)}-\frac{3\varkappa_1-2}{(\varkappa_1-1)^2}\ln\varkappa_1+\frac{1}{\varkappa_1-1}\right]\\ & +\frac{1}{\varkappa_1^2}q_{2\sigma}q_{2\tau}\left[6J^{(1)}+\frac{(\varkappa_1^2+6\varkappa_1-6)\,(\varkappa_1-2)}{2(\varkappa_1-1)^2}\ln\varkappa_1\right.\\ & \left.-\frac{2\varkappa_1^2+9\varkappa_1-12}{2(\varkappa_1-1)}\right]+\frac{1}{2\varkappa_1}(p_{2\sigma}q_{2\tau}+p_{2\tau}q_{2\sigma})\\ & \times\left[6J^{(1)}+\frac{2\varkappa_1^2-9\varkappa_1+6}{(\varkappa_1-1)^2}\ln\varkappa_1+\frac{1}{\varkappa_1-1}-5\right],\end{aligned}$$

where the following notation has been introduced:

$$\sinh^2 y = -\tfrac{1}{4}(\varkappa_1+\varkappa_2),$$
$$b = 1-y\coth y,$$
$$h(y) = \frac{1}{y}\int_0^y u\coth u\,du,$$
$$F(x) = \int_0^x \ln(1+u)\frac{du}{u},$$

and

$$A_0 = 1-\ln\frac{L^2}{M^2}.$$

In evaluating the traces of the matrices appearing in $P^{(1)}$, we must keep in mind that $Y_2$ is flanked by the matrices $i\hat{p}_1-m$ and $i\hat{p}_2-m$, and that both the matrix $i\hat{p}_2$ appearing on the left, and the matrix $i\hat{p}_1$ appearing on the right can be replaced by $-m$. This substitution, together with the summation over $\mu$, enables us to rewrite the numerator of the integrand in $Y_1$ in the form

$$\begin{aligned}Y_1^{\nu\lambda} \equiv{}& \gamma_\mu[i(\hat{p}_2-\hat{k})-m]\gamma_\nu[i(\hat{f}_1-\hat{k})-m]\gamma_\lambda[i(\hat{p}_1-\hat{k})-m]\gamma_\mu\\ ={}& -2i\hat{k}\gamma_\lambda\hat{k}\gamma_\nu\hat{k}-2i\hat{p}_1\hat{k}\gamma_\nu\hat{k}\gamma_\lambda+2i\hat{k}\gamma_\lambda\hat{f}_1\gamma_\nu\hat{k}+4mk^2\delta_{\nu\lambda}-2i\gamma_\nu\hat{k}\gamma_\lambda\hat{k}\hat{p}_2\\ &+2\hat{p}_1\hat{k}\gamma_\nu(i\hat{f}_1-m)\gamma_\lambda+2\gamma_\nu(i\hat{f}_1-m)\gamma_\lambda\hat{k}\hat{p}_2\\ &-4(p_1p_2)\gamma_\nu[i(\hat{f}_1-\hat{k})-m]\gamma_\lambda,\end{aligned}$$

after which the evaluation of the traces presents no further difficulties. (Here and in subsequent formulas for $P^{(n)}$ we have replaced in accordance with the rules of subsection 25.5 the vector $\hat{e}_1$ by $\gamma_\lambda$ and $\hat{e}_2$ by $\gamma_\nu$.)

In order to be able to express the traces of the matrices in terms of the invariants $\varkappa_1$ and $\varkappa_2$ it is useful to employ the following relations:

$$\begin{aligned}&p_1^2 = p_2^2 = -1,\\ &k_1^2 = k_2^2 = 0,\\ &f_1^2 = \varkappa_1-1,\\ &f_2^2 = \varkappa_2-1,\\ &2p_1k_1 = 2p_2k_2 = 2f_1k_1 = 2f_1k_2 = \varkappa_1,\\ &2p_1k_2 = 2p_2k_1 = 2f_2k_1 = 2f_2k_2 = -\varkappa_2,\\ &2p_1f_1 = 2p_2f_1 = \varkappa_1-2,\end{aligned}$$

$$\begin{aligned} 2p_1f_2 &= 2p_2f_2 = \varkappa_2-2,\\ 2p_1p_2 &= \varkappa_1+\varkappa_2-2,\\ 2f_1f_2 &= -2 \end{aligned}$$

(in these relations we have set $m = 1$).

We finally obtain the expression for $P^{(1)}$

$$\begin{aligned} i\pi^2P^{(1)} = 4(J^{(1)}_{\sigma\tau}-J^{(0)}_{\sigma\tau})&\left(\frac{1}{\varkappa_1}p_{2\sigma}f_{1\tau}+\frac{1}{\varkappa_2}p_{2\sigma}f_{2\tau}\right)\\ &+(J^{(0)}_{\sigma}-J^{(1)}_{\sigma})\,p_{2\sigma}\left(2-\frac{4}{\varkappa_1}-\frac{4}{\varkappa_2}\right)+J^{(0)}_{\sigma}f_{1\sigma}\left(3+\frac{2}{\varkappa_1}+\frac{2\varkappa_2}{\varkappa_1}\right)\\ &+J^{(0)}_{\sigma}f_{2\sigma}\left(5-\frac{2}{\varkappa_1}+\frac{4\varkappa_1}{\varkappa_2}\right)+(J^{(1)}_{\sigma}p_{1\sigma}+J^{(2)}_{\sigma}p_{2\sigma})\left(1-\frac{2}{\varkappa_2}\right)\\ &-(J^{(1)}_{\sigma}+J^{(2)}_{\sigma})\left[f_{1\sigma}\left(1+\frac{4}{\varkappa_1}+\frac{4}{\varkappa_2}\right)+f_{2\sigma}\left(2+\frac{2\varkappa_1}{\varkappa_2}\right)\right]\\ &-J_{\sigma}(p_{1\sigma}+p_{2\sigma})\left(4\varkappa_1+3\varkappa_2-\frac{12}{\varkappa_2}-\frac{4}{\varkappa_1}-\frac{2\varkappa_2}{\varkappa_1}+\frac{2\varkappa_1^2}{\varkappa_2}\right)\\ &-J^{(3)}_{\sigma}f_{1\sigma}\left(\frac{2}{\varkappa_2}-\frac{\varkappa_2}{\varkappa_1}\right)-J^{(3)}_{\sigma}f_{2\sigma}\left(1-\frac{2}{\varkappa_1}\right)\\ &-J_{\sigma}f_{1\sigma}\left(6-4\varkappa_1-\varkappa_2-\frac{4}{\varkappa_1}+\frac{12}{\varkappa_2}+\frac{2\varkappa_1}{\varkappa_2}-\frac{2\varkappa_1^2}{\varkappa_2}+\frac{\varkappa_2^2}{\varkappa_1}\right)\\ &+J^{(0)}\left(5-2\varkappa_1-\varkappa_2-\frac{2}{\varkappa_1}-\frac{2}{\varkappa_2}+\frac{4\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right)\\ &\qquad+J(-2+\varkappa_1+\varkappa_2)\left(3+\varkappa_2-\frac{4}{\varkappa_1}-\frac{4}{\varkappa_2}+\frac{\varkappa_1}{\varkappa_2}\right). \end{aligned} \tag{52.9}$$

The integrals $2J_\sigma p_{1\sigma}$ and $2J_\sigma f_{1\sigma}$ appearing in this expression are related to the integrals $J^{(0)}$, $J^{(1)}$, $J^{(2)}$, $J^{(3)}$, $J$ by the expressions

$$\begin{aligned} 2J_\sigma p_{1\sigma} = 2J_\sigma p_{2\sigma} &= J^{(0)}-J^{(1)} = J^{(0)}-J^{(2)},\\ 2J_\sigma f_{1\sigma} &= J^{(0)}-J^{(3)}+\varkappa J. \end{aligned}$$

We now proceed to evaluate $P^{(2)}$:

$$\begin{aligned} P^{(2)} = \frac{1}{16\pi^2}\,\mathrm{Sp}\,\frac{1}{i}\Bigg\{\gamma_\nu\frac{i\hat{f}_1-m}{m^2\varkappa_1}\int\gamma_\mu\frac{i(\hat{f}_1-\hat{k})-m}{k^2-2kf_1+m^2\varkappa_1}\gamma_\lambda\frac{i(\hat{p}_1-\hat{k})-m}{k^2-2kp_1}\gamma_\mu\frac{d^4k}{k^2}&\\ \times(i\hat{p}_1-m)\left[\gamma_\lambda\frac{i\hat{f}_1-m}{m^2\varkappa_1}\gamma_\nu+\gamma_\nu\frac{i\hat{f}_2-m}{m^2\varkappa_2}\gamma_\lambda\right](i\hat{p}_2-m)\Bigg\}&. \end{aligned}$$

The value of $P^{(2)}$ will obviously be unaltered if we carry out the substitution $p_1 \rightleftarrows p_2$, $k_1 \rightleftarrows k_2$ ($\varkappa_1$ and $\varkappa_2$ remain unaltered) and rewrite all the matrices following the symbol Sp in the inverse order. If in addition we interchange the indices $\nu$ and $\lambda$, we obtain $P^{(2)} = P^{(3)}$.

On noting that the regularized value of $Y_3$ has the form

$$Y_3 = \gamma_\nu \frac{i\hat{f}_1 - m}{m^2 \varkappa_1} \frac{(2\pi)^4}{ie^2} \Lambda^{(3)}_{\lambda R}(p_1, p_1 + k_1; k_1),$$

where in accordance with (47.56)

$$\Lambda^{(3)}_{\lambda R}(p_1, p_1 + k_1; k_1) = \frac{e^2}{8\pi^2} (A\gamma_\lambda + iB\hat{k}_1 \gamma_\lambda + iCp_\lambda + D\hat{k}_1 p_\lambda),$$

$$A = -\ln\left(\frac{\lambda}{m}\right)^2 - 2 + \frac{\varkappa_1 - 2}{2(\varkappa_1 - 1)} \ln \varkappa_1 + \frac{1}{\varkappa_1}\{F(\varkappa_1 - 1) - F(-1)\},$$

$$B = \frac{\ln \varkappa_1}{\varkappa_1 - 1},$$

$$C = \frac{1}{1 - \varkappa_1} + \frac{3\varkappa_1 - 2}{(\varkappa_1 - 1)^2} \ln \varkappa_1,$$

$$D = \frac{1}{1 - \varkappa_1} + \frac{2}{\varkappa_1} - \frac{(2\varkappa_1 - 1)(\varkappa_1 - 2)}{\varkappa_1(\varkappa_1 - 1)} \ln \varkappa_1 - \frac{2}{\varkappa_1^2}\{F(\varkappa_1 - 1) - F(-1)\},$$

we obtain

$$\begin{aligned}
P^{(2)} + P^{(3)} &= 2P^{(2)} \\
&= -\frac{1}{4} \operatorname{Sp}\left\{\gamma_\nu \frac{i\hat{f}_1 - m}{m^2 \varkappa_1} (A\gamma_\lambda + iB\hat{k}_1 \gamma_\lambda + iCp_\lambda + D\hat{k}_1 p_\lambda)(i\hat{p}_1 - m)\right. \\
&\quad \times \left.\left(\gamma_\lambda \frac{i\hat{f}_1 - m}{m^2 \varkappa_1} \gamma_\nu + \gamma_\nu \frac{i\hat{f}_2 - m}{m^2 \varkappa_2} \gamma_\lambda\right)(i\hat{p}_2 - m)\right\} \\
&= \frac{1}{\varkappa_1}\left\{2A\left(3 + \varkappa_2 - \frac{4}{\varkappa_1} - \frac{4}{\varkappa_2} + \frac{\varkappa_1}{\varkappa_2}\right) - 2B(\varkappa_1 + \varkappa_2)\right. \\
&\quad + C\left(-8 + \varkappa_1 + \frac{8}{\varkappa_1} + \frac{8}{\varkappa_2} - \frac{4\varkappa_1}{\varkappa_2} + \frac{\varkappa_1^2}{\varkappa_2}\right) \\
&\quad \left. - D\left(4 - \varkappa_1 + \varkappa_2 - \varkappa_1^2 - \varkappa_1 \varkappa_2 + \frac{4\varkappa_1}{\varkappa_2}\right)\right\}. \qquad (52.10)
\end{aligned}$$

Finally, we obtain $P^{(4)}$. The regularized value of $Y_4$ has the form

$$Y_4 = \gamma_\nu \frac{i\hat{f}_1 - m}{m^2 \varkappa_1} \frac{(2\pi)^4}{e^2} \Sigma^{(2)}_R(f_1) \frac{i\hat{f}_1 - m}{m^2 \varkappa_1} \gamma_\lambda,$$

where $\Sigma_R^{(2)}(p)$ is defined by formula (47.24). We write $\Sigma_R^{(2)}(f_1)$ in the form

$$\frac{(2\pi)^4}{e^2}\Sigma_R^{(2)}(f_1) = \pi^2 i\{B_1(i\hat{f}_1+m)+B_2 m\},$$

where

$$B_1 = 2\left(1+\ln\left(\frac{\lambda}{m}\right)^2\right)+\frac{\varkappa_1}{\varkappa_1-1}+\frac{\varkappa_1(\varkappa_1-2)}{(\varkappa_1-1)^2}\ln\varkappa_1,$$

$$B_2 = -\frac{\varkappa_1}{\varkappa_1-1}+\frac{\varkappa_1(3\varkappa_1-2)}{(\varkappa_1-1)^2}\ln\varkappa_1.$$

On utilizing this notation we obtain in accordance with (52.6′) the expression for $P^{(4)}$

$$\begin{aligned}P^{(4)} &= \frac{1}{16\varkappa_1^2 m^4}\mathrm{Sp}\Big\{\gamma_\nu(i\hat{f}_1-m)(B_1(i\hat{f}_1+m)+B_2 m)(i\hat{f}_1-m)\gamma_\lambda\\ &\quad\times(i\hat{p}_1-m)\left(\gamma_\lambda\frac{i\hat{f}_1-m}{m^2\varkappa_1}\gamma_\nu+\gamma_\nu\frac{i\hat{f}_2-m}{m^2\varkappa_2}\gamma_\lambda\right)(i\hat{p}_2-m)\Big\}\\ &= \frac{1}{2\varkappa_1^2}B_2\left(8+\varkappa_2-\frac{8}{\varkappa_1}-\frac{8}{\varkappa}+\frac{4\varkappa_1}{\varkappa_2}-\frac{\varkappa_1^2}{\varkappa_2}\right)\\ &\quad+\frac{1}{2\varkappa_1}B_1\left(3+\varkappa_2-\frac{4}{\varkappa_1}-\frac{4}{\varkappa_2}+\frac{\varkappa_1}{\varkappa_2}\right).\end{aligned}\tag{52.11}$$

Finally, on adding $P^{(1)}$, $P^{(2)}$, $P^{(3)}$, and $P^{(4)}$ we obtain

$$\begin{aligned}P_1(\varkappa_1,\varkappa_2) &= \sum_{n=1}^{4}P^{(n)} = (1-2y\coth 2y)\cdot U_0\ln\lambda-2y\coth 2y\,[2h(y)\\ &\quad-h(2y)]\cdot U_0+h(y)\left(-4y\sinh 2y\frac{2-\coth 2y}{\varkappa_1\varkappa_2}+2y\coth y\right)\\ &\quad+\ln|\varkappa_1|\Big\{4y\coth 2y\Big[\frac{4\coth^2 y}{\varkappa_1\varkappa_2}+\frac{\varkappa_1-6}{2\varkappa_2}\frac{1}{\cosh 2y}+\frac{4}{\varkappa_1^2}\\ &\quad-\frac{1}{\varkappa_1}-\frac{\varkappa_2}{2\varkappa_1}-\frac{\varkappa_1}{\varkappa_2}-1\Big]+1+\frac{3}{\varkappa_2}+\frac{8}{\varkappa_1}+\frac{3\varkappa_2}{2\varkappa_1}-\frac{8}{\varkappa_1^2}\\ &\quad-\frac{7}{\varkappa_1\varkappa_2}+\frac{3\varkappa_2}{2\varkappa_1^2}+\frac{2\varkappa_1-\varkappa_2^2-\varkappa_1^2\varkappa_2}{2\varkappa_1^2\varkappa_2(\varkappa_1-1)}-\frac{2\varkappa_1^2+\varkappa_2}{2\varkappa_2(\varkappa_1-1)^2}\Big\}-\frac{4y^2}{\varkappa_1+\varkappa_2}\times\end{aligned}$$

$$\times\left[\frac{2}{\varkappa_1}-\frac{7}{4}\varkappa_1-\frac{3}{4}\frac{\varkappa_2^2}{\varkappa_1}\right]-4y\tanh y\left(\frac{1}{2}-\frac{1}{\varkappa_1}\right)+4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2$$
$$-\frac{12}{\varkappa_1}-\frac{3}{2}\frac{\varkappa_1}{\varkappa_2}-\frac{2\varkappa_1}{\varkappa_2^2}+\frac{1}{\varkappa_1-1}\left(\frac{\varkappa_1}{\varkappa_2}+\frac{1}{2}\right)+\frac{2}{\varkappa_1}[F(\varkappa_1-1)$$
$$-F(-1)]\left(-1+\frac{2}{\varkappa_1}-\frac{3}{\varkappa_2}+\varkappa_1+\frac{\varkappa_2}{2}+\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1^2}+\frac{\varkappa_1^2}{\varkappa_2}\right). \tag{52.12}$$

The photon "mass" $\lambda$ appears in the quantity $U_1$ defining the radiation corrections to the Compton effect. In order to eliminate it, it is necessary, as was done in the investigation of the radiation corrections to the scattering of an electron by an external field, to take into account in the Compton effect the emission of an additional soft photon. We refer to such a process as the double Compton effect.

In the laboratory system of coordinates ($p_1 = 0$) the cross section for the double Compton effect can be written in accordance with the general result of subsection 51.4 in the form

$$d\sigma_D = -\frac{\alpha}{\pi}\,d\sigma_0\,\frac{U_D}{U_0}, \tag{52.13}$$

where $d\sigma_0$ is the cross section for the basic Compton effect,

$$\frac{U_D}{U_0} = 2(1-2y\coth 2y)\left(\ln\frac{2\Delta\varepsilon}{\lambda}-\frac{1}{2}\right)+4y\coth 2y\,[h(2y)-1] \tag{52.14}$$

and $\Delta\varepsilon$ is the maximum photon energy.

The total cross section for the scattering of a photon by an electron taking into account both the radiation corrections and the emission of an additional soft photon is given by

$$d\sigma = d\sigma_0\left\{1-\frac{\alpha}{\pi}\,\frac{U_1+U_D}{U_0}\right\}. \tag{52.15}$$

As can be easily seen, this quantity does not contain the photon "mass."

### 52.2. *Limiting Cases of Low and High Energies*

The foregoing general formulas can be considerably simplified in the limiting cases of low and high photon energies.

In the region of low energies, when $\omega_1 \approx \omega_2 = \omega \ll m$ the scattering cross section has the form

$$d\sigma = \frac{r_0^2}{2}\,do(1-2\omega+2\omega^2\ldots)\left(U_0-\frac{\alpha}{\pi}(U_1+U_D)\right), \tag{52.16}$$

where

$$
\begin{aligned}
U_0 &= 1+\cos^2\vartheta,\\
U_1 &= -\frac{4}{3}\omega^2(1-\cos\vartheta)U_0\ln\frac{\lambda}{m}\\
&\quad+\left(1+\cos\vartheta+\cos^2\vartheta-\frac{1}{3}\cos^3\vartheta\right)4\omega^2\ln\frac{\omega}{m},\\
U_D &= -\frac{4}{3}\omega^2(1-\cos\vartheta)U_0\ln\frac{2\Delta\varepsilon}{\lambda}
\end{aligned}
\tag{52.16'}
$$

and $\vartheta$ is the scattering angle.

In the region of high energies we consider three cases depending on the value of the parameter $(\varkappa_1+\varkappa_2)/\varkappa_2$ (in the laboratory system of coordinates this parameter is equal to $(\varkappa_1+\varkappa_2)/\varkappa_2=\frac{1}{2}\varkappa_1(1-\cos\vartheta)$):

$$
1)\left|\frac{\varkappa_1+\varkappa_2}{\varkappa_2}\right|\ll 1,\qquad 2)\left|\frac{\varkappa_1+\varkappa_2}{\varkappa_2}\right|\sim 1,\qquad 3)\left|\frac{\varkappa_1+\varkappa_2}{\varkappa_2}\right|\gg 1.
$$

In the first case $\varkappa_1\approx\varkappa_2\gg 1$; here in the laboratory system we have $\omega_2\approx\omega_1\gg m$, $1-\cos\vartheta\ll m/\omega_1$. In the second case $\varkappa_2\gg 1$, $|\varkappa_1+\varkappa|\gg 1$, and in the laboratory system $\omega_2\sim\frac{1}{2}\omega_1\gg m$, $1-\cos\vartheta\sim m/\omega_1$. In the third case $|\varkappa_1+\varkappa_2|\gg 1$, $|\varkappa_1|\gg\varkappa_2$, and in the laboratory system $\omega_2\sim m$, $1-\cos\vartheta\gg m/\omega_1$.

In the first case

$$
U_0=2,
$$

$$
\begin{aligned}
U_1 &= 4(1-2y\coth 2y)\ln\frac{\lambda}{m}-8y\coth 2y[2h(y)-h(2y)]+4yh(y)\coth y\\
&\quad+\ln|\varkappa_1|(4y\tanh y-1)-2y^2-4y\tanh y+3-\ln^2|\varkappa_1|-\frac{\pi^2}{6}.
\end{aligned}
\tag{52.17}
$$

In the second case

$$
U_0=-\left(\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right),
$$

$$
\begin{aligned}
U_1 &= U_0\left\{(1-2y)\left(\frac{3}{2}+2\ln\frac{\lambda}{m}\right)+2y^2-\frac{\pi^2}{6}\right\}+\left(1+\frac{\varkappa_2}{2\varkappa_1}+\frac{\varkappa_1}{\varkappa_2}\right)\\
&\quad\times\left\{\left[\ln\left(1+\frac{\varkappa_2}{\varkappa_1}\right)\right]^2-\ln\left(1+\frac{\varkappa_1}{\varkappa_2}\right)+2\ln\frac{\varkappa_2}{|\varkappa_1|}\right\}\\
&\quad+\left(1+\frac{\varkappa_2}{\varkappa_1}+\frac{\varkappa_1}{2\varkappa_2}\right)\left\{\left[\ln\left|1+\frac{\varkappa_1}{\varkappa_2}\right|\right]^2-\ln\left(1+\frac{\varkappa_2}{\varkappa_1}\right)-\ln\left|\frac{\varkappa_2}{\varkappa_1}\right|+\pi^2\right\}.
\end{aligned}
\tag{52.18}
$$

Finally, in the third case

$$U_0 = -\frac{\varkappa_1}{\varkappa_2},$$

$$U_1 = U_0\left\{2(1-2y)\ln\frac{\lambda}{m} + \ln\varkappa_2\left[2y - \frac{3}{2}\,\frac{\varkappa_1+1}{\varkappa_2} + \frac{1}{2\varkappa_2(\varkappa_2-1)}\right]\right.$$
$$\left. -\frac{\pi^2}{3} - \frac{2}{\varkappa_2}\left[F(\varkappa_2-1)-F(-1)\right]\left(\frac{1}{\varkappa_2}+\frac{\varkappa_2}{2}\right) + \frac{3}{2} + \frac{2}{\varkappa_2}\right\}. \qquad (52.19)$$

The quantity $U_D$ is determined in the second and in the third cases by the formula

$$U_D = U_0\left\{2(1-2y)\left[\ln\frac{2\Delta\varepsilon}{\lambda} - \frac{1}{2}\right] + 4y\left(y + \frac{\pi^2}{24y} - 1\right)\right\}, \qquad (52.20)$$

while in the first case the general formula (52.14) should be used.

The ratio $-(\alpha/\pi)\,(U_1/U_0)$ attains its maximum value at $\vartheta = 0$; this maximum value is equal to 0.04 for $\omega = 50$ MeV and to 0.09 for $\omega = 1000$ MeV.

In the high energy region due to the existence of terms containing $\ln^2(\omega/m)$ the correction to the cross section for the scattering of a photon by an electron becomes comparable with the basic cross section. Therefore, formula (52.15) ceases to have meaning, and, as in the case of the investigation of corrections to the scattering of an electron by an external-field, it becomes necessary to take into account higher order approximations of perturbation theory. Such a procedure leads to the multiplication of the scattering cross section given by the Klein-Nishina formula by an appropriate exponential factor. In this case doubly logarithmic terms come from diagrams of type $Y_1$ and $Y_1'$ (cf. Fig. 111) with additional virtual photon lines (second of references (1)).

It should be noted that in the high energy region the multiple Compton effect involving the emission of a large number of secondary photons becomes important. In the center of the inertia system of the electron and the photon these processes have the character of the ordinary Compton effect accompanied by emission of additional photons of energy small compared to the energy of the principal particles. Therefore, such additional photons have little effect on the momenta of the principal particles. However, in the laboratory system these "additional" photons can have energies exceeding the energy of the "principal" secondary photon.

The following expression for the energy distribution of the secondary electrons can be taken as characteristic for such processes:

$$dw(\varepsilon) = \frac{1}{\ln\frac{\omega_1}{m}}\frac{d\varepsilon/\varepsilon}{\frac{\alpha}{\pi}\ln^2\frac{\omega_1}{\varepsilon}}\left[\left(\frac{2\alpha}{\pi}\ln^2\frac{\omega_1}{\varepsilon}+1\right)\exp\left(-\frac{\alpha}{\pi}\ln^2\frac{\omega_1}{\varepsilon}\right)\right.$$

$$\left.+\left(\frac{\alpha}{\pi}\ln\frac{\omega_1}{\varepsilon}\ln\frac{\omega_1}{m}+1\right)\exp\left(-\frac{\alpha}{\pi}\ln\frac{\omega_1}{\varepsilon}\ln\frac{\omega_1}{m}\right)\right].$$

Up to terms of order $\alpha$ the total cross section for all such multiple processes is given by the ordinary zero order Klein-Nishina formula.

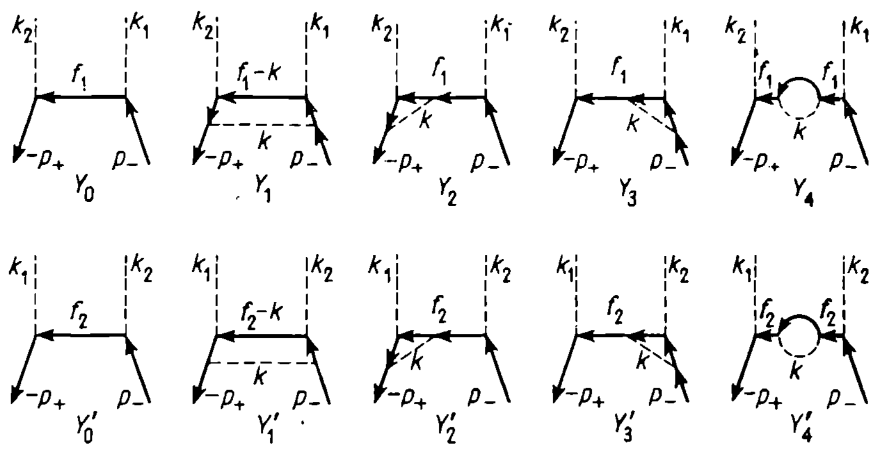


Fig. 111.

For the differential cross section this assertion does not hold for very small scattering angles. (We note that the additional photons are in this case very soft compared to the primary secondary photon). The total scattering cross section for small angles of deflection of the original photon is given by the following formulas (second of references (1)):

$$d\sigma = d\sigma_0 \exp\left(\frac{\alpha}{2\pi}\ln^2\frac{m}{\omega_1\theta^2}\right) \quad \text{for} \quad \sqrt{\frac{m}{\omega_1}} \gg \theta \gg \frac{m}{\omega_1},$$

$$d\sigma = d\sigma_0 \exp\left(\frac{\alpha}{2\pi}\ln^2\frac{\omega_1}{m}\right) \quad \text{for} \quad \theta \ll \frac{m}{\omega_1}.$$

However, at large energies these angles make a negligible contribution to the total scattering cross section.

A similar situation arises in the case of the scattering of a positron by an electron for angles of scattering close to 180° in the center of mass system (third of references (1)). The total cross section for all the processes accompanied by the emission of any number of additional photons has in this region the form

$$d\sigma = d\sigma_0 \exp \frac{2\alpha}{\pi} \ln^2 \frac{2}{\pi-\theta} \qquad \text{for} \qquad \frac{m}{\varepsilon} \ll \pi-\theta \ll \frac{\pi}{2},$$

$$d\sigma = d\sigma_0 \exp \frac{2\alpha}{\pi} \ln^2 \frac{\varepsilon}{m} \qquad \text{for} \qquad \pi-\theta \ll \frac{m}{\varepsilon},$$

where $\varepsilon$ is the energy of the positron or the electron in the center of mass system. For high energies this region makes a small contribution to the total cross section. Therefore, the sum of the total cross sections for processes accompanied by the emission of additional photons is expressed by the zero order approximation formula (up to terms of order $\alpha$).

This also applies to the differential cross section for the scattering of a positron by an electron for angles of scattering which are not too close to 180°, and to the cross section for electron-electron scattering for all values of the scattering angle.

### 52.3. *Radiation Corrections to Two-Photon Pair Annihilation*

If we know the radiation corrections to the Compton effect we can easily obtain the radiation corrections to the process of two-photon pair annihilation (84). Indeed, diagrams representing two-photon annihilation including radiation corrections (Fig. 111) do not differ topologically from diagrams representing the Compton effect with corrections (Fig. 110). The difference consists merely of the fact that instead of the two electron states in the Compton effect, one electron and one positron state participate in the process of two-photon annihilation and that, moreover, in the Compton effect one photon is absorbed and another is emitted, while in the process of pair annihilation both photons are emitted. Therefore, if in the formula which gives the cross section for the scattering of a photon by an electron including radiation corrections we change the sign of the four-momenta of the incident photon and of the scattered electron, we shall obtain the cross section for two-photon pair annihilation taking radiation corrections into account.

Thus, we must carry out in formula (52.15) the substitution $p_1 \to p_-$, $k_1 \to -k_1$, $p_2 \to -p_+$, and $k_1 \to k_2$, where $p_-$ and $p_+$ are the momenta of the electron and the positron, and $k_1$ and $k_2$ are the momenta of the two photons, and at the same time we must replace the invariants $\varkappa_1, \varkappa_2$ by $\varkappa_1 = -2p_-k_1 = -2p_+k_2$, and $\varkappa_2 = -2p_+k_1 = -2p_-k_2$. As a result of this we obtain the following formula for the cross section for two-photon pair annihilation including radiation corrections:

$$
\begin{aligned}
d\sigma = d\sigma_0 \Bigg\{ 1 - \frac{\alpha}{\pi} \Bigg\{ 2(1-2x\coth 2x)\ln\frac{\lambda}{m} - 4x\coth 2x \Bigg[ h(2x) - 2h(x) \\
+ \frac{\pi^2}{4x} \Bigg] - G(\varkappa_1, \varkappa_2) - G(\varkappa_2, \varkappa_1) \Bigg\} \Bigg\}, \qquad (52.21)
\end{aligned}
$$

where $d\sigma_0$ is the cross section for the basic process of two-photon pair annihilation and

$$
\begin{aligned}
U_0 G(\varkappa_1, \varkappa_2) = {} & [h(2x) - h(x)] \left[ \frac{4}{\varkappa_1\varkappa_2} x \sinh 2x (1 + 2\cosh^2 x) + 2x \tanh x \right] \\
& + \ln|\varkappa_1| \Bigg\{ 4x \coth 2x \left[ \frac{4}{\varkappa_1^2} - \frac{1}{\varkappa_1} - \frac{\varkappa_2}{2\varkappa_1} - \frac{\varkappa_1}{\varkappa_2} - 1 - \frac{4}{\varkappa_1\varkappa_2} \sinh^2 x \right] \\
& - \frac{2x}{\sinh 2x} \left( \frac{\varkappa_1 - 6}{\varkappa_2} \right) + \frac{3\varkappa_2}{2\varkappa_1^2}(1+\varkappa_1) + \frac{3}{\varkappa_2} + 1 - \frac{7}{\varkappa_1\varkappa_2} + \frac{8}{\varkappa_1} - \frac{8}{\varkappa_1^2} \\
& - \frac{\varkappa_2^2 - 2\varkappa_1 + \varkappa_1^2\varkappa_2}{2\varkappa_1^2\varkappa_2(\varkappa_1 - 1)} - \frac{2\varkappa_1^2 + \varkappa_2}{2\varkappa_2(\varkappa_1 - 1)^2} \Bigg\} - 4x \coth x \left( \frac{1}{2} - \frac{1}{\varkappa_1} \right) \\
& + \frac{\dfrac{\pi^2}{4} - x^2}{\cosh^2 x} \left( \frac{2}{\varkappa_1} - \frac{7\varkappa_1}{4} - \frac{3\varkappa_1^2}{4\varkappa_2} \right) + 4\left( \frac{1}{\varkappa_1} + \frac{1}{\varkappa_2} \right)^2 - \frac{12}{\varkappa_1} - \frac{3\varkappa_1}{2\varkappa_2} \\
& - \frac{2\varkappa_1}{\varkappa_2^2} + \frac{1}{\varkappa_1 - 1} \left( \frac{\varkappa_1}{\varkappa_2} + \frac{1}{2} \right) + \frac{2}{\varkappa_1} [F(\varkappa_1 - 1) - F(-1)] \\
& \times \left( \frac{\varkappa_1^2}{\varkappa_2} + \frac{\varkappa_2}{\varkappa_1^2} + \frac{\varkappa_1}{\varkappa_2} + \varkappa_1 + \frac{\varkappa_2}{2} + \frac{2}{\varkappa_1} - \frac{3}{\varkappa_2} - 1 \right),
\end{aligned}
$$

$$U_0 = \left(\frac{\varkappa_1}{\varkappa_2}+\frac{\varkappa_2}{\varkappa_1}\right)+4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)-4\left(\frac{1}{\varkappa_1}+\frac{1}{\varkappa_2}\right)^2,$$

$$4\cosh^2 x = \varkappa_1+\varkappa_2 .$$

This expression contains the photon "mass". In order to eliminate it we must add to the cross section $d\sigma$ the cross section for three-photon pair annihilation, in which in addition to the photons $k_1$ and $k_2$ a soft photon $k_3$ is emitted whose energy does not exceed $\Delta\varepsilon$.

The cross section for this process is equal to

$$d\sigma_3 = d\sigma_0\frac{\alpha}{\pi}\left\{2(2x\coth 2x-1)\left[\ln\frac{2\Delta\varepsilon}{\lambda}-\frac{1}{2}\right] +4x\coth 2x[1-h(2x)]\right\}. \qquad (52.22)$$

We state the formulas for the total cross section in the limiting cases of low and high energies.

In the nonrelativistic case we have

$$d\sigma = d\sigma_0\left\{1+\frac{\pi\alpha}{v}-\frac{\alpha}{\pi}\left(5-\frac{\pi^2}{4}\right)\right\}, \qquad \frac{\pi\alpha}{v}\ll 1, \qquad (52.23)$$

where $v$ is the positron velocity, $v \ll 1$; the electron velocity is equal to zero. (The divergence of this expression for $v \to 0$ is associated with the inapplicability of the Born approximation.)

In the relativistic case we have

$$d\sigma = d\sigma_0\left\{1-\frac{\alpha}{\pi}\left[2(1-2x\coth 2x)\ln\frac{2\Delta\varepsilon}{m}+X\right]\right\}, \qquad (52.24)$$

where $X$ has the following values. If the angle between directions of motion of the two photons is 180° and $\varkappa_1 \gg \varkappa_2 \approx 1$, then

$$X = (\ln\varkappa_1)^2-\ln\varkappa_1+3+\frac{\pi^2}{3}.$$

If the angle between the directions of motion of the two photons is 90° and $\varkappa_1 = \varkappa_2 \gg 1$, then we have

$$X = \frac{3}{2}(\ln\varkappa_1)^2-\frac{5}{2}\ln 2\varkappa_1+5.97+\frac{1}{2\varkappa_1}[18.5(\ln 2\varkappa_1)^2 -67.25\ln 2\varkappa_1+41.2].$$

### 52.4. *Radiation Corrections to Bremsstrahlung*

We now investigate the radiation corrections to bremsstrahlung from an electron in the Coulomb field of a nucleus (67).

The diagrams representing the basic process of emission and the radiation corrections to it are shown in Fig. 112.

Since the matrix elements corresponding to the diagrams $Y_i$, $Y_i'$ contain an infrared divergence, we must in addition to the radiation corrections also take into account double bremsstrahlung, in which a soft photon ($\boldsymbol{k}'$, $\omega'$) is emitted together with the photon ($\boldsymbol{k}$, $\omega$).

We do not reproduce here the calculation of the cross section taking these effects into account, but confine ourselves merely to stating the final results for several of the more interesting limiting cases.

We write the radiation corrections to the bremsstrahlung cross section and the cross section for double bremsstrahlung in the form

$$d\sigma_R = -\frac{\alpha}{\pi}\,\delta_R\,d\sigma_0, \qquad d\sigma_D = -\frac{\alpha}{\pi}\,\delta_D\,d\sigma_0, \tag{52.25}$$

where $d\sigma_0$ is the cross section for the basic process.Then in the limiting case $\omega\varepsilon_1 \ll m^2$ (we use the same notation as in § 51) we have

$$\begin{aligned}\delta_R = {} & 2(1-x\coth 2x)\left(\ln\frac{\lambda}{m}+1\right)+x\tanh x\\ & +4x\coth 2x\,[h(2x)-h(x)]+\frac{2x}{\sinh 2x}\,\frac{v^2\sin^2\frac{\theta}{2}}{1-v^2\sin^2\frac{\theta}{2}}\\ & +2(1-x\coth x)\left(1-\frac{1}{3}\coth^2 x\right)-\frac{2}{9};\end{aligned} \tag{52.26}$$

$$\delta_D = 2(1-2\coth 2x)\ln\frac{2\Delta\varepsilon}{\lambda}+\frac{1}{v}\ln\frac{1-v}{1+v}+\frac{1-v^2}{v\sin\frac{\theta}{2}}\cosh 2x\,G(v,\theta),$$

where

$$G(v,\theta)=\int_{\cos\frac{\theta}{2}}^{1}\frac{du}{(1-v^2u^2)\sqrt{u^2-\cos^2\frac{\theta}{2}}}\ln\frac{1+vu}{1-vu}$$

and $x$ is related to $\varrho = q^2 = (-2+\varkappa_1+\varkappa_2)m^2+2(\varepsilon_1\varepsilon_2-p_1p_2\cos\theta)$ by the expression $\varrho = 4m^2\sinh^2 x$ ($\Delta\varepsilon$ is the maximum energy of the soft photon).

As should have been expected, this expression agrees with the radiation correction to the cross section for elastic scattering (cf. formula (51.18)).

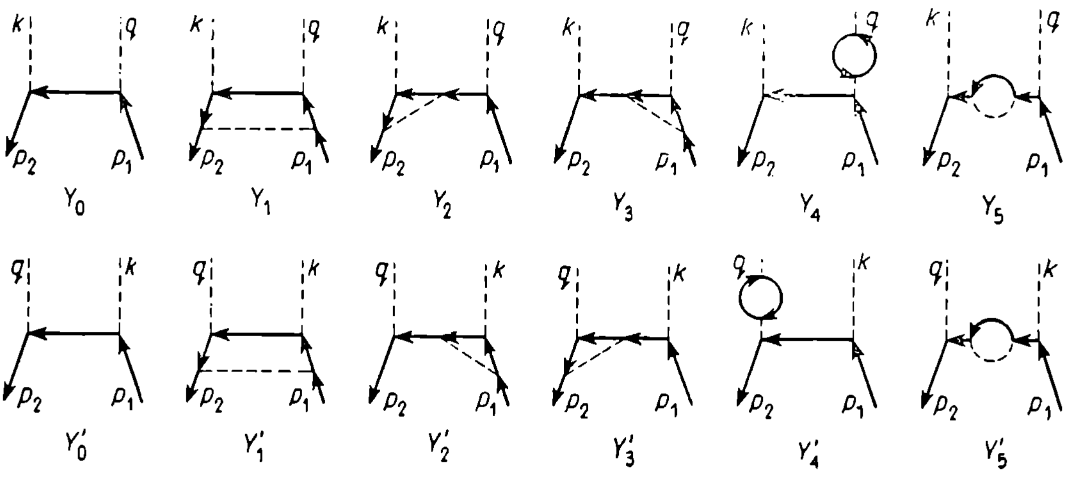


Fig. 112.

Further, we consider the relativistic case when $\varepsilon_1 \gg m$, $\varepsilon_2 \gg m$, but the energy loss is small, i.e., $\omega \ll \varepsilon_1, \varepsilon_2$. If, in addition, the scattering angles are small, $\theta, \theta_1, \theta_2 \sim (m/\varepsilon) \ll 1$, then

$$\delta_R = 2(1-2x\coth 2x)\left(\ln\frac{\lambda}{m}+1\right)+x\tanh x$$
$$+4x\coth 2x[h(2x)-h(x)]+2(1-x\coth x)\left(1-\frac{1}{3}\coth^2 x\right)-\frac{2}{9}; \tag{52.27}$$

$$\delta_D = 2(1-2x\coth 2x)\ln\frac{2\Delta\varepsilon}{\lambda}+2\ln\frac{2\varepsilon}{m}(2x\coth 2x-1)$$
$$-4x\coth 2x\,[h(2x)-h(x)].$$

In the extreme relativistic case when not only the conditions $\varepsilon_1, \varepsilon_2, \omega \gg m$ are satisfied, but also the condition $\ln|\varkappa_1|$, $\ln\varkappa_2$, $\ln\varrho$,

$\ln(\varrho-\varkappa_1-\varkappa_2) \gg 1$ holds, but $\ln(\varrho/\varkappa_1), \ln(\varrho/\varkappa_2), \ln[(\varrho-\varkappa_1-\varkappa_2)/|\varkappa_1|]$, $\ln[(\varrho-\varkappa_1-\varkappa_2)/|\varkappa_2|] \sim 1$, the quantities $\delta_R$ and $\delta_D$ are equal to

$$\begin{aligned}\delta_R &= 2(1-2y)\ln\frac{\lambda}{m}+2y^2-3y-\frac{4}{3}x,\\ \delta_D &= 2(1-2y)\ln\frac{2\Delta\varepsilon}{\lambda}+2y^2-2y\ln\frac{(\varrho-\varkappa_1-\varkappa_2)m^2}{4\varepsilon_1\varepsilon_2}-\ln\frac{\varepsilon_1\varepsilon_2}{m^2},\end{aligned} \tag{52.28}$$

where

$$x=\frac{1}{2}\ln\varrho, \qquad y=\frac{1}{2}\ln(\varrho-\varkappa_1-\varkappa_2).$$

Finally, in the nonrelativistic case we have

$$\begin{aligned}\delta_R &= \frac{2}{3}\left[-(p_1-p_2)^2\left(\ln\frac{\lambda}{m}+\frac{1}{5}\right)+2(p_1-p_2)k\ln\frac{\omega}{m}\right],\\ \delta_D &= \frac{2}{3}(p_1-p_2)^2\left(\frac{5}{6}-\ln\frac{2\Delta\varepsilon}{\lambda}\right).\end{aligned} \tag{52.29}$$

For $p_1 \to 0$ the corrections to the cross section vanish.

The variation of the cross section in the energy range for which $\alpha y^2/\pi > 1$ (1) is similar to that discussed earlier. Moreover, in this energy range multiple bremsstrahlung begins to play a role. However, the energy of one of the bremstrahlen is always considerably larger than the total energy of all the other photons. Because of this the energy lost by the electron in the form of radiation is expressed by the usual zero order approximation formula.

The number distribution of emitted photons is given by the Poisson formula. The total cross section for scattering accompanied by emission of any number of photons of arbitrary energy agrees up to terms of order $\alpha$ with the zero order approximation to the cross section for scattering unaccompanied by bremsstrahlung.

### 52.5. *Radiation Corrections to Photoproduction and Single Photon Annihilation of Pairs*

Radiation corrections to the photoproduction of pairs in the Coulomb field of a nucleus can be obtained from the radiation corrections to bremsstrahlung by means of the substitution $p_1 \to p_+$, $p_2 \to p_-$, and $k \to -k$. We must keep in mind here that additional poles appear in the integrals over the momenta of the virtual photon.

As usual, we must add to the cross section for photoproduction the cross section for photoproduction accompanied by the emission of a soft photon.

The total cross section for the photoproduction of a pair in the field of a nucleus taking radiation corrections of order $\alpha$ into account can be conveniently written in the form

$$d\sigma = d\sigma_0\left(1-\frac{\alpha}{\pi}\delta\right) \tag{52.30}$$

where $d\sigma_0$ is the cross section for the basic effect and $\delta$ is a quantity which determines the corrections. We give here the expression for $\delta$ in several limiting cases (81), (26).

Near the photoproduction threshold, when $|p_+|$, $|p_-| \ll m$, $\delta$ is given by the formula

$$\delta = -\frac{\pi^2}{|\boldsymbol{v}_+-\boldsymbol{v}_-|}+2.4-\frac{\cos\theta-\cos\theta_+\cos\theta_-}{\dfrac{p_+}{p_-}\sin^2\theta_+ + \dfrac{p_-}{p_+}\sin^2\theta_-} \tag{52.31}$$

where $\theta_+$ and $\theta_-$ are the angles between $\boldsymbol{k}$ and $\boldsymbol{p}_+, \boldsymbol{p}_-$; $\theta$ is the angle between $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, and $\boldsymbol{v}_\pm$ are the velocities of the positron and the electron. Here the first term takes into account the interaction between the created electron and positron in the first Born approximation. Therefore, the applicability of formula (52.31) is restricted by the condition $\alpha/|\boldsymbol{v}_+-\boldsymbol{v}_-| \ll 1$.

In the relativistic case of equal energies ($\varepsilon_+ \approx \varepsilon_- \gg m$) and of small angles ($\theta_\pm$, $\theta \ll m/\varepsilon$) $\delta$ is determined by the formula

$$\delta = -\frac{\pi^2}{\sqrt{-2-2\dfrac{p_+p_-}{m^2}}}+\frac{13}{2}-\frac{\pi^2}{4}. \tag{52.32}$$

Here the first term is analogous to the first term in formula (52.31), since the small quantity $\sqrt{(-2-2p_+p_-/m^2}$ represents the relative velocity of the electron and the positron in the center of mass system for the pair.

Finally, in the ultrarelativistic case when $\ln(kp_+/m^2) \gg 1$, $\ln(p_-p_+/m^2) \gg 1$, $\ln(q^2/m^2) \gg 1$, but $\ln(q^2/kp_\pm) \sim 1$, $\ln(p_-p_+/kp_\pm) \sim 1$, $\delta$ has the form

$$\delta = \left(\ln\frac{2p_+p_-}{m^2}-1\right)\ln\frac{\varepsilon_+\varepsilon_-}{(\Delta\varepsilon)^2}-\frac{13}{6}\ln\frac{2p_+p_-}{m^2} \tag{52.33}$$

The one-photon annihilation of a pair in the Coulomb field of a nucleus is the process inverse with respect to the photoproduction of a pair. Therefore, it can be easily shown that the radiation corrections to the one-photon annihilation are determined by the same formulas as the corrections to photoproduction, if we interpret the values of $p_+$, $p_-$ and $k$ appearing in them as the momenta of the particles being annihilated and the momentum of the emitted photon.

## § 53. Radiation Corrections to Atomic Levels

### 53.1. *Radiation Shift of Atomic Levels*

So far we have been considering radiation corrections to various scattering processes. But the interaction of the particles with the vacuum also affects their stationary states. This effect manifests itself, firstly, in the radiation shift of levels and, secondly, in giving rise to level widths.

We begin by discussing the radiation shift of an atomic level. In order to do this we utilize the effective potential energy of the electron introduced in § 50.

If we neglect the magnetic interaction between the electrons, then in the absence of an external magnetic field the effective potential energy of the electron has the form

$$e\delta\varphi = a\frac{e}{4\pi}\left\{\frac{4}{3m^2}\left(\ln\frac{m}{\lambda}-\frac{3}{8}-\frac{1}{5}\right)\Box\varphi+\frac{1}{m}i\beta\boldsymbol{\alpha}\boldsymbol{E}\right\}, \qquad (53.1)$$

where $\varphi$ is the potential of the self-consistent electric field in the atom and $\boldsymbol{E}$ is the intensity of this field. If the electron is in the state $\psi_n$ of energy $E_n$, then the radiation shift of the level will be given by the expectation value of $e\delta\varphi$ in the state $\psi_n$:

$$\begin{aligned}\delta E_n' &= (e\delta\varphi)_{nn} = (\psi_n, e\delta\varphi\psi_n)\\ &= \frac{1}{137}\frac{e}{4\pi}\left\{\frac{4}{3m^2}\left(\ln\frac{m}{\lambda}-\frac{3}{8}-\frac{1}{5}\right)\langle\Delta\varphi\rangle+\frac{i}{m}\langle\beta\boldsymbol{\alpha}\boldsymbol{E}\rangle\right\},\end{aligned} \qquad (53.2)$$

where $\langle L\rangle$ is the expectation value of $L$ in the state $\psi_n$.

This expression contains the photon "mass" $\lambda$, and, as has been explained in § 30, this indicates that the expression does not take into account the interaction between the electron and the long wavelength photons. In other words, $\delta E_n'$ determines only that part of the level shift which is due to the interaction between the electron and the short wavelength photons. We must, therefore, separately determine the level shift due to the interaction between the electron and the long wavelength photons, and add it to the shift determined by formula (53.2) (19).

The process in which we are interested can be described in the following manner: an electron in the state $\psi_n$ of energy $E_n$ emits a virtual photon of energy $\omega$ and makes a transition to the state $\psi_{n'}$, it then absorbs the photon $\omega$ and returns to the initial state $\psi_n$. The change in the electron energy associated with this process is given, as is well known, in the second perturbation theory approximation to which we here confine ourselves, by

$$\Delta E_n = \sum_{n';\,\omega,\,e} \frac{V_{nn'}V_{n'n}}{E_n - E_{n'} - \omega}. \tag{53.3}$$

where $V_{nn'}$ is the matrix element of the energy of interaction between the electron and the photon, and the summation is carried out over all the states of the atomic electron $n'$ and of the emitted photon (summation over the photon states means integration over $4\pi\omega^2 d\omega/(2\pi)^3$, and summation over the photon polarizations $e$).

Since we are interested in the interaction between an atomic electron and long wavelength photons, we can utilize the nonrelativistic approximation in which the matrix element of the energy of interaction between the electron and the photon is given by the formula

$$V_{nn'} = -\frac{e}{\sqrt{2\omega}}(\boldsymbol{e}\boldsymbol{v})_{nn'},$$

where $\boldsymbol{v} = (1/im)\nabla$ is the operator for the electron velocity. On substituting this expression into (53.3), and on noting that

$$\sum_{\boldsymbol{e}} |(\boldsymbol{e}\boldsymbol{v})_{nn'}|^2 = \tfrac{2}{3}|\boldsymbol{v}_{nn'}|^2,$$

we rewrite (53.3) in the form

$$\Delta E_n = \frac{2}{3}\frac{e^2}{4\pi^2}\int_0^K \omega\, d\omega \sum_{n'} \frac{|\boldsymbol{v}_{nn'}|^2}{E_n - E_n' - \omega}, \tag{53.4}$$

with the integration over $\omega$ being taken here between the limits from $\omega = 0$ to some large value $K$.

We must renormalize the mass in this expression. For a free electron $\Delta E_n$ is equal to its electromagnetic mass $\delta m$ (or, more accurately, to that part of $\delta m$ which is due to the interaction of the electron with photons of energy less than $K$). But we have already included $\delta m$ in the electron mass $m$, and we interpret the energy of the level $n$ to be the total energy with $m$ subtracted from it. We must, therefore, subtract from (53.4) the corresponding expression for the free electron.

Since for a free electron only the diagonal elements of the velocity differ from zero, formula (53.4) in this case assumes the form

$$\Delta E = -\frac{2}{3}\frac{e^2}{4\pi^2}\int_0^K \omega\, d\omega \frac{\boldsymbol{v}^2}{\omega}.$$

On subtracting this expression from (53.4) and on noting that

$$\sum_{n'} |\boldsymbol{v}_{nn'}|^2 = (\boldsymbol{v}^2)_{nn},$$

we obtain the following expression for the radiation shift of the level $E_n$, due to the interaction of the electron with photons whose energy does not exceed $K$:

$$\delta E_n'' = \Delta E_n - \Delta E = \frac{2}{3}\frac{e^2}{4\pi^2}\int_0^K d\omega \sum_{n'} \frac{|\boldsymbol{v}_{nn'}|^2}{E_n - E_{n'} - \omega}(E_n - E_{n'}). \tag{53.5}$$

We now transform this formula. We first carry out the integration over $\omega$. On assuming that $K$ is considerably greater than all the differences between atomic energy levels $E_n - E_n'$, we obtain

$$\delta E_n'' = \frac{2}{3}\frac{e^2}{4\pi^2}\sum_m |\boldsymbol{v}_{nm}|^2 \ln \frac{K}{|E_m - E|}(E_m - E_n). \tag{53.6}$$

Further, on introducing the quantity $\varepsilon_0$ defined by the formula

$$\ln \varepsilon_0 = \frac{\sum_m |\boldsymbol{v}_{nm}|^2 (E_m - E_n) \ln |E_m - E_n|}{\sum_m |\boldsymbol{v}_{nm}|^2 (E_m - E_n)}, \tag{53.6'}$$

we rewrite $\delta E_n''$ in the form

$$\delta E_n'' = \frac{2}{3}\frac{e^2}{4\pi^2}\sum_m |\boldsymbol{v}_{nm}|^2(E_m-E_n)\ln\frac{K}{\varepsilon_0}. \tag{53.7}$$

The sum appearing in this expression, as is well known (14), is equal to

$$\sum_m |\boldsymbol{v}_{nm}|^2(E_m-E_n) = -\frac{1}{m^2}\int \psi_n^* \nabla V(\boldsymbol{r})\nabla\psi_n\,dv$$
$$= \frac{1}{2m^2}\int \varDelta V(\boldsymbol{r})|\psi_n(\boldsymbol{r})|^2\,dv,$$

where $V = e\varphi$. Therefore, we have finally

$$\varDelta E_n'' = \frac{e^2}{12\pi^2 m^2}\, e(\psi_n, \varDelta\varphi\psi_n)\ln\frac{K}{\varepsilon_0}. \tag{53.8}$$

So far we have made no assumptions about the magnitude of $K$. We now choose $K$ in such a way that the minimum energy of those photons, the interaction with which is still taken into account by formula (53.2), will be equal to $K$. But in accordance with (30.24) $\ln\lambda = \ln 2K - \frac{5}{6}$ and the sum of $\delta E_n'$ and $\delta E_n''$, which determines the total radiation shift of the level $E_n$, will contain neither $\lambda$ nor $K$.

In the usual units this shift is given by the formula (109)

$$\delta E_n = \delta E_n' + \delta E_n'' = \frac{e^2}{(4\pi)^2}\, e\left\{\frac{4}{3m^2}\left(\ln\frac{m}{2\varepsilon_0} - \frac{3}{8} + \frac{5}{6} - \frac{1}{5}\right)(\psi_n, \varDelta\varphi\psi_n)\right.$$
$$\left. + \frac{i}{m}(\psi_n, \beta\boldsymbol{\alpha}\boldsymbol{E}\psi_n)\right\}. \tag{53.9}$$

We now evaluate the level shift for a hydrogen-like atom. In this case (in CGSE units) we have

$$e\varphi = -\frac{Ze^2}{r}$$

and

$$\frac{\hbar^2}{m^2c^2}\langle\varDelta e\varphi\rangle = \frac{\hbar^2}{m^2c^2}(\psi_n, \varDelta e\varphi\psi_n) = \frac{\hbar^2}{m^2c^2}4\pi e^2 Z|\psi_n(0)|^2, \tag{53.10}$$

but since

$$|\psi_n(0)|^2 = \begin{cases} \left(\dfrac{Z}{na}\right)^3 \dfrac{1}{\pi}, & l = 0, \\ 0, & l \neq 0, \end{cases}$$

where $n$ and $l$ are the principal and the orbital quantum numbers, and $a = \hbar^2/me^2$, we have

$$\frac{\hbar^2}{m^2c^2}\langle \Delta e\varphi \rangle = \begin{cases} 8\dfrac{a^2}{n^3} Z^4 Ry, & l = 0, \\ 0, & l \neq 0, \end{cases}$$

where $Ry = \frac{1}{2}\alpha^2 mc^2$. Further, we can show that

$$\frac{e\hbar}{mc}\langle i\beta\alpha E \rangle = \begin{cases} -4\alpha^2 \dfrac{Z^4}{n^3} \dfrac{Ry}{(l+1)(2l+1)}, & j = l + \dfrac{1}{2}, \\ 4\alpha^2 \dfrac{Z^4}{n^3} \dfrac{Ry}{l(2l+1)}, & j = l - \dfrac{1}{2}. \end{cases} \tag{53.10'}$$

We see that the radiation shift of an $s$-level is in order of magnitude equal to $\alpha^2 E_0$, where $E_0$ is the energy of the ground state.

We now obtain the shift of the hydrogen atom levels. In the case of hydrogen the quantity $\ln (mc^2/\varepsilon_0)$ is equal to 7.6876. On utilizing expressions (53.10) and (53.10′) we obtain the following values for the radiation shift of the hydrogen levels (in frequency units):

$$\begin{aligned} \Delta\nu(2s_{\frac{1}{2}}) &= 1034 \text{ Mc/sec}, \\ \Delta\nu(2p_{\frac{1}{2}}) &= -17 \text{ Mc/sec}, \\ \Delta\nu(2p_{\frac{3}{2}}) &= 8 \text{ Mc/sec}. \end{aligned} \tag{53.11}$$

As is well known, the undisplaced states $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ have the same energy. But the radiation shifts of these levels are different and the $2s_{\frac{1}{2}}$ level turns out to lie higher than the $2p_{\frac{1}{2}}$ level by approximately 1051Mc/sec.

Formulas (53.9), (53.10), (53.10′) determine the radiation shift of a level of order $\alpha^2 E_0$. It can be shown that in the next approximation of perturbation theory we would obtain a level shift for a hydrogen-like atom equal to (101)

$$\delta E_n^{(2)} = \frac{Z^5 \alpha^4}{n^3} Ry \left(1 + \frac{11}{128} - \frac{1}{2}\ln 2 + \frac{5}{192}\right). \tag{53.12}$$

For $n = 2$ and $Z = 1$ this gives 7 Mc/sec. Taking this correction into account the difference in the $2s_{\frac{1}{2}}$ and the $2p_{\frac{1}{2}}$ energy levels in hydrogen turns out to be equal to 1057.19 Mc/sec, while the experimental value for this quantity is equal to 1057.77±0.1 Mc/sec (167).

In concluding this subsection we note that by utilizing equation (49.12) we can obtain the radiation shift of the levels of positronium. The difference between the ground levels of ortho- and parapositronium turns out in this case to be equal to (100)

$$\Delta W = \alpha^2 Ry\left\{\frac{7}{6} - \left(\frac{16}{9} + \ln 2\right)\frac{\alpha}{\pi}\right\} = 2.0337 \cdot 10^5\,\text{Mc/sec}.$$

The experimentally obtained value for this quantity is (91):

$$\Delta W = (2.0333 \pm 0.0004) \cdot 10^5\,\text{Mc/sec}.$$

### 53.2. *Radiation Shift of the Levels of $\mu$-Mesohydrogen*

Polarization of the vacuum plays an insignificant role in the radiation shift of the levels of an atomic electron. This is associated with the fact that the Coulomb field of a nucleus is distorted by the polarization of the electron-positron vacuum at distances of the order of a Compton wavelength of the electron $\hbar/mc$ while the dimensions of the atomic electron orbits are considerably greater than $\hbar/mc$. For this reason polarization of the vacuum introduces into the radiation shift of atomic levels a contribution of order $\alpha^3$, which amounts to approximately 3% of the total radiation shift.

A different situation exists in the case of a $\mu$-mesic atom in which $\mu$-mesons play the role of electrons. Since the $\mu$-meson mass is large compared to the electron mass, the dimensions of the orbits in a $\mu$-mesic atom are considerably smaller than the dimensions of the electron orbits in an ordinary atom, and the polarization of the electron-positron vacuum plays a predominant role in the radiation shift of the levels of a $\mu$-mesic atom.

We shall see that the radiation shift of the levels of a $\mu$-mesic atom of hydrogen, or of $\mu$-mesohydrogen, is approximately given by $\alpha E_0$, where $E_0$ is the energy of the ground state of this atom, while for ordinary atoms this shift amounts to only $\alpha^2$ of the energy of the ground state.

We now obtain the radiation shift of the levels of $\mu$-mesohydrogen (70) (or, more accurately speaking, of a hydrogen-like $\mu$-mesic atom).

In order to do this we use formula (50.15) for the radiation correction to Coulomb's law due to the polarization of the electron-positron vacuum:

$$\delta\varphi(r) = \frac{aZe}{6\pi^2 r}\int_1^\infty e^{-2mr\zeta}\left(1+\frac{1}{2\zeta^2}\right)\frac{(\zeta^2-1)^{1/2}}{\zeta^2}d\zeta .$$

The radiation shift of the energy of a $\mu$-meson which is in a state $\psi_n(\boldsymbol{r})$ of energy $E_n$ is given by

$$\delta E_n = \int e\delta\varphi(r)|\psi_n(\boldsymbol{r})|^2 d\boldsymbol{r}$$

$$= \frac{aZe^2}{6\pi^2}\int_1^\infty \left(1+\frac{1}{2\zeta^2}\right)\frac{(\zeta^2-1)^{1/2}}{\zeta^2}d\zeta\int\frac{e^{-2mr\zeta}}{r}|\psi_n(\boldsymbol{r})|^2 d\boldsymbol{r}.$$

The ratio of this expression to the energy of the ground state $E_0 = (e^2Z^2/4\pi a_\mu)$ ($a_\mu$ is the radius of the normal orbit of the $\mu$-meson in mesohydrogen, equal to $a_\mu = 4\pi/m_\mu e^2$) can be written in the following form:

$$\frac{\delta E_n}{E_0} = \frac{2a}{3\pi}\int_1^\infty \left(1+\frac{1}{2\zeta^2}\right)\frac{(\zeta^2-1)^{1/2}}{\zeta^2}J_{nl}(2\varepsilon\zeta)d\zeta, \qquad (53.13)$$

where

$$J_{nl}(y) = \left(\frac{\varepsilon}{m}\right)^3\int_0^\infty re^{-yr}R_{nl}^2\left(\frac{\varepsilon}{m}r\right)dr,$$

$R_{nl}(r)$ is the normalized radial function and $\varepsilon = (m/m_\mu aZ)$ ($m_\mu$ is the reduced mass of the $\mu$-meson, $l$ is the orbital quantum number). Approximate integration of (53.13) leads to the following values for the shift in $eV$ for different values of $Z$.

TABLE 15

| $nl$ \ $Z$ | 1 | 6 | 20 |
|---|---|---|---|
| $1s$ | 1.8 | 320 | 20000 |
| $2s$ | 0.2 | 47 | 3250 |
| $2p$ | 0.014 | 27 | 2550 |

### 53.3. *Natural Line Width*

The interaction of an atomic electron with the electromagnetic field leads not only to a change in the electron energy, but also to the level acquiring a width, since due to this interaction there always exists a finite probability for the transition of the atom to the ground state.

We now show how this width, which is called the natural width, can be introduced.

We start with the Dirac equation containing the mass operator $M^{(3)}(x, x')$ in the presence of an external field (cf. formulas (43.45), (49.5)):

$$\left\{\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ieA_\mu^{(e)}(x)\right)+m\right\}\psi(x)+\int M^{(e)}(x, x')\,\psi(x')\,dx' = 0, \qquad (53.14)$$

where $A_\mu^{(e)}(x)$ is the potential of the external field including the added term due to the polarization of the vacuum.

As we already know (cf. § 43), the mass operator takes into account the interaction of the electron with the vacuum of the electromagnetic field, and in the first approximation with respect to $\alpha$ is given by the formula

$$M^{(e)}(x,x') = ie^2D^c(x,x')\gamma_\mu S_c^{(e)}(x,x')\gamma_\mu, \qquad (53.15)$$

where $S_c^{(e)}(x, x')$ is the Green's function for the Dirac equation for an electron situated in an external field.

If we substitute into (53.15) the expression (35.3) for $S_c^{(e)}(x, x')$ then the mass operator (in first approximation) assumes the form

$$M^{(e)}(x,x')$$
$$= \frac{e^2}{i(2\pi)^5}\int \frac{d^4k}{k^2-i0}\gamma_\mu \sum_n e^{ik\boldsymbol{r}}\psi_n(\boldsymbol{r}')\overline{\psi}_n(\boldsymbol{r}')e^{-ik\boldsymbol{r}'} \int_{-\infty}^{\infty} \frac{e^{i(\omega-k_0)(t-t')}}{E_n(1-i0)-\omega}\,d\omega, \qquad (53.16)$$

where $\psi_n(x) = \psi_n(\boldsymbol{r})\exp(-iE_n t)$ are the "stationary" electron wave functions, satisfying the unperturbed Dirac equation

$$\left\{\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ieA_\mu^{(e)}(x)\right)+m\right\}\psi(x) = 0$$

(here, as before, we must interpret $A_\mu^{(e)}(x)$ as the external potential including the added term due to the polarization of the vacuum).

We rewrite equation (53.14) in the form

$$i\frac{\partial\psi}{\partial t} = \mathrm{H}^{(0)}\psi + \int \beta M^{(e)}(x, x')\psi(x')d^4x' = 0, \qquad (53.17)$$

where $H^{(0)}$ is the Hamiltonian corresponding to the unperturbed Dirac equation, and we seek solutions of (53.17) which depend exponentially on the time

$$\psi(x) = \psi(\boldsymbol{r})e^{-iEt}.$$

On substituting this expression into (53.17) we obtain the equation

$$\mathrm{H}^{(0)}\psi(\boldsymbol{r}) + \int \beta M_E^{(e)}(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}')d\boldsymbol{r}' = E\psi(\boldsymbol{r}), \qquad (53.18)$$

or, in abbreviated form,

$$\mathrm{H}_E\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}), \qquad (53.18')$$

where

$$\mathrm{H}_E = \mathrm{H}^{(0)} + \beta \mathrm{M}_E^{(e)},$$

$$\mathrm{M}_E^{(e)}\psi(\boldsymbol{r}) = \int M_E^{(e)}(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}')d\boldsymbol{r}',$$

$$M_E^{(e)}(\boldsymbol{r}, \boldsymbol{r}') = \int\limits_{-\infty}^{\infty} e^{iE\tau}M^{(e)}(\boldsymbol{r}, \boldsymbol{r}'; \tau)d\tau. \qquad (53.18'')$$

The usual perturbation theory can be applied to equation (53.18), and this yields

$$E = E_n^{(0)} + (\beta \mathrm{M}_E^{(e)})_{nn} + \sum_{n' \neq n} \frac{(\beta \mathrm{M}_E^{(e)})_{nn'}(\beta \mathrm{M}_E^{(e)})_{n'n}}{E_n^{(0)} - E_{n'}^{(0)}} + \dots, \qquad (53.19)$$

where $E_n^{(0)}$ is the energy of the unperturbed state determined by the Dirac equation without the mass operator, and

$$(\beta \mathrm{M}_E^{(e)})_{nn'} = \int \psi_{n'}^{(0)*}(\boldsymbol{r})\beta \mathrm{M}_E^{(e)}\psi_n^{(0)}(\boldsymbol{r})d\boldsymbol{r} = \int \bar{\psi}_{n'}^{(0)}(\boldsymbol{r})\mathrm{M}_E^{(e)}\psi_n^{(0)}(\boldsymbol{r})d\boldsymbol{r}.$$

We confine ourselves to considering only the first order perturbation theory approximation

$$E \approx E_n^{(0)} + (\beta \mathrm{M}_E^{(e)})_{nn}. \qquad (53.20)$$

On utilizing expression (53.16) for $\mathrm{M}^{(e)}(x, x')$, and formulas (53.18''), we obtain from the above expression

$$E \approx E_n^{(0)} + \frac{e^2}{(2\pi)^4 i}\int \frac{d^4k}{k^2 - i0} \sum_{n'} \frac{(n|\gamma_\mu e^{i\boldsymbol{kr}}|n')(n'|\gamma_\mu e^{-i\boldsymbol{kr}}|n)}{(1 - i0)E_{n'}^{(0)} - E_n^{(0)} + k_0}$$

$$= E_n^{(0)} + \frac{e^2}{(2\pi)^4 i}\sum_{n'}\int dk \mathrm{Q}(E_n^{(0)}, E_{n'}^{(0)}, |\boldsymbol{k}|)(n|\gamma_\mu e^{i\boldsymbol{kr}}|n')(n'|\gamma_\mu e^{-i\boldsymbol{kr}}|n), \qquad (53.21)$$

where

$$(n|\gamma_\mu e^{i\mathbf{kr}}|n') = \int \bar{\psi}_n(\mathbf{r})\gamma_n e^{i\mathbf{kr}}\psi_{n'}(\mathbf{r})\,d\mathbf{r},$$

$$Q(E_n^{(0)}, E_{n'}^{(0)}, |\mathbf{k}|) = \int\limits_{-\infty}^{\infty} \frac{dk_0}{[(1-i0)E_{n'}^{(0)} - E_n^{(0)} + k_0]\,[k^2 - k_0^2 - i0]}. \tag{53.21'}$$

We now evaluate $Q(E_n^{(0)}, E_{n'}^{(0)}, |\mathbf{k}|)$. Direct integration yields

$$Q(E_n^{(0)}, E_{n'}^{(0)}, |\mathbf{k}|) = \frac{\pi i}{|\mathbf{k}|}\,\frac{1}{E_n^{(0)} - E_{n'}^{(0)}(1-i0) - |\mathbf{k}|\,E^0_{n'}/|E^0_{n'}|}.$$

On the other hand, the quantity $1/(x-i0)$ appearing in the integral can be replaced by

$$\frac{1}{x-i0} = \pi i\delta(x) + \mathcal{P}\frac{1}{x};$$

and, therefore,

$$Q(E_n^{(0)}, E_{n'}^{(0)}, |\mathbf{k}|) = \frac{\pi^2}{|\mathbf{k}|} E^0_{n'}/|E^0_{n'}|\,\delta[E_n^{(0)} - E_{n'}^{(0)} - |\mathbf{k}|\,E^0_{n'}/|E^0_{n'}|]$$

$$+\frac{\pi i}{|\mathbf{k}|}\mathcal{P}\frac{1}{E_n^{(0)} - E_{n'}^{(0)} - |\mathbf{k}|\,E^0_{n'}/|E^0_{n'}|}. \tag{53.22}$$

Substitution of this expression into (53.21) shows that the quantity $\Delta E = E - E_n^{(0)}$, which is the level shift due to the interaction of the electron with the electromagnetic vacuum, is complex. On setting

$$\Delta E = \delta E_n - i\frac{\Gamma_n}{2}, \tag{53.23}$$

we find in accordance with (53.21) and (53.22) the expression for the real and the imaginary parts of $\Delta E$.

$$\delta E_n = \frac{e^2}{16\pi^3}\mathcal{P}\int \frac{d\mathbf{k}}{|\mathbf{k}|}\sum_{n'} \frac{(n|\gamma_\mu e^{i\mathbf{kr}}|n')\,(n'|\gamma_\mu e^{-i\mathbf{kr}}|n)}{E_n^{(0)} - E_{n'}^{(0)} - \dfrac{E_{n'}^{(0)}}{|E_{n'}^{(0)}|}|\mathbf{k}|},$$

$$\Gamma_n = \frac{e^2}{8\pi^2}\int \frac{d\mathbf{k}}{|\mathbf{k}|}\sum_{n'} \frac{E_{n'}^{(0)}}{|E_{n'}^{(0)}|}(n|\gamma_\mu e^{i\mathbf{kr}}|n')\,(n'|\gamma_\mu e^{-i\mathbf{kr}}|n)$$

$$\times\,\delta\left(E_n^{(0)} - E_{n'}^{(0)} - \frac{E_{n'}^{(0)}}{|E_{n'}^{(0)}|}|\mathbf{k}|\right). \tag{53.24}$$

The real part of $\Delta E$ determines the radiation shift of the energy of an atom found in subsection 53.1, while the imaginary part determines the level width.

The imaginary part has a simple physical meaning: it obviously represents the total probability of the electron making a transition from the initial state by emitting a photon.

We note that for the determination of the radiation level shift we could have from the outset utilized formula (53.24), but it is more convenient to use this formula only for evaluating the contribution made by long wavelength photons, as was done in subsection 53.1, while the contribution made by short wavelength photons can be more simply obtained with the aid of the effective potential energy of the electron.

### 53.4. *Photon Scattering near Resonance*

In § 35 in considering the scattering of a photon by a bound electron we saw that in the case of resonance, when the photon energy coincides with the difference between the energy levels of the electron, the theory yielding the first approximation based on the second order scattering matrix becomes inapplicable. The usual method of taking radiation corrections into account is also inapplicable here, since it depends on the expansion of the scattering matrix into a series in powers of $\alpha$, while it can be easily seen that in the case under consideration the expansion is made in powers of the quantity $\alpha/(E_n^{(0)}-E_i^{(0)}-\omega)$ which becomes infinite at resonance.

We must, therefore, take radiation corrections into account at an earlier stage of the calculations which in the simplified derivation of formula (35.14) corresponds to the introduction of complex frequencies.

From the derivation of formula (35.14) for the cross section for photon scattering we can easily conclude that the resonance denominator in the matrix element is related to the structure of the function $S_c^{(e)}(x_1, x_2)$, which in accordance with (35.13) contains $E_n+\omega$ in the denominator. Therefore, we should expect that the replacement of the function $S_c^{(e)}(x_1, x_2)$ by the exact electron Green's function, which takes into account the interaction between the electron and the electromagnetic field, must lead to finite results.

We denote this function in the presence of an external electromagnetic field by $G_e^{(e)}(x_1, x_2)$. It satisfies equation (49.5):

$$G_e^{(e)}(x_1, x_2) = S_c^{(e)}(x_1, x_2) - i\int S_c^{(e)}(x_1, x_3) M^{(e)}(x_3, x_4) G_e^{(e)}(x_4, x_2)\, d^4x_3\, d^4x_4. \tag{53.25}$$

By utilizing expression (35.3) for $S_c^{(e)}(x_1, x_2)$ we can easily show that the Green's function $G_e^{(e)}(x, x')$ also satisfies the equation

$$\left\{\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ieA_\mu^{(e)}\right) + m\right\} G_e^{(e)}(x, x') + \int M^{(e)}(x, x'') G_e^{(e)}(x'', x')\, d^4x'' = -i\,\delta(x-x'). \tag{53.26}$$

We now show that if we introduce into the matrix element for the scattering of a photon by a bound electron in place of the function $S_c^{(e)}(x_1, x_2)$ the function $G_e^{(e)}(x_1, x_2)$, we shall obtain a finite expression at resonance in accordance with the results of § 35 (128).

We expand the function $G_e^{(e)}(x_1, x_2)$ into a series in terms of the complete system of functions $\psi_n(\boldsymbol{r})$ which are the spatial parts of the wave functions for an electron in the field $A_\mu^{(e)}(x)$.

We expand the time-dependent coefficients in the above expansion into a Fourier integral. Such a combined expansion of $G_e^{(e)}(x_1, x_2)$ has the form

$$G_e^{(e)}(x_1, x_2) = \frac{1}{2\pi i}\sum_n \int d\omega\, e^{i\omega(t_1-t_2)} f_{nn}(\omega)\psi_n(\boldsymbol{r}_1)\overline{\psi}_n(\boldsymbol{r}_2) + \frac{1}{2\pi i}\sum_{n\neq m}\int d\omega\, e^{i\omega(t_1-t_2)} f_{nm}(\omega)\psi_n(\boldsymbol{r}_1)\overline{\psi}_m(\boldsymbol{r}_2). \tag{53.27}$$

We now obtain the equations satisfied by the expansion coefficients $f_{nm}(\omega)$. In order to do this we substitute the expansion (53.27) into the integral equation (53.25). On noting that

$$\int \overline{\psi}_n(\boldsymbol{r}_1)\beta S_c^{(e)}(x_1, x_3)\, d\boldsymbol{r}_1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\omega(t_1-t_3)}}{E_n(1-i0)+\omega}\, d\omega\, \overline{\psi}_n(\boldsymbol{r}),$$

we obtain the system of equations for $f_{nm}(\omega)$

$$f_{nn}(\omega) = \frac{1}{E_n(1-i0)+\omega}\left\{1 - H_{nn}(-\omega) f_{nn}(\omega) - \sum_{m \neq n} H_{nm}(-\omega) f_{mn}(\omega)\right\},$$

$$f_{nm}(\omega) = \frac{1}{E_n(1-i0)+\omega}\left\{-H_{nm}(-\omega) f_{mm}(\omega) - \sum_{p \neq m} H_{np}(-\omega) f_{pm}(\omega)\right\}, \tag{53.28}$$

where

$$H_{nm}(\omega) = \int_{-\infty}^{\infty} H_{nm}(t) e^{i\omega t}\, dt$$

and

$$H_{nm}(t) = \int \bar{\psi}_n(\mathbf{r}_3) M^{(e)}(\mathbf{r}_3, \mathbf{r}_4; t)\, \psi_m(\mathbf{r}_4)\, d\mathbf{r}_3\, d\mathbf{r}_4 .$$

Introducing the notation

$$\frac{1}{f_{nn}(\omega)} = Q_{nn}, \qquad \frac{f_{mn}(\omega)}{f_{nn}(\omega)} = R_{mn}, \qquad m \neq n,$$

we rewrite the system (53.28) in the form

$$-R_{nm} = \frac{H_{nm}(-\omega)}{E_n+\omega} + \sum_{p \neq m} \frac{H_{np}(-\omega) R_{pm}}{E_n+\omega}, \tag{53.29}$$

$$Q_{nn} = E_n + \omega + H_{nn}(-\omega) + \sum_{m \neq n} H_{nm}(-\omega) R_{mn}. \tag{53.29'}$$

In principle the system (53.29) enables us to determine $R_{nm}$, as a result of which $Q_{nn}$ can be obtained in accordance with equation (53.29').

By assuming that the quantity $H_{nm}(-\omega)$ is small, and by using the method of successive approximations we obtain

$$R_{nm} = -\frac{1}{E_n+\omega}\left\{H_{nm}(-\omega) + \sum_{p \neq m} \frac{H_{np}(-\omega) H_{pm}(-\omega)}{-\omega - E_p} + \cdots\right\},$$

$$Q_{nm} = \omega + E_n + H_{nn}(-\omega) + \sum_{m \neq n} \frac{H_{nm}(-\omega)\, H_{nm}(-\omega)}{-\omega - E_m} \tag{53.30}$$

In the first approximation we have

$$Q_{nn} \approx \omega + E_n + H_{nn}(-\omega),$$
$$R_{mn} \approx -\frac{H_{mn}(-\omega)}{\omega + E_m + H_{mm}(-\omega)}, \tag{53.30'}$$

whence it follows that

$$f_{nn} = \frac{1}{\omega + E_n + H_{nn}(-\omega)},$$
$$f_{mn} = \frac{-H_{mn}(-\omega) + \cdots}{[\omega + E_m + H_{mm}(-\omega)]\,[\omega + E_n + H_{nn}(-\omega)]}. \tag{53.31}$$

We see $|f_{mn}| \ll |f_{nn}|$; therefore in future we can neglect the nondiagonal elements $f_{mn}$.

We now obtain $G_e^{(e)}(x_1, x_2)$. On substituting (53.31) into (53.27) and on neglecting the quantities $f_{mn}$ we obtain

$$G_e^{(e)}(x_1, x_2) \approx \frac{1}{2\pi i} \sum_n \psi_n(\boldsymbol{r}_1)\bar{\psi}_n(\boldsymbol{r}_2) \int_{-\infty}^{\infty} \frac{e^{i\omega(t_1 - t_2)}}{E_n + \omega + H_{nn}(-\omega)} d\omega. \tag{53.32}$$

This exprcssion differs from expression (35.3) for the function $S_c^{(e)}(x_1, x_2)$ by the denominators of the fractions appearing in the integral. The quantity $H_{nn}(-\omega)$ which appears in $G_e^{(e)}(x_1, x_2)$ obviously coincides with the quantity $(\beta M_E)_{nn}$, which determines the radiation shift and the level width (cf. (53.23)):

$$H_{nn}(-\omega) = \delta E_n - \frac{i}{2}\Gamma_n.$$

Since this quantity is complex the denominators of the fractions in (53.32) cannot vanish. On substituting expression (53.32) instead of $S_c^{(e)}(x_1, x_2)$ into formula (35.6) we obtain the photon scattering cross section of the form (35.14).

## § 54. Photon-Photon Scattering and the Lagrangian for the Electromagnetic Field

### 54.1. *Photon-Photon Scattering Tensor of the Fourth Rank*

Interaction of electromagnetic fields with the vacuum of the electron-positron field must lead to interaction between these fields, in particular, to interaction between photons. Indeed, let us consider two photons of momenta $k_1$ and $k_2$ whose energy is insufficient to create a pair. Such photons can create two virtual pairs. As a result two photons will appear of momenta $k_3$ and $k_4$ related to the momenta of the original photons $k_1$ and $k_2$ by the single condition—the law of conservation of energy-momentum $k_1+k_2 = k_3+k_4$.

This process represents photon-photon scattering (56), (3), (103). It obviously cannot be described by means of the classical field equations, since in classical electrodynamics electromagnetic waves are propagated independently without affecting each other.

In order that the field equations should be able to describe the process of photon-photon scattering, and also other processes involving interaction between electromagnetic fields, they must be non-linear, in contrast to the linear Maxwell's equations. From this it follows that the Lagrangian density of the electromagnetic field cannot be a quadratic function of the fields, i.e., it must differ from the function $L_0 = \frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{H}^2)$.

In the lowest order of perturbation theory to which we confine ourselves here, the interaction between electromagnetic fields is described by a part of the scattering matrix $S^{(4)}$.

In accordance with (24.14) the matrix $S^{(4)}$ has the form

$$S^{(4)} = \frac{e^4}{4!}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4 T\big\{N\big(\overline{\psi}(x_1)\hat{\mathrm{A}}(x_1)\psi(x_1)\big)$$
$$\times N\big(\overline{\psi}(x_2)\hat{\mathrm{A}}(x_2)\psi(x_2)\big)N\big(\overline{\psi}(x_3)\hat{\mathrm{A}}(x_3)\psi(x_3)\big)N\big(\overline{\psi}(x_4)\hat{\mathrm{A}}(x_4)\psi(x_4)\big)\big\}.$$

Since we are interested in processes in which only electromagnetic fields but no real electrons take part, then in this approximation we must replace the product of the electron operators $\overline{\psi}(x)\psi(x)$ by pairings between them. The total number of pairings is, obviously, equal to six (they are represented schematically by Fig. 113). All these pairings produce equal contributions, so that the part of the scattering matrix $S^{(4)}$ which is of interest to us has the form

$$S^{(4)} = -\frac{e^4}{4}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4 N\big(\mathrm{A}_\mu(x_1)\mathrm{A}_\nu(x_2)\mathrm{A}_\lambda(x_3)\mathrm{A}_\sigma(x_4)\big)$$
$$\times \mathrm{Sp}\,\{\gamma_\mu S_c(x_2-x_1)\gamma_\nu S_c(x_3-x_2)\gamma_\lambda S_c(x_4-x_3)\gamma_\sigma S_c(x_1-x_4)\}. \quad (54.1)$$

Here only the quantities $\mathrm{A}_\mu(x)$ are operators.

In momentum space this expression assumes the form

$$S^{(4)} = -\frac{i}{4}\left(\frac{e^2}{4\pi}\right)^2 \int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4 (2\pi)^4\, \delta(k_1+k_2+k_3+k_4)$$

$$\times N\left(\frac{A_\mu(k_1)}{(2\pi)^4}\,\frac{A_\nu(k_2)}{(2\pi)^4}\,\frac{A_\lambda(k_3)}{(2\pi)^4}\,\frac{A_\sigma(k_4)}{(2\pi)^4}\right) T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4), \qquad (54.2)$$

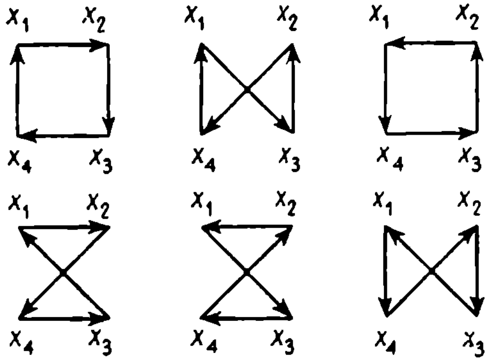


Fig. 113.

where

$$A_\mu(k) = \int A_\mu(x) e^{-ikx}\, d^4x$$

and

$$T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$$

$$= \frac{1}{i\pi^2}\int d^4p\, \mathrm{Sp}\,\{\gamma_\mu(i\hat{p}+m)^{-1}\gamma_\nu(i\hat{p}-i\hat{k}_2+m)^{-1}\gamma_\lambda(i\hat{p}-i\hat{k}_2-i\hat{k}_3+m)^{-1}\gamma_\sigma$$

$$\times (i\hat{p}-i\hat{k}_2-i\hat{k}_3-i\hat{k}_4+m)^{-1}\}. \qquad (54.3)$$

In place of the tensor $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ we introduce the tensor $J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ which is fully symmetrized with respect to the simultaneous interchange of the tensor indices and of the variables $k_i$:

$$J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$$

$$+T_{\mu\nu\sigma\lambda}(k_1, k_2, k_4, k_3)+T_{\mu\lambda\nu\sigma}(k_1, k_3, k_2, k_4). \qquad (54.3')$$

It is clear that in expression (54.2) we can replace the tensor $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ by $\frac{1}{3}J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$:

$$S^{(4)} = -\frac{i}{12}\left(\frac{e^2}{4\pi}\right)^2 \int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4 (2\pi)^4\, \delta(k_1+k_2+k_3+k_4)$$

$$\times N\left(\frac{A_\mu(k_1)}{(2\pi)^4}\,\frac{A_\nu(k_2)}{(2\pi)^4}\,\frac{A_\lambda(k_3)}{(2\pi)^4}\,\frac{A_\sigma(k_4)}{(2\pi)^4}\right) J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4). \qquad (54.4)$$

We call the symmetrized tensor $J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ the photon-photon scattering tensor of the fourth rank.

In accordance with the results of § 46 the tensor $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ diverges for large momenta of the virtual electrons. In order to show this we note that for large values of $p$ the tensor $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ behaves in the same way as $T_{\mu\nu\lambda\sigma}(0, 0, 0, 0)$:

$$T_{\mu\nu\lambda\sigma}(0, 0, 0, 0) = \frac{1}{i\pi^2}\int \frac{d^4p}{(p^2+m^2)^4} \times \operatorname{Sp}\{\gamma_\mu(i\hat{p}-m)\gamma_\nu(i\hat{p}-m)\gamma_\lambda(i\hat{p}-m)\gamma_\sigma(i\hat{p}-m)\}.$$

On substituting into this expression

$$\tfrac{1}{4}\operatorname{Sp}\{\gamma_\mu(i\hat{p}-m)\gamma_\nu(i\hat{p}-m)\gamma_\lambda(i\hat{p}-m)\gamma_\sigma(i\hat{p}-m)\}$$
$$= (p^2+m^2)^2(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}+\delta_{\mu\lambda}\,\delta_{\nu\sigma})+8p_\mu p_\nu p_\lambda p_\sigma$$
$$-2(p^2+m^2)\,(p_\mu p_\nu\,\delta_{\lambda\sigma}+p_\nu p_\lambda\,\delta_{\mu\sigma}+p_\mu p_\sigma\,\delta_{\nu\lambda}+p_\lambda p_\sigma\,\delta_{\mu\nu}),$$

we obtain

$$T_{\mu\nu\lambda\sigma}(0, 0, 0, 0) = \frac{4}{i\pi^2}(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\lambda}\,\delta_{\nu\sigma}+\delta_{\mu\sigma}\,\delta_{\sigma\lambda})\int\frac{d^4p}{(p^2+m^2)^2}$$
$$-\frac{8}{i\pi^2}\int\frac{d^4p}{(p^2+m^2)^3}(p_\mu p_\nu\,\delta_{\lambda\sigma}+p_\nu p_\lambda\,\delta_{\mu\sigma}+p_\mu p_\sigma\,\delta_{\nu\lambda}+p_\lambda p_\sigma\,\delta_{\mu\nu})$$
$$+\frac{32}{i\pi^2}\int\frac{d^4p}{(p^2+m^2)^4}\,p_\mu p_\nu p_\lambda p_\sigma.$$

But

$$\int p_\mu p_\nu f(p^2)\,d^4p = \tfrac{1}{4}\,\delta_{\mu\nu}\int p^2 f(p^2)\,d^4p,$$

$$\int p_\mu p_\nu p_\lambda p_\sigma f(p^2)\,d^4p = \tfrac{1}{24}\,(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\lambda}\,\delta_{\nu\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda})\int p^4 f(p^2)\,d^4p.$$

Therefore

$$T_{\mu\nu\lambda\sigma}(0, 0, 0, 0) = \frac{1}{i\pi^2}\left\{\frac{4}{3}(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-2\delta_{\mu\lambda}\,\delta_{\nu\sigma})\int\frac{d^4p}{(p^2+m^2)^2}\right.$$
$$+\frac{4}{3}(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-2\delta_{\mu\lambda}\,\delta_{\nu\sigma})m^2\int\frac{d^4p}{(p^2+m^2)^3}$$
$$\left.+\frac{4}{3}(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\lambda\nu}+\delta_{\mu\lambda}\,\delta_{\nu\sigma})m^4\int\frac{d^4p}{(p^2+m^2)^4}\right\}.$$

The first integral appearing in this expression diverges logarithmically.

Insofar as the symmetrized tensor $J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ is concerned the divergent integrals contained in its individual terms cancel, but, nevertheless, the tensor $J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ must be regularized just as the divergent tensor $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$, since from considerations of gauge invariance the tensors $T_{\mu\nu\lambda\sigma}(0, 0, 0, 0)$ and $J_{\mu\nu\lambda\sigma}(0, 0, 0, 0)$ must obviously vanish for $k_1 = k_2 = k_3 = k_4 = 0$, while the expression $J_{\mu\nu\lambda\sigma}(0, 0, 0, 0)$ differs from zero. As has been already explained in § 46, this loss of gauge invariance is associated with the existence of divergences in quantum electrodynamics.

In order to obtain the regularized value of the tensor $J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ we must subtract from it its value for $k_1 = k_2 = k_3 = k_4 = 0$. We denote the regularized value of $J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ obtained in this manner by $I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$:

$$I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4,) = J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) - J_{\mu\nu\lambda\sigma}(0, 0, 0, 0). \quad (54.5)$$

The regularized matrix $S^{(4)}$ has the form

$$S^{(4)} = -\frac{i}{12}\left(\frac{e^2}{4\pi}\right)^2 \int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4 (2\pi)^4 \delta(k_1+k_2+k_3+k_4)$$
$$\times N\left(\frac{A_\mu(k_1)}{(2\pi)^4}\,\frac{A_\nu(k)_2}{(2\pi)^4}\,\frac{A_\lambda(k_3)}{(2\pi)^4}\,\frac{A_\sigma(k_4)}{(2\pi)^4}\right) I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4). \quad (54.6)$$

## 54.2. *Photon-Photon Scattering*

We now establish the connection between the tensor $I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4$ and the photon-photon scattering cross section.

We consider the process in which the electron-positron vacuum absorbs photons of momenta $k_1$ and $k_2$ and polarizations $e_1$ and $e_2$, and emits photons of momenta $k_3$ and $k_4$ and polarizations $e_3$ and $e_4$. This process corresponds to the six diagrams shown in Fig. 114. Diagrams 1, 2, 3 and 4, 5, 6 differ only by the sense of going around the electron loop, so that the matrix elements corresponding to them are the same.

The total matrix element for photon-photon scattering is determined in accordance with subsection 25.5 by the following general formula:

$$S^{(4)}_{i\to f} = -\frac{e^4}{2}\,\frac{\delta(k_1+k_2-k_3-k_4)}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}$$
$$\times \Big\{ e_{1_\mu} e_{2_\nu} e_{3\lambda} e_{4\sigma} \int \mathrm{Sp}\{\gamma_\mu (i\hat{p}+m)^{-1} \gamma_\nu (i\hat{p}+i\hat{k}_2+m)^{-1}$$
$$\times \gamma_\lambda (i\hat{p}+i\hat{k}_2-i\hat{k}_3+m)^{-1} \gamma_\sigma (i\hat{p}-i\hat{k}_1+m)^{-1}\}\, d^4p$$

$$+e_{1\mu}e_{2\nu}e_{4\sigma}e_{3\lambda}\int \mathrm{Sp}\{\gamma_\mu(i\hat{p}+m)^{-1}\gamma_\nu(i\hat{p}+i\hat{k}_2+m)^{-1}$$
$$\times\gamma_\sigma(i\hat{p}+i\hat{k}_2-i\hat{k}_4+m)^{-1}\gamma_\lambda(i\hat{p}-i\hat{k}_1+m)^{-1}\}\,d^4p$$
$$+e_{1\mu}e_{3\lambda}e_{2\nu}e_{4\sigma}\int \mathrm{Sp}\{\gamma_\mu(i\hat{p}+m)^{-1}\gamma_\lambda(i\hat{p}-i\hat{k}_3+m)^{-1}$$
$$\times\gamma_\nu(i\hat{p}-i\hat{k}_3+i\hat{k}_2+m)^{-1}\gamma_\sigma(i\hat{p}-i\hat{k}_1+m)^{-1}\}\,d^4p\Big\}. \qquad (54.7)$$

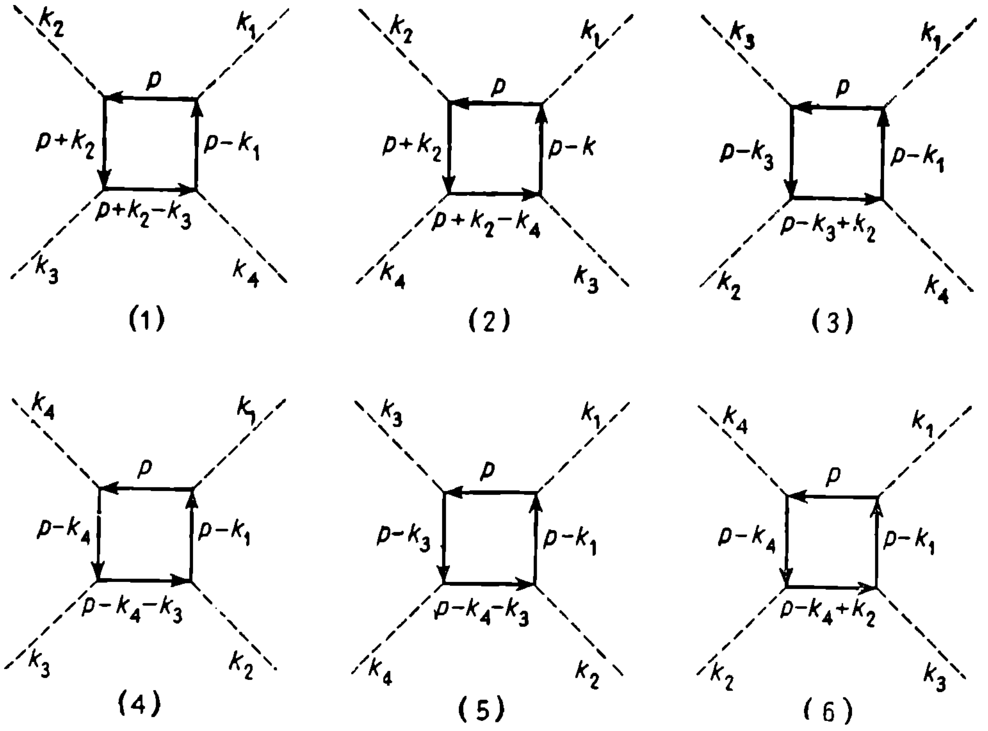


Fig. 114.

Utilizing expression (54.5) for the regularized photon-photon scattering tensor we can rewrite this expression in the form

$$S^{(4)}_{i\to f}=-\frac{i}{2}\left(\frac{e^2}{4\pi}\right)^2\frac{(2\pi)^4\delta(k_3+k_4-k_1-k_2)}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}\,e_{1\mu}e_{2\nu}e_{3\lambda}e_{4\sigma}$$
$$\times I_{\mu\nu\lambda\sigma}(-k_1,-k_2,k_3,k_4). \qquad (54.8)$$

In order to determine the photon-photon scattering cross section which in accordance with subsection 28.3 is an invariant, it is convenient to go over to the center of inertia system of the photons, in which the total three-dimensional momentum of the photons before and after scattering is equal to zero: $\boldsymbol{k}_1+\boldsymbol{k}_2 = \boldsymbol{k}_3+\boldsymbol{k}_4 = 0$. We denote the four-momenta of the photons in this system by

$$k_1\,(\boldsymbol{k}, i\omega),\ k_2\,(-\boldsymbol{k}, i\omega),\ k_3\,(\boldsymbol{k}', i\omega),\ k_4\,(-\boldsymbol{k}', i\omega),$$

where $|\boldsymbol{k}| = |\boldsymbol{k}'|$. (It follows from the law of conservation of energy that the frequencies of all four photons are the same in this system.) On taking into account the fact that the flux density of photons $k_2$ with respect to the photons $k_1$ is equal to 2, we obtain the following expression for the differential photon-photon scattering cross section:

$$d\sigma = \frac{1}{(2\pi)^2}\left(\frac{e^2}{4\pi}\right)^4 \frac{1}{16\omega^2}\,|e_{1\mu}e_{2\nu}e_{3\lambda}e_{4\sigma}I_{\mu\nu\lambda\sigma}(-k_1, -k_2, k_3, k_4)|^2\, do \tag{54.9}$$

where $do = 2\pi \sin\theta d\theta$ is the solid angle which contains the momentum of the photon $k_3$ ($\theta$ is the scattering angle, i.e., the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$).

Thus, for the evaluation of $d\sigma$ we must know the value of $I_{\mu\nu\lambda\sigma}(-k_1, -k_2, k_3, k_4)$, but we can obtain a simple analytic expression for the photon-photon scattering tensor only in the two limiting cases of low and high frequencies, when $\omega \ll m$ and $\omega \gg m$. In these two cases, which we shall now consider, we can also obtain simple closed expressions for the photon-photon scattering cross section.

In the low frequency case $\omega \ll m$ the photon-photon scattering tensor can be expanded into a series in powers of the frequency. The first term in this expansion corresponding to $\omega = 0$ vanishes for the regularized tensor; therefore, the expansion begins with the term containing all four frequencies linearly, i.e., which is proportional to $\omega^4$ in the center of inertia system. On the basis of this result, and of formula (54.9) we can easily conclude that the photon-photon scattering cross section is proportional to $\omega^6$ for $\omega \ll m$. On the other hand, in accordance with (54.9) it must contain the eighth power of the electron charge $e$. On taking into account the fact that the scattering cross section must have the dimensions of an area, and that we have at our disposal only the constants $e$, $\hbar$, $c$, $m$, we obtain unambiguously the following expression for the integrated photon-photon scattering cross section for $\omega \ll m$:

$$\sigma = a\left(\frac{e^2}{\hbar c}\right)^4\left(\frac{\hbar\omega}{mc^2}\right)^6\left(\frac{\hbar}{mc}\right)^2, \quad \hbar\omega \ll mc^2, \tag{54.10}$$

where $a$ is a dimensionless constant.

In the high frequency domain $\omega \gg m$ we can in practice neglect the electron mass $m$ in the expression for $I_{\mu\nu\lambda\sigma}$. Therefore, in this case the mass $m$ will not appear in the expression for the scattering cross section. But from the quantities $e$, $\hbar$, $c$, $\omega$ we can construct only one expression, viz., $e^8c^2/\omega^2$, which contains the eighth power of the charge and which has the dimensions of an area. Therefore, the photon-photon scattering cross section for $\omega \gg m$ must have the form

$$\sigma = a'\left(\frac{e^2}{\hbar c}\right)^4 \frac{c^2}{\omega^2}, \qquad \hbar\omega \gg mc^2, \tag{54.11}$$

where $a'$ is a dimensionless constant.

### 54.3. *Connection between the Photon-Photon Scattering Cross Section and the Radiation Corrections to the Lagrangian of the Electromagnetic Field*

We now undertake a more detailed investigation of photon-photon scattering in the low frequency range $\omega \ll m$. In this case the fields $\boldsymbol{F}(\boldsymbol{E}, \boldsymbol{H})$ associated with the photons satisfy the conditions

$$\frac{\hbar}{mc}\,|\operatorname{grad} \boldsymbol{F}| \ll |\boldsymbol{F}|, \qquad \frac{\hbar}{mc^2}\left|\frac{\partial \boldsymbol{F}}{\partial t}\right| \ll |\boldsymbol{F}|. \tag{54.12}$$

We call such fields slowly varying.

Slowly varying fields cannot in practice create real electron-positron pairs (if we exclude from consideration pair production by a large number of soft photons); therefore when conditions (54.12) are satisfied we can speak of a purely electromagnetic field.

The state of a slowly varying electromagnetic field can be described by a Lagrangian which depends only on the components of the field, and is independent of their derivatives with respect to the coordinates and the time. Since the Lagrangian must be a relativistic invariant, the fields can appear in the Lagrangian only in the form of the two combinations $\boldsymbol{E}^2-\boldsymbol{H}^2$ and $(\boldsymbol{EH})^2$, which are the only two independent field invariants.

If the fields are weak, then the Lagrangian density of the field $L$ can be expanded into a series in powers of $\boldsymbol{E}^2-\boldsymbol{H}^2$ and $(\boldsymbol{EH})^2$:

$$L = L_0 + L_1, \tag{54.13}$$

where

$$\begin{aligned} L_0 &= \tfrac{1}{2}(E^2-H^2), \\ L_1 &= \alpha(E^2-H^2)^2+\beta(EH)^2+\dots \\ &= a(F_{ik}F_{ik})^2+bF_{ik}F_{kl}F_{lm}F_{mi}+\dots \end{aligned} \tag{54.14}$$

and $\alpha, \beta, \dots, a, b, \dots$ are constants interrelated by

$$\alpha = 4a+2b \quad \text{and} \quad \beta = 4b, \dots . \tag{54.14'}$$

The terms collected in $L_1$ are radiation corrections to the principal Lagrangian $L_0$. These corrections are associated with such effects as photon-photon scattering. Indeed, the Lagrangian $L = L_0+L_1$ leads to non-linear equations for the electrodynamics of the vacuum, which, naturally, can describe photon-photon scattering.

We retain in the expansion of $L_1$ only the first two terms, and show how they may be related to the photon-photon scattering cross section.

We first recall that in the first approximation the scattering matrix has the form

$$\mathrm{S}^{(1)} = i\int \mathrm{L}'(x)\, d^4x, \tag{54.15}$$

where $L'(x) = j_\mu(x)\, A_\mu(x)$ is the part of the Lagrangian which determines the interaction between the electromagnetic and the electron-positron fields. We have seen previously that interactions between electromagnetic fields are described by a part of the scattering matrix $\mathrm{S}^{(4)}$. But the same interactions can, obviously, also be described by the equivalent first order matrix $\mathrm{S}^{(1)}_{\text{eff}}$, without taking into account any virtual electron-positron fields, if we interpret $L'$ in (54.15) as an additional term $L_1$ to the Lagrangian of the electromagnetic field. Thus, we have

$$\mathrm{S}^{(1)}_{\text{eff}} \equiv i\int L_1(x)\, d^4x = \mathrm{S}^{(4)}. \tag{54.16}$$

Such a treatment is obviously valid only for slowly varying electromagnetic fields to the investigation of which we now confine ourselves.

On utilizing relation (54.14) we can obtain the form of the photon-photon scattering tensor, or, more accurately, its dependence on the photon frequencies. In order to do this we compare the matrix

$$\begin{aligned} \mathrm{S}^{(4)} = -\frac{i}{12}\left(\frac{e^2}{4\pi}\right)^2 \int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4\, N\Bigg\{ &\frac{\mathrm{A}_\mu(k_1)}{(2\pi)^4}\cdot\frac{\mathrm{A}_\nu(k_2)}{(2\pi)^4} \\ &\times \frac{\mathrm{A}_\lambda(k_3)}{(2\pi)^4}\cdot\frac{\mathrm{A}_\sigma(k_4)}{(2\pi)^4}\Bigg\} I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)(2\pi)^4\, \delta(k_1+k_2+k_3+k_4) \end{aligned}$$

with the first order effective matrix

$$S_{\text{eff}}^{(1)} = i\int N\left[a\left(F_{ik}(x)\,F_{ki}(x)\right)^2 + b\left(F_{ik}(x)\,F_{kl}(x)\,F_{lm}(x)\,F_{mi}(x)\right)\right] d^4x.$$

On substituting into this expression

$$F_{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int F_{\mu\nu}(k)\,e^{ikx}\,d^4k,$$

where

$$F_{\mu\nu}(k) = i\left((k_\mu A_\nu(k) - k_\nu A_\mu(k)\right),$$

and on noting that

$$\int\left(F_{ik}(x)\,F_{ki}(x)\right)d^4x = \frac{1}{3}\int d^4k_1\int d^4k_2\int d^4k_3\int d^4k_4\,\frac{A_\mu(k_1)}{(2\pi)^4}\,\frac{A_\nu(k_2)}{(2\pi)^4}$$

$$\times\frac{A_\lambda(k_3)}{(2\pi)^4}\,\frac{A_\sigma(k_4)}{(2\pi)^4}\,S_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)\,(2\pi)^4\,\delta(k_1+k_2+k_3+k_4),$$

where $S_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ is the symmetrized tensor:

$$\begin{aligned}S_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = 4\{&k_{2_\mu}k_{1_\nu}k_{4\lambda}k_{3\sigma} + k_{3_\mu}k_{4_\nu}k_{1\lambda}k_{2\sigma} + k_{4_\mu}k_{3_\nu}k_{2\lambda}k_{1\sigma}\\ &-\delta_{\mu\nu}k_{4\lambda}k_{3\sigma}(k_1k_2) - \delta_{\lambda\sigma}k_{2\mu}k_{1_\nu}(k_3k_4) - \delta_{\mu\lambda}k_{4_\nu}k_{2\sigma}(k_1k_3)\\ &-\delta_{\nu\sigma}k_{3_\mu}k_{1\lambda}(k_2k_4) - \delta_{\mu\sigma}k_{3_\nu}k_{2\lambda}(k_1k_4) - \delta_{\nu\lambda}k_{1\sigma}k_{4_\mu}(k_2k_3)\\ &+\delta_{\mu\nu}\delta_{\lambda\sigma}(k_1k_2)(k_3k_4) + \delta_{\mu\lambda}\delta_{\nu\sigma}(k_1k_3)(k_2k_4) + \delta_{\mu\sigma}\delta_{\nu\lambda}(k_1k_4)(k_2k_3)\},\end{aligned}$$

and (54.17)

$$\int F_{ik}(x)\,F_{kl}(x)\,F_{lm}(x)\,F_{mi}(x)\,d^4x$$

$$= \frac{1}{3}\int d^4k_1\int d^4k_2\int d^4k_3\int d^4k_4\,(2\pi)^2\,\delta(k_1+k_2+k_3+k_4)$$

$$\times\frac{A_\mu(k_1)}{(2\pi)^4}\,\frac{A_\nu(k_2)}{(2\pi)^4}\,\frac{A_\lambda(k_3)}{(2\pi^4)}\,\frac{A_\sigma(k_4)}{(2\pi)^4}\,R_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4),$$

where $R_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ is the symmetrized tensor:

$$\begin{aligned}R_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = {}&k_{4_\mu}k_{1_\nu}k_{2\lambda}k_{3\sigma} + k_{2_\mu}k_{3_\nu}k_{4\lambda}k_{1\sigma} + k_{3_\mu}k_{1_\nu}k_{4\lambda}k_{2\sigma}\\ &+k_{2_\mu}k_{4_\nu}k_{1\lambda}k_{3\sigma} + k_{4_\mu}k_{3_\nu}k_{1\lambda}k_{2\sigma} + k_{3_\mu}k_{4_\nu}k_{2\lambda}k_{1\sigma} + \delta_{\mu\nu}[k_{2\lambda}k_{1\sigma}(k_3k_4)\\ &+k_{1\lambda}k_{2\sigma}(k_3k_4) - k_{2\lambda}k_{3\sigma}(k_1k_4) - k_{4\lambda}k_{1\sigma}(k_2k_3) - k_{4\lambda}k_{2\sigma}(k_1k_3)\\ &-k_{1\lambda}k_{3\sigma}(k_2k_4)] + \delta_{\lambda\sigma}[k_{4_\mu}k_{3_\nu}(k_1k_2) + k_{3_\mu}k_{4_\nu}(k_1k_2) - k_{4_\mu}k_{1_\nu}(k_2k_3)\\ &-k_{2_\mu}k_{3_\nu}(k_1k_4) - k_{3_\mu}k_{1_\nu}(k_2k_4) - k_{2_\mu}k_{4_\nu}(k_1k_3)] + \delta_{\mu\lambda}[k_{1_\nu}k_{3\sigma}(k_2k_4)\end{aligned}$$

$$
\begin{aligned}
&+k_{3_\nu}k_{1\sigma}(k_2k_4)-k_{4_\nu}k_{3\sigma}(k_1k_2)-k_{4_\nu}k_{1\sigma}(k_2k_3)-k_{1_\nu}k_{2\sigma}(k_3k_4)\\
&-k_{3_\nu}k_{2\sigma}(k_1k_4)]+\delta_{\nu\lambda}[k_{2_\mu}k_{3\sigma}(k_1k_4)+k_{3_\mu}k_{2\sigma}(k_1k_4)-k_{2_\mu}k_{1\sigma}(k_3k_4)\\
&-k_{4_\mu}k_{3\sigma}(k_1k_2)-k_{3_\mu}k_{1\sigma}(k_2k_4)-k_{4_\mu}k_{2\sigma}(k_1k_3)]+\delta_{\nu\sigma}[k_{2_\mu}k_{4\lambda}(k_1k_3)\\
&+k_{4_\mu}k_{2\lambda}(k_1k_3)-k_{3_\mu}k_{4\lambda}(k_1k_2)-k_{2_\mu}k_{1\lambda}(k_3k_4)-k_{4_\mu}k_{1\lambda}(k_2k_3)\\
&-k_{3_\mu}k_{2\lambda}(k_1k_4)]+\delta_{\mu\sigma}[k_{1_\nu}k_{4\lambda}(k_2k_3)+k_{4_\nu}k_{1\lambda}(k_2k_3)-k_{1_\nu}k_{2\lambda}(k_3k_4)\\
&-k_{3_\nu}k_{4\lambda}(k_1k_2)-k_{3_\nu}k_{1\lambda}(k_2k_4)-k_{4_\nu}k_{2\lambda}(k_1k_3)]+\delta_{\mu\nu}\,\delta_{\lambda\sigma}[(k_1k_4)\,(k_2k_3)\\
&+(k_1k_3)\,(k_2k_4)]+\delta_{\mu\lambda}\,\delta_{\nu\sigma}[(k_1k_2)\,(k_3k_4)+(k_1k_4)\,(k_2k_3)]\\
&\qquad\qquad\qquad+\delta_{\mu\sigma}\,\delta_{\nu\lambda}[(k_1k_2)\,(k_3k_4)+(k_1k_3)\,(k_2k_4)],
\end{aligned}
\tag{54.18}
$$

we obtain

$$
I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)=\frac{4}{\left(\dfrac{e^2}{4\pi}\right)^2}\{aS_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)+bR_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)\}. \tag{54.19}
$$

We now obtain the photon-photon scattering cross section.

On substituting (54.19) into (54.9) we obtain the expression for the photon-photon scattering cross section in the center of inertia system

$$
d\sigma=\frac{1}{(8\pi)^2}\left(\frac{e^2}{4\pi}\right)^4\frac{1}{\omega^2}|M|^2\,do, \tag{54.20}
$$

where

$$
M=e_\mu^{(\lambda_1)}e_\nu^{(\lambda_2)}\,e_\varkappa^{(\lambda_3)}e_\sigma^{(\lambda_4)}I_{\mu\nu\varkappa\sigma}(-k_1, -k_2, k_3, k_4)=\frac{4}{(e^2/4\pi)^2}(aS+bR), \tag{54.20'}
$$

$$
\begin{aligned}
S&=e_\mu^{(\lambda_1)}e_\nu^{(\lambda_2)}e_\varkappa^{(\lambda_3)}e_\sigma^{(\lambda_4)}\,S_{\mu\nu\varkappa\sigma}(-k_1, -k_2, k_3, k_4),\\
R&=e_\mu^{(\lambda_1)}e_\nu^{(\lambda_2)}e_\varkappa^{(\lambda_3)}e_\sigma^{(\lambda_4)}\,R_{\mu\nu\varkappa\sigma}(-k_1, -k_2, k_3, k_4)
\end{aligned}
$$

(the index $\lambda_i$ is used to denote the two different polarizations of the photon $k_i$).

In accordance with (54.17) and (54.18) we have

$$
\begin{aligned}
S=4\{&2(e^{(\lambda_1)}k_3)\,(e^{(\lambda_2)}k_3)\,(e^{(\lambda_3)}k_1)\,(e^{(\lambda_4)}k_1)\\
&-(e^{(\lambda_1)}e^{(\lambda_3)})\,(e^{(\lambda_2)}k_3)\,(e^{(\lambda_4)}k_1)\,(k_1k_2)+(e^{(\lambda_2)}e^{(\lambda_4)})\,(e^{(\lambda_1)}k_3)\,(e^{(\lambda_3)}k_1)\,[(k_2k_3)\\
&-(k_3k_4)]-(e^{(\lambda_2)}e^{(\lambda_3)})\,(e^{(\lambda_1)}k_3)\,(e^{(\lambda_4)}k_1)\,(k_2k_3)\\
&+(e^{(\lambda_1)}e^{(\lambda_2)})\,(e^{(\lambda_3)}e^{(\lambda_4)})\,(k_1k_2)\,(k_3k_4)\\
&+(e^{(\lambda_1)}e^{(\lambda_3)})\,(e^{(\lambda_2)}e^{(\lambda_4)})\,(k_1k_3)\,(k_2k_4)\\
&\qquad\qquad+(e^{(\lambda_1)}e^{(\lambda_4)})\,(e^{(\lambda_2)}e^{(\lambda_3)})\,(k_2k_4)\,(k_1k_3)\},
\end{aligned}
$$

$$
\begin{aligned}
R = {} & 2(e^{(\lambda_1)}k_3)\,(e^{(\lambda_2)}k_3)\,(e^{(\lambda_3)}k_1)\,(e^{(\lambda_4)}k_1) - 2(e^{(\lambda_1)}e^{(\lambda_2)})\,(e^{(\lambda_3)}k_1)\,(e^{(\lambda_4)}k_1)\,(k_3k_4)\\
& -2(e^{(\lambda_1)}e^{(\lambda_4)})\,(e^{(\lambda_1)}k_3)\,(e^{(\lambda_2)}k_3)\,(k_1k_2)\\
& +(e^{(\lambda_1)}e^{(\lambda_3)})\,(e^{(\lambda_2)}k_3)\,(e^{(\lambda_4)}k_3)\,[(k_3k_4)-(k_2k_3)]\\
& +(e^{(\lambda_1)}e^{(\lambda_4)})\,(e^{(\lambda_3)}k_1)\,(e^{(\lambda_2)}k_3)\,[(k_2k_3)-2(k_3k_4)]\\
& +(e^{(\lambda_2)}e^{(\lambda_3)})\,(e^{(\lambda_1)}k_3)\,(e^{(\lambda_4)}k_1)\,[(k_2k_3)-2(k_3k_4)]\\
& +(e^{(\lambda_2)}e^{(\lambda_4)})\,(e^{(\lambda_1)}k_3)\,(e^{(\lambda_3)}k_1)\,[(k_2k_3)+(k_3k_4)]\\
& +(e^{(\lambda_1)}e^{(\lambda_2)})\,(e^{(\lambda_3)}e^{(\lambda_4)})\,[(k_1k_4)\,(k_2k_3)+(k_1k_3)\,(k_2k_4)]\\
& +(e^{(\lambda_3)}e^{(\lambda_1)})\,(e^{(\lambda_2)}e^{(\lambda_4)})\,[(k_1k_2)\,(k_3k_4)+(k_1k_4)\,(k_2k_3)]\\
& \qquad\qquad +(e^{(\lambda_1)}e^{(\lambda_4)})\,(e^{(\lambda_3)}e^{(\lambda_4)})\,[(k_1k_2)\,(k_3k_4)+(k_1k_3)\,(k_2k_3)]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M = {} & \frac{4\omega^4}{\left(\dfrac{e^2}{4\pi}\right)^2}\,\{(8a+2b)\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_3)}\boldsymbol{n})\,(\boldsymbol{e}^{(\lambda_4)}\boldsymbol{n})\\
& +4b[(e^{(\lambda_1)}e^{(\lambda_2)})\,(\boldsymbol{e}^{(\lambda_3)}\boldsymbol{n})\,(\boldsymbol{e}^{(\lambda_4)}\boldsymbol{n})+(e^{(\lambda_3)}e^{(\lambda_4)})\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{n}')]\\
& +[4a(1-\cos\theta)-b(3+\cos\theta)]\,[(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{e}^{(\lambda_3)})\,(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_4)}\boldsymbol{n})\\
& +(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{e}^{(\lambda_4)})\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_3)}\boldsymbol{n})]\\
& +[4a(1+\cos\theta)-b(3-\cos\theta)]\,[(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{e}^{(\lambda_4)})\,(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_3)}\boldsymbol{n})\\
& +(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{e}^{(\lambda_3)})\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{n}')\,(\boldsymbol{e}^{(\lambda_4)}\boldsymbol{n})]+2[8a+b(1+\cos^2\theta)]\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{e}^{(\lambda_2)})\,(\boldsymbol{e}^{(\lambda_3)}\boldsymbol{e}^{(\lambda_4)})\\
& +[4a(1+\cos\theta)^2+b(5-2\cos\theta+\cos^2\theta)]\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{e}^{(\lambda_4)})\,(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{e}^{(\lambda_3)})\\
& +[4a(1-\cos\theta)^2+b(5+2\cos\theta+\cos^2\theta)]\,(\boldsymbol{e}^{(\lambda_1)}\boldsymbol{e}^{(\lambda_3)})\,(\boldsymbol{e}^{(\lambda_2)}\boldsymbol{e}^{(\lambda_4)})\},
\end{aligned}
\tag{54.21}
$$

where $\boldsymbol{n}$ and $\boldsymbol{n}'$ are unit vectors in the directions $\boldsymbol{k}$ and $\boldsymbol{k}'$ and $\theta$ is the scattering angle $\cos\theta = \boldsymbol{n}\cdot\boldsymbol{n}'$.

On substituting this expression into (54.20) and on averaging over the photon polarizations we obtain

$$
d\sigma = \frac{1}{(2\pi)^2}\,(48a^2+40ab+11b^2)\,\omega^4(3+\cos^2\theta)^2\,do. \tag{54.22}
$$

On noting that $\alpha = 4a+2b$ and $\beta = 4b$ we can rewrite this expression in the form

$$
d\sigma = \frac{1}{(8\pi)^2}(48\alpha^2-8\alpha\beta+3\beta^2)\,\omega^1(3+\cos^2\theta)\,do. \tag{54.22'}
$$

We see that the frequency and the angular dependence of the photon-photon scattering cross section for $\omega \ll m$ does not depend on specific values of the constants $a$ and $b$, while the magnitude of the cross section is determined by these constants.

Before evaluating the constants $a$ and $b$ we shall give the values of $M$ for different photon polarizations.

We denote by $e_{\parallel}^{(i)}$ the polarization of the photon $k_i$, lying in the scattering plane $(\boldsymbol{n}, \boldsymbol{n}')$, and by $e_{\perp}^{(i)}$ the polarization perpendicular to the plane $(\boldsymbol{n}, \boldsymbol{n}')$. On utilizing formula (54.21) for $M$ we can easily show that

$$
\begin{aligned}
M(e_{\perp}^{(1)}, e_{\perp}^{(2)}, e_{\perp}^{(3)}, e_{\perp}^{(4)}) &= M(e_{\parallel}^{(1)}, e_{\parallel}^{(2)}, e_{\parallel}^{(3)}, e_{\parallel}^{(4)}) \\
&= \frac{16}{\left(\dfrac{e^2}{4\pi}\right)^2}(2a+b)(3+\cos^2\theta), \\
M(e_{\perp}^{(1)}, e_{\perp}^{(2)}, e_{\parallel}^{(3)}, e_{\parallel}^{(4)}) &= M(e_{\parallel}^{(1)}, e_{\parallel}^{(2)}, e_{\perp}^{(3)}, e_{\perp}^{(4)}) \\
&= \frac{8}{\left(\dfrac{e^2}{4\pi}\right)^2}(8a+3b-b\cos^2\theta),
\end{aligned}
\qquad (54.23)
$$

$$
\begin{aligned}
&M(e_{\perp}^{(1)}, e_{\parallel}^{(2)}, e_{\perp}^{(3)}, e_{\parallel}^{(4)}) = M(e_{\parallel}^{(1)}, e_{\perp}^{(2)}, e_{\parallel}^{(3)}, e_{\perp}^{(4)}) \\
&= -\frac{16a}{\left(\dfrac{e^2}{4\pi}\right)^2}(1-2\cos\theta+\cos^2\theta)+\frac{4b}{\left(\dfrac{e^2}{4\pi}\right)^2}(3+6\cos\theta-\cos^2\theta), \\
&M(e_{\perp}^{(1)}, e_{\parallel}^{(2)}, e_{\parallel}^{(3)}, e_{\perp}^{(4)} = M(e_{\parallel}^{(1)}, e_{\perp}^{(2)}, e_{\perp}^{(3)}, e_{\parallel}^{(4)}) \\
&= -\frac{16a}{\left(\dfrac{e^2}{4\pi}\right)^2}(1+2\cos\theta+\cos^2\theta)+\frac{4b}{\left(\dfrac{e^2}{4\pi}\right)^2}(3-6\cos\theta-\cos^2\theta).
\end{aligned}
$$

We now proceed to evaluate the coefficients $a$ and $b$. We utilize the formula

$$
\begin{aligned}
\frac{1}{ABCD} &= 3!\int_0^1 dy_1 \int_0^{1-y_1} dy_2 \int_0^{1-y_1-y_2} dy_3 \big(Ay_1+By_2+Cy_3+D(1-y_1-y_2-y_3)\big)^{-4} \\
&= 3! \int\limits_{y_1+y_2+y_3+y_4=1,\ y_i \geqslant 0} d^4y(Ay_1+By_2+Cy_4+Dy_4)^{-4},
\end{aligned}
$$

where $d^4y = dy_1\,dy_2\,dy_3\,dy_4$, and we set in it

$$A = p^2+m^2, \qquad B = (p-k_2)^2+m^2,$$

$$C = (p-k_2-k_3)^2+m^2, \qquad D = (p+k_1)^2+m^2,$$

$$Ay_1+By_2+Cy_3+Dy_4 = [p-k_2y_2-(k_2+k_3)y_3+k_1y_4]^2+m^2f_1,$$

$$f_1 = 1+2\frac{k_2k_3}{m^2}y_1y_3+2\frac{k_2k_4}{m^2}y_2y_4.$$

With the aid of this formula we can write the tensor $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ in the form

$$T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = \frac{1}{i\pi^2}\int d^4y\int \frac{d^4q}{(q^2+m^2f_1)^4}\mathrm{Sp}\{\gamma_\mu[i\hat{q}+i\hat{l}-m]\gamma_\nu$$

$$\times [i\hat{q}+i\hat{l}-i\hat{k}_2-m]\gamma_\lambda[i\hat{q}+i\hat{l}-i\hat{k}_2-i\hat{k}_3-m]\gamma_\sigma[i\hat{q}+i\hat{l}+i\hat{k}_1-m]\}, \qquad (54.24)$$

where

$$l = k_2y_2+(k_2+k_3)y_3-k_1y_4.$$

In $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ we pick out the terms containing $\delta_{\mu\nu}\,\delta_{\lambda\sigma}$, $\delta_{\mu\lambda}\,\delta_{\nu\sigma}$, $\delta_{\mu\sigma}\,\delta_{\nu\lambda}$. We can show that the sum of these terms has the following form:

$$X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = \frac{24}{\pi^2 i}\int d^4y\int \frac{d^4q}{(q^2+m^2f_1)^4}\Big\{\frac{1}{3}(\delta_{\mu\nu}\,\delta_{\lambda\sigma}$$

$$+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-2\delta_{\mu\lambda}\,\delta_{\nu\sigma})[(q^2+m^2f_1)^2+(q^2+m^2f_1)m^2f_1]$$

$$+\frac{1}{3}(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}+\delta_{\mu\lambda}\,\delta_{\nu\sigma})m^2f_1+m^2(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-\delta_{\mu\lambda}\,\delta_{\nu\sigma})$$

$$\times [(q^2+m^2f_1)(5-5f_1+6(\alpha-\beta)y_2-2\alpha-\beta)]+m^2[4(1-f_1)^2$$

$$+8(1-f_1)(\alpha-\beta)y_2-8(1-f_1)\alpha-4(1-f_1)\beta+8\alpha^2y_1y_3+8\beta^2y_2y_4$$

$$+2f_1(\alpha-\beta)y_2+f_1(1-f_1)-2\alpha f_1-\beta f_1]+2\,\delta_{\mu\nu}\,\delta_{\sigma\lambda}q^2m^2[\beta-2\alpha y^2-1+f_1]$$

$$+2\,\delta_{\mu\sigma}\,\delta_{\nu\lambda}q^2m^2(2\beta y_2-1+f_1)-2\,\delta_{\mu\lambda}\,\delta_{\nu\sigma}q^2\beta\Big\}, \qquad (54.24')$$

where

$$\alpha = -\frac{1}{2}\frac{k_2k_3}{m^2}, \qquad \beta = -\frac{1}{2}\frac{k_3k_4}{m^2},$$

$$\gamma = -\frac{1}{2}\frac{k_2k_4}{m^2}, \qquad f_1 = 1-4\alpha y_1y_3-4\beta y_2y_4.$$

The regularized value of $X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ is equal to

$$\begin{aligned}
&X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)-X_{\mu\nu\lambda\sigma}(0, 0, 0, 0)\\
&= 4(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-\delta_{\mu\lambda}\,\delta_{\nu\sigma}\int d^4y\Bigg\{\frac{3}{f_1}[5(1-f_1)+6(\alpha-\beta)y_3-2\alpha-\beta]\\
&\quad+\frac{1}{f_1^2}[4(1-f_1)^2+10(1-f_1)(\alpha-\beta)y_2-2\alpha-\beta-2(\alpha-\beta)y_2-6(1-f_1)\alpha\\
&\quad-3(1-f_1)\beta+8\alpha^2y_1y_3+8\beta^2y_2y_4]\Bigg\}+16\delta_{\mu\nu}\,\delta_{\lambda\sigma}\int\frac{d^4y}{f_1}(\beta-2\alpha y_2-1+f_1)\\
&\quad+16\delta_{\mu\sigma}\,\delta_{\nu\lambda}\int\frac{d^4y}{f_1}(2\beta y_2-1+f_1)-16\delta_{\mu\lambda}\,\delta_{\nu\sigma}\beta\int\frac{d^4y}{f_1}\\
&\quad-8(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-2\delta_{\mu\lambda}\,\delta_{\nu\sigma})\int d^4y\ln f_1.
\end{aligned}$$

Since we are interested in the low frequency case, we can take

$$\begin{gathered}
\alpha\ll 1,\quad |\beta|\ll 1,\quad |\gamma|\ll 1,\\
\ln f_1=-(1-f_1)-\frac{(1-f_1)^2}{2},\\
\frac{1}{f_1}=1+(1-f_1),\\
\frac{1}{f_1^2}=1+2(1-f_1),
\end{gathered}$$

as a result of which $X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)-X_{\mu\nu\lambda\sigma}(0, 0, 0, 0)$ assumes the form

$$\begin{aligned}
&X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)-X_{\mu\nu\lambda\sigma}(0, 0, 0, 0)\\
&\quad= 4(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-\delta_{\mu\lambda}\,\delta_{\nu\sigma})\int d^4y[14(1-f_1)-4\alpha-8\beta\\
&\quad+13(1-f_1)^2+24(1-f_1)y_2(\alpha-\beta)-10(1-f_1)(\alpha+\beta)+8\alpha^2y_1y_3\\
&\quad+8\beta^2y_2y_4]+16\delta_{\mu\nu}\,\delta_{\lambda\sigma}\int d^4y\Bigg[\beta-\frac{\alpha}{2}-1+f_1+\beta(1-f_1)\\
&\quad-2\alpha(1-f_1)y_2-(1-f_1)^2\Bigg]+16\delta_{\mu\sigma}\,\delta_{\nu\lambda}\int d^4y\Bigg[\frac{\beta}{2}-(1-f_1)\\
&\quad+2\beta(1-f_1)y_2-(1-f_1)^2\Bigg]-16\delta_{\mu\lambda}\,\delta_{\nu\sigma}\int d^4y\beta\big(1+(1-f_1)\big)\\
&\quad+8(\delta_{\mu\nu}\,\delta_{\lambda\sigma}+\delta_{\mu\sigma}\,\delta_{\nu\lambda}-2\delta_{\mu\lambda}\,\delta_{\nu\sigma})\int d^4y\left[(1-f_1)+\frac{1}{2}(1-f_1)^2\right].
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)-X_{\mu\nu\lambda\sigma}(0, 0, 0, 0)+X_{\mu\nu\sigma\lambda}(k_1, k_2, k_4, k_3)\\
&\quad -X_{\mu\nu\sigma\lambda}(0, 0, 0, 0)+X_{\mu\lambda\nu\sigma}(k_1, k_3, k_2, k_4)-X_{\mu\lambda\nu\sigma}(0, 0, 0, 0)\\
&\quad =24\delta_{\mu\nu}\delta_{\lambda\sigma}\int d^4y[2(1-f_1)+2(1-f_2)-3(1-f_3)-\beta]\\
&\quad +24\delta_{\mu\sigma}\delta_{\nu\lambda}\int d^4y[2(1-f_1)-3(1-f_2)+2(1-f_3)-\alpha]\\
&\quad +24\delta_{\mu\lambda}\delta_{\nu\sigma}\int d^4y[-3(1-f_1)+2(1-f_2)+2(1-f_3)-\gamma]\\
&\quad +4\delta_{\mu\lambda}\delta_{\lambda\sigma}\int d^4y[10(1-f_1)^2+10(1-f_2)^2-15(1-f_3)^2+6(1-f_3)\gamma\\
&\quad -6(1-f_1)\beta-6(1-f_2)\beta-10(1-f_1)\alpha-10(1-f_2)\gamma+10(1-f_3)\alpha\\
&\quad +24(1-f_1)(\alpha-\beta)y_2+24(1-f_2)(\gamma-\beta)y_2-24(1-f_3)(\alpha-\gamma)y_2\\
&\quad -8\alpha(1-f_1)y_2-8\gamma(1-f_2)y_2+8\alpha^2y_1y_3+8\beta^2y_2y_4]\\
&\quad +4\delta_{\mu\sigma}\delta_{\nu\lambda}\int d^4y[10(1-f_1)^2-15(1-f_2)^2+10(1-f_3)^2+8\beta(1-f_1)y_2\\
&\quad +8\gamma(1-f_3)y_2+6(1-f_2)\beta-10(1-f_1)(\alpha+\beta)-10(1-f_2)(\alpha+\beta)\\
&\quad +24(1-f_3)(\alpha-\gamma)y_2-24(1-f_2)(\gamma-\beta)y^2+16\alpha^2y_1y_3\\
&\quad +10(1-f_3)\gamma+24(1-f_1)(\alpha-\beta)y_2]+4\delta_{\mu\lambda}\delta_{\nu\sigma}\int d^4y[-15(1-f_1)^2\\
&\quad +10(1-f_2)^2+10(1-f_3)^2+4(1-f_3)\gamma-4\beta(1-f_1)\\
&\quad +10(1-f_1)(\alpha+\beta)-10(1-f_2)(\gamma+\beta)-10(1-f_3)(\alpha+\gamma)\\
&\quad -24(1-f_1)(\alpha-\beta)y_2+24(1-f_2)(\gamma-\beta)y_2+24(1-f_3)(\alpha-\gamma)y_2\\
&\quad -8\alpha(1-f_3)y_2+8\beta(1-f_2)y_2].
\end{aligned}$$

The integrals appearing in this expression can be evaluated with the aid of

$$\int d^4y=\frac{1}{6}, \qquad \int y_i\, d^4y=\frac{1}{24},$$

$$\int y_iy_k\, d^4y=\frac{1}{120}, \qquad i\neq k, \qquad \int y_i^2\, d^4y=\frac{1}{60}.$$

On utilizing these formulas we finally obtain

$$\begin{aligned}
&X_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)-X_{\mu\nu\lambda\sigma}(0, 0, 0, 0)+X_{\mu\nu\sigma\lambda}(k_1, k_2, k_4, k_3)\\
&\quad -X_{\mu\nu\sigma\lambda}(0, 0, 0, 0)+X_{\mu\lambda\nu\sigma}(k_1, k_3, k_2, k_4)-X_{\mu\lambda\nu\sigma}(0, 0, 0, 0)\\
&\quad =\frac{4}{90m^4}\{\delta_{\mu\nu}\delta_{\lambda\sigma}[14(k_2k_3)^2+14(k_2k_3)(k_3k_4)-3(k_3k_4)^2]\\
&\quad +\delta_{\mu\sigma}\delta_{\nu\lambda}[14(k_3k_4)^2+14(k_2k_3)(k_3k_4)-3(k_2k_3)^2]\\
&\qquad -\delta_{\mu\lambda}\delta_{\nu\sigma}[3(k_2k_3)^2+3(k_3k_4)^2+20(k_2k_3)(k_3k_4)]\}. \qquad (54.25)
\end{aligned}$$

A comparison of (54.25) with (54.19) yields

$$4a+b=-\left(\frac{e^2}{4\pi}\right)^2\frac{1}{30m^4},\qquad 2b=\left(\frac{e^2}{4\pi}\right)^2\frac{7}{45m^4},$$

where it follows that

$$a=-\frac{5}{180}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{m^4},\quad b=\frac{14}{180}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{m^4}. \tag{54.26}$$

On substituting these values of $a$ and $b$ into (54.14) we obtain the expression for the radiation corrections to the Lagrangian density of the electromagnetic field of order $(e^2/4\pi)^2$ (56)

$$\begin{aligned} L_1 &= \frac{1}{180}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{m^4}[14F_{ik}F_{kl}F_{lm}F_{mi}-5(F_{ik}F_{ki})^2 \\ &= \frac{2}{45}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{m^4}[(\mathbf{E}^2-\mathbf{H}^2)^2+7(\mathbf{EH})^2]. \end{aligned} \tag{54.27}$$

The differential photon scattering cross section will in this case be given in accordance with (54.22) by

$$d\sigma=\frac{1}{(2\pi)^2}\frac{139}{90^2}\alpha^4\frac{1}{m^2}\left(\frac{\omega}{m}\right)^6(3+\cos^2\theta)^2\,do. \tag{54.28}$$

The total cross section is equal to

$$\sigma=\alpha^4\frac{1}{\pi}\frac{139}{90^2}\cdot\frac{56}{5}\frac{1}{m^2}\left(\frac{\omega}{m}\right)^6,\quad \text{where}\quad \omega\leqslant m. \tag{54.28'}$$

If we know the Lagrangian density we can obtain the electromagnetic energy density

$$w=\mathbf{E}\frac{\partial L}{\partial\mathbf{E}}-L=\frac{1}{2}(\mathbf{E}^2+\mathbf{H}^2)+w', \tag{54.29}$$

where

$$w'=\alpha(\mathbf{E}^2-\mathbf{H}^2)(3\mathbf{E}^2+\mathbf{H}^2)+\beta(\mathbf{EH})^2.$$

The second term gives the radiation corrections to the classical energy density

$$w_0=\tfrac{1}{2}(\mathbf{E}^2+\mathbf{H}^2).$$

The nonlinear electrodynamic effects can be described with the aid of a field-dependent dielectric permittivity and magnetic permeability of the vacuum. In order to obtain these quantities we determine the electric displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$:

$$\boldsymbol{D} = \frac{\partial L}{\partial \boldsymbol{E}}, \qquad \boldsymbol{B} = -\frac{\delta L}{\partial \boldsymbol{H}}.$$

On substituting (54.27) into these expressions we obtain

$$\begin{aligned} D_i &= E_i + \frac{1}{45\pi}\frac{e^4}{4\pi}\frac{1}{m^4}\{2(E^2-H^2)E_i + 7H_i(\boldsymbol{EH})\}, \\ B_i &= H_i + \frac{1}{45}\frac{e^4}{4\pi}\frac{1}{m^4}\{2(E^2-H^2)H_i - 7E_i(\boldsymbol{EH})\}. \end{aligned} \tag{54.30}$$

In macroscopic electrodynamics $\boldsymbol{D}$ and $\boldsymbol{B}$ are related to $\boldsymbol{E}$ and $\boldsymbol{H}$ by the following expressions

$$D_i = \varepsilon_{ik}E_k, \qquad B_i = \mu_{ik}H_k.$$

A comparison of these formulas with (54.30) shows that the dielectric permittivity and the magnetic permeability tensors of the vacuum in the case of weak slowly varying fields are equal to

$$\begin{aligned} \varepsilon_{ik} &= \delta_{ik} + \frac{1}{45}\frac{e^4}{4\pi^2}\frac{1}{m^4}\{2(E^2-H^2)\delta_{ik} + 7H_iH_k\}, \\ \mu_{ik} &= \delta_{ik} + \frac{1}{45}\frac{e^4}{4\pi^2}\frac{1}{m^4}\{2(E^2-H^2)\delta_{ik} - 7E_iE_k\}. \end{aligned} \tag{54.30'}$$

### 54.4. *Exact Expression for the Lagrangian of the Electromagnetic Field*

In the preceding subsection we have obtained the radiation corrections to the Lagrangian of the electromagnetic field proportional to the fourth power of the fields. We now determine the Lagrangian for any arbitrarily strong electromagnetic field which satisfies the single condition (54.12).

For the solution of this problem it is convenient to employ the concept of the electron-positron vacuum as a system of electrons filling the negative energy levels (cf. § 18). We shall in the future call such electrons vacuum electrons.

Our starting point will be the assumption that the addition $w'$ to the classical energy density of the electromagnetic field

$$w_0 = \tfrac{1}{2}(E^2+H^2),$$

due to the existence of the electron-positron vacuum, coincides with the energy density of the vacuum electrons, i.e., of electrons filling the negative energy levels, after subtracting their potential energy in the external electric field. In other words, if $(A, i\varphi)$ is the four-potential of the electromagnetic field and $\psi_p^{(-)}$ is the set of wave functions describing the states of a vacuum electron in such a field

$$i\frac{\partial \psi_p^{(-)}}{\partial t} = \left(\boldsymbol{\alpha}, \frac{1}{i}\nabla + eA\right)\psi_p^{(-)} + (\beta m + e\varphi)\psi_p^{(-)},$$

then

$$w' = i\sum_p \left(\psi_p^{(-)*}, \frac{\partial}{\partial t}\psi_p^{(-)}\right) - \sum_p (\psi_p^{(-)*}, e\varphi\psi_p^{(-)}), \tag{54.31}$$

where $(\psi, \chi) \equiv \psi_\lambda \chi_\lambda$ ($\lambda$ is the spinor index) and the summation is taken over all the individual states of the vacuum electron $p$ (the wave functions $\psi_p^{(-)}$ are, naturally, assumed to be normalized per unit volume). The first term in this expression represents the total energy density of the vacuum electrons, which we denote by

$$w_m = i\sum_p \left(\psi_p^{(-)*}, \frac{\partial}{\partial t}\psi_p^{(-)}\right). \tag{54.32}$$

We now show that the second term—the potential energy density of the vacuum electrons—is related to the first expression by

$$\sum_p (\psi_p^{(-)*}, e\varphi\psi_p^{(-)}) = \frac{\partial w_m}{\partial E}, \tag{54.33}$$

where $E$ is the electric field intensity.

We first note that if the quantum mechanical energy operator $H$ depends on a parameter $\lambda$, then as a result of an infinitesimal adiabatic change of $\lambda$ by $d\lambda$ the energy eigenvalue $\varepsilon_p = H_{pp}$ undergoes a change equal to

$$d\varepsilon_p = \left(\frac{\partial H}{\partial \lambda}\right)_{pp} d\lambda.$$

This relation is obtained immediately if we differentiate the equation

$$\int \left(\psi_p^*(\lambda), \left(\mathrm{H}(\lambda) - \varepsilon_p(\lambda)\right)\psi_p(\lambda)\right) dr = 0$$

with respect to $\lambda$.

In particular, if $H = H_0 + \lambda e\varphi$, ($\varphi$ does not depend on $\lambda$), then

$$\lambda(e\varphi)_{pp} = \lambda\left(\frac{\partial H}{\partial \lambda}\right)_{pp}\cdot$$

In other words, if the scalar potential contains a constant factor $\lambda$, then the following relation holds:

$$\lambda\int\sum_p(\psi_p^{(-)*}, e\varphi\psi_p^{(-)})\,d\boldsymbol{r} = \lambda\frac{\partial}{\partial\lambda}\int w_m\,d\boldsymbol{r},$$

where the integration is taken over the whole volume. In the case of constant fields in which we are interested we can regard the electric field $E$ as the factor $\lambda$ and, moreover, we can go over from the total energy to the energy density, i.e., we can omit the integral sign; we thus arrive at relation (54.33).

It follows from (54.31) that

$$w' = w_m - E\frac{\partial w_m}{\partial E}\cdot \qquad (54.34)$$

On comparing this equation with $w = E(\partial L/\partial E) - L$, we conclude that the additional term $L_1$ in the Lagrangian density of the electromagnetic field compared to the classical Lagrangian $L_0 = \frac{1}{2}(E^2 - H^2)$ agrees, except for its sign, with the total energy density of the electron-positron vacuum in the presence of an external field:

$$L_1 = -w_m. \qquad (54.35)$$

Since, on the other hand, $L_1$ is a function of only the two independent invariants $\boldsymbol{E}^2 - \boldsymbol{H}^2$ and $(\boldsymbol{EH})^2$, the problem can be reduced to finding the energy $w_m$ in special fields, viz., it is sufficient to obtain the value of $w_m$ in a constant and homogeneous magnetic field $\boldsymbol{H}(0, 0, H)$ and an electrostatic field parallel to it with the potential

$$\varphi = \varphi_0 e^{igz} + \varphi_0^* e^{-igz}, \qquad (54.36)$$

where $\varphi_0$ and $g$ are constants (for small values of $g$ we obtain a homogeneous field of intensity $E = ig\ (\varphi_0 - \varphi_0^*)$.

This choice of electric field eliminates the difficulty associated with the possibility of pair production in a constant arbitrarily weak electric field extending over all space. It is clear that if the difference of potentials of the form (54.36) between any two points does not exceed the quantity $2mc^2$, then the field (54.36) will not produce any pairs, We obtain first of all the value of $w_m$ in a constant and homogeneous magnetic field $\boldsymbol{H}(0, 0, H)$. In future we shall denote this quantity by $w_m^{(0)}$.

In accordance with subsection 12.5, the possible values of the energy of a vacuum electron are equal to

$$\varepsilon_{n\mu}(p_z) = -\sqrt{(m^2+eH(2n-\mu+1)+p_z^2)},$$

where $n = 0, 1, 2, \dots, \mu = \pm 1$ and $p_z$ is the component of the electron momentum along the magnetic field; the corresponding wave functions have the form (cf. 12.13))

$$\psi^{(-)}_{n\mu p_y p_z}(\boldsymbol{r}) = \frac{e^{ip_y y + ip_z z}}{\sqrt{2\varepsilon_{n_\mu}(p_z)\big(\varepsilon_{n_\mu}(p_z)-m\big)}} \begin{pmatrix} \big(-\varepsilon_{n_\mu}(p_z)+m\big)v_n(\xi) \\ 0 \\ p_z v_n(\xi), \\ -i(2eH_n)^{\frac{1}{2}} v_{n-1}(\xi) \end{pmatrix}, \qquad \mu = 1,$$

$$\psi^{(-)}_{n\mu p_z p_y}(\boldsymbol{r}) = \frac{e^{ip_y y + ip_z z}}{\sqrt{2\varepsilon_{n_\mu}(p_z)\big(\varepsilon_{n_\mu}(p_z)-m\big)}} \begin{pmatrix} 0 \\ \big(-\varepsilon_{n_\mu}(p_z)+m\big)v_n(\xi) \\ i(2eH_{(n+1)})^{\frac{1}{2}} v_{n+1}(\xi) \\ -p_z v_n(\xi) \end{pmatrix}, \quad \mu = -1,$$

where

$$\xi = \sqrt{eH}\left(x + \frac{p_y}{eH}\right).$$

They satisfy the normalization conditions

$$\int \psi^{(-)*}_{n\mu p_y y_z}(\boldsymbol{r})\, \psi^{(-)}_{n'\mu' p_y p_z}(\boldsymbol{r})\, d\boldsymbol{r} = \delta_{nn'}\delta_{\mu\mu'},$$

$$\int_{-\infty}^{\infty} \psi^{(-)*}_{n\mu p_y p_z}(\boldsymbol{r})\, \psi^{(-)*}_{n\mu p_y p_z}(\boldsymbol{r})\, dp_y = eH$$

(it is assumed that the normalizing volume is equal to unity). Therefore, the energy density of the vacuum can be written in the form

$$w_m^{(0)} = -\sum_{n=0}^{\infty} \sum_{\mu=\pm 1} \int \frac{dp_y\, dp_z}{(2\pi)^2} \left(\psi^{(-)*}_{n\mu p_y p_z}(\boldsymbol{r}), \psi^{(-)}_{n\mu p_y p_z}(\boldsymbol{r})\right) \varepsilon_{n_\mu}(p_z)$$

$$= -\frac{eH}{(2\pi)^2} \sum_{\mu=\pm 1} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \varepsilon_{n_\mu}(p)\, dp, \tag{54.37}$$

where $p \equiv p_z$.

On noting that $\varepsilon_{n,-1}(p) = \varepsilon_{n+1,+1}(p)$, we rewrite (54.37) in the form

$$w_m^{(0)} = -\frac{eH}{(2\pi)^2} \int_{-\infty}^{\infty} dp \left\{ \varepsilon_0(p) + 2 \sum_{n=1}^{\infty} \varepsilon_n(p) \right\}, \qquad (54.37')$$

where

$$\varepsilon_n(p) = \sqrt{p^2+m^2+2eHn}, \qquad n = 0, 1, 2, \ldots .$$

On using Euler's summation formula

$$\frac{1}{2}F(0) + \sum_{k=1}^{N} F(kb) - \frac{1}{2}F(bN)$$

$$= \frac{1}{b}\left\{ \int_0^{bN} F(x)\,dx + \sum_{m=1}^{\infty} \frac{B_m b^{2m}}{(2m)!} [F^{(2m-1)}(bN) - F^{(2m-1)}(0)] \right\}, \qquad (54.38)$$

where $B_m$ are the Bernoulli numbers: $B_1 = \frac{1}{6}$, $B_2 = -\frac{1}{30}$, $B_3 = \frac{1}{42}$, ..., and on setting $b = 2eH$ in this formula, we obtain

$$w_m^{(0)} = -\frac{1}{(2\pi)^2} \int dp \left[ \int_{-\infty}^{\infty} F(x)\,dx - \sum_{m=1}^{\infty} b^{2m} \frac{B_m}{(2m)!} F^{(2m-1)}(0) \right],$$

where

$$F(x) = \sqrt{p^2+m^2+x}.$$

We must now regularize the quantity $w_m^{(0)}$. This regularization can obviously be carried out in the following manner. We must, first of all, neglect the integral $\int_0^{\infty} F(x)\,dx$ since it does not contain the magnetic field intensity and therefore represents the energy of the free vacuum electrons, and, moreover, we must neglect the first term with $m = 1$ in the infinite sum since it leads to an energy proportional to $H^2$, which has already been included in the unperturbed field energy $w_0$. In neglecting this term we are essentially carrying out a renormalization of the field intensity associated with the renormalization of the charge.

Thus, the regularized expression for the energy density $w_m^{(0)}$ has the form

$$w_{mR}^{(0)} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \sum_{n=2}^{\infty} b^{2n} \frac{B_n}{(2n)!} F^{(2n-1)}(0)\, dp .$$

On noting that

$$F^{(2n-1)}(0) = \frac{\Gamma(2n-\frac{3}{2})}{2\sqrt{\pi}} \frac{1}{(p^2+m^2)^{\frac{4n-3}{2}}}$$

and that

$$\int_{-\infty}^{\infty} \frac{dp}{(p^2+m^2)^q} = \sqrt{\pi}\, m^{1-2q} \frac{\Gamma(q-\frac{1}{2})}{\Gamma(q)},$$

we obtain

$$w_{mR}^{(0)} = \frac{1}{2(\pi)^2} \sum_{n=2}^{\infty} b^{2n} B_n (m^2)^{2-2n} \frac{\Gamma(2n-2)}{2(2n)!}, \tag{54.39}$$

and since

$$\Gamma(2n-2) = \int_0^{\infty} d\eta\, e^{-\eta} \eta^{2n-3},$$

we have

$$w_{mR}^{(0)} = \frac{bm^2}{16\pi^2} \int_0^{\infty} \frac{d\eta}{\eta^2} e^{-\eta} \sum_{n=2}^{\infty} \frac{2^{2n} B_n}{(2n)!} \left( \frac{b\eta}{2m^2} \right)^{2n-1}.$$

On comparing this expansion with the expansion

$$\coth x = \frac{1}{x} + \frac{x}{3} + \sum_{k=2}^{\infty} \frac{2^{2k} B_k}{(2k)!} x^{2k-1}, \tag{54.39'}$$

we finally obtain

$$w_{mR}^{(0)} = \frac{m^4}{8\pi^2} \int_0^{\infty} \frac{d\eta}{\eta^3} e^{-\eta} \left\{ \eta H^* \coth(\eta H^*) - 1 - \frac{\eta^2 H^{*2}}{3} \right\}, \tag{54.40}$$

where

$$H^* = \frac{b}{2m^2} = \frac{eH}{m^2}.$$

Now in addition to the magnetic field we also turn our attention to the electrostatic field parallel to it, the potential for which is given by formula (54.36).

We expand $w_m$ into a series in powers of $E^2$ or, what amounts to the same thing, in powers of $\varphi^2$:

$$w_m = \sum_{k=1}^{\infty} w_m^{(2k)}. \tag{54.41}$$

It can be easily shown by repeating the arguments which lead to (54.37) that

$$w_m^{(2k)} = \frac{eH}{(2\pi)^2} \sum_{\mu=\pm 1} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp\varepsilon_{n\mu}^{(2k)}(p), \tag{54.41'}$$

where $\varepsilon_{n\mu}^{(2k)}(p)$ is the correction to the electron energy $\varepsilon_{n_\mu}(p)$ in the magnetic field proportional to $E^{2k}$.

We first of all consider $w_m^{(2)}$. By using perturbation theory we can show that

$$\int_{-\infty}^{\infty} dp\varepsilon_{n\mu}^{(2)}(p) = -e^2|\varphi_0|^2 \frac{g^2}{4} \int \frac{dpK^2}{(p^2+K^2)^{5/2}},$$

where

$$K^2 = m^2 + eH(2n-\mu+1),$$

and where we have expanded $\varphi$ into a series in powers of $g$ and have kept only the second order terms.

On setting in Euler's formula (54.38)

$$F(x) = -e^2|\varphi_0|^2 \frac{g^2}{4} \frac{m^2+x}{(p^2+m^2+x)^{5/2}},$$

we obtain

$$w_m^{(2)} = -\frac{1}{8\pi^2} \frac{e^2E^2}{12} \left[ \int_0^{\infty} \frac{dx}{m^2+x} - \sum_{k=1}^{\infty} b^{2k} \frac{B_k}{(2k)!} \left( \frac{d^{2k-1}}{dx^{2k-1}} \frac{1}{m^2+x} \right)_{x=0} \right],$$

where $E$ is the intensity of the electric field equal to $E^2 = 2g^2|\varphi_0|^2$.

Regularization of this expression reduces to neglecting the first term containing the integral

$$\int_0^{\infty} \frac{dx}{m^2+x}$$

and proportional to $E^2$. By neglecting this term we, in fact, carry out a renormalization of the charge, just as we have done earlier in evaluating the energy $w_m^{(0)}$ in the magnetic field.

By utilizing the expansion (54.39′) for $\coth x$ we finally obtain the following expression for the regularized value of $w_m^{(2)}$:

$$w_{mR}^{(2)} = -\frac{m^4}{8\pi^2}\frac{E^{*2}}{3}\int_0^\infty \frac{d\eta}{\eta}\{\eta H^* \coth(\eta H^*) - 1\}, \tag{54.42}$$

where $E^* = 2E/m^2$.

We now consider further terms in (54.41).

The assertion can be made that for small $g$ the quantity $\varepsilon_{n\mu}^{(r)}$ has the form

$$\varepsilon_{n\mu}^{(r)}(p) = g^r e^r |\varphi_0|^r G_r,$$

where $G_r$ is a function of $K$ and $P$. Therefore

$$w_m^{(r)} = \frac{1}{8\pi^2} g^r e^r |\varphi_0|^r \left[\int_0^\infty dx \int_{-\infty}^\infty G_r\,dp - \sum_{n=1}^\infty b^{2n}\frac{B_n}{(2n)!}\left(\frac{d^{2n-1}}{dx^{2n-1}}\int_{-\infty}^\infty G_r\,dp\right)\right]$$

and

$$w_m = \frac{1}{8\pi^2}\sum_{k=2}^\infty g^{2k}e^{2k}|\varphi_0|^{2k}\int_0^\infty dx\int_{-\infty}^\infty G_{2k}\,dp - \frac{1}{8\pi^2}\sum_{k=2}^\infty g^{2k}e^{2k}|\varphi_0|^{2k}$$

$$\times \sum_{n=1}^\infty b^{2n}\frac{B_n}{(2n)!}\left(\frac{d^{2n-1}}{dx^{2n-1}}\int_{-\infty}^\infty G_{2k}\,dp\right) + w_m^{(2)} + w_m^{(0)}. \tag{54.43}$$

But the integral of $G_r$ over $p$ must have the dimensions of the $(2r-2)$th power of the momentum and, therefore, can be only of the form

$$\int_{-\infty}^\infty G_r\,dp = \frac{f_r}{K^{2r-2}} = \frac{f_r}{(m^2+x)^{r-1}}, \tag{54.44}$$

where $f_r$ are dimensionless constants.

In order to determine $f_r$ we consider in (54.43) the term proportional to $E^2$ and not containing $\boldsymbol{H}$:

$$\int_0^\infty dx\int_{-\infty}^\infty G_r\,dp = \frac{f_r}{(r-2)m^{2r-4}}.$$

We compare it with the term proportional to $H^r$ in expression (54.39). Since $w'_m$ is a function only of $E^2-H^2$ and $(EH)^2$, the coefficient of $E^r$ must differ from the coefficient of $H^r$ by the factor $(-1)^{r/2}$. We already know this latter coefficient which is equal to $(2e)^{2r}B_r m^{4-4r}$ $\Gamma(2r-2)/2(2r)!$, so that

$$f_{2r}=(-1)^r 2^{3r+1}\frac{\Gamma(2r-1)}{(2r)!}B_r.$$

We now evaluate the first term in (54.43):

$$\frac{1}{8\pi^2}\sum_{k=2}^{\infty}(eE)^{2k}2^{2k}B_k(-1)^k(m^2)^{2-2k}\frac{\Gamma(2k-1)}{(2k)!}$$
$$=\frac{m^4}{8\pi^2}\int_0^{\infty}\frac{d\eta}{\eta^3}e^{-\eta}\left\{\eta E^*\cot\eta E^*-1+\frac{\eta^2E^{*2}}{3}\right\}. \qquad (54.45)$$

Further, we consider the second term in (54.43). On noting that

$$\frac{d^{2n-1}}{dx^{2n-1}}\left.\frac{f_k}{(m^2+x)^{k-1}}\right|_{x=0}=-f_k(m^2)^{2-k-2n}\frac{\Gamma(2n+k-2)}{(k-2)!},$$

we obtain

$$\frac{1}{8\pi^2}\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}b^{2n}\frac{B_n}{(2n)!}e^{2k}g^{2k}|\varphi_0|^{2k}\left(\frac{d^{2n-1}}{dx^{2n-1}}\int_{-\infty}^{\infty}G_{2k}\,dp\right)_{x=0}$$
$$=\frac{1}{8\pi^2}\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}b^{2n}\frac{B_n}{(2n)!}(eE)^{2k}\frac{B_k}{(2k)!}(-1)^{k+1}(m^2)^{2-2k-n}\Gamma(2n+2k-2)$$
$$=\frac{m^4}{8\pi^2}\int_0^{\infty}\frac{d\eta e^{-\eta}}{\eta^2}\sum_{k=2}^{\infty}2^{2k}\frac{B_k(-1)^{k+1}}{(2k)!}m^{-4k}\eta^{2k}(eE)^{2k}$$
$$\times H^*\sum_{n=1}^{\infty}2^{2n}(\eta H^*)^{2n-1}\frac{B_n}{(2n)!}=-\frac{m^4}{8\pi^2}\int_0^{\infty}\frac{d\eta}{\eta^3}$$
$$\times e^{-\eta}\left(\eta E^*\cot\eta E^*-1+\frac{\eta^2E^{*2}}{3}\right)(\eta H^*\coth\eta H^*-1). \qquad (54.45')$$

Therefore

$$\sum_{k=2}^{\infty}w_m^{(2k)}=\frac{m^4}{8\pi^2}\int^{\infty}\frac{d\eta}{\eta^3}e^{-\eta}\left\{\eta E^*\cot\eta E^*-1+\frac{\eta E^{*2}}{3}\right\}\eta H^*\coth\eta H^*.$$

On adding this expression to expressions (54.42) and (54.40) for $w_m^{(2)}$ and $w_m^{(0)}$, we obtain

$$w_m = \frac{m^4}{8\pi^2}\int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta}\{\eta E^* \cot\eta E^* \eta H^* \coth\eta H^* - 1 + \tfrac{1}{3}\eta^2(E^{*2}-H^{*2})\}. \tag{54.46}$$

On noting that

$$\cot\alpha \coth\beta = i\,\frac{\cos\sqrt{\beta^2-\alpha^2+2i\alpha\beta}+\cos\sqrt{\beta^2-\alpha^2-2i\alpha\beta}}{\cos\sqrt{\beta^2-\alpha^2+2i\alpha\beta}-\cos\sqrt{\beta^2-\alpha^2-2i\alpha\beta}},$$

and on utilizing (54.35), we obtain the final expression[1] for $L_1$

$$L_1 = \frac{e^2}{8\pi^2}\int_0 \frac{d\eta}{\eta^3} e^{-\eta}$$
$$\times\left\{i\eta(\boldsymbol{EH})\frac{\cos\left(\eta\sqrt{\boldsymbol{E}^{*2}-\boldsymbol{H}^{*2}+2i(\boldsymbol{E}^*\boldsymbol{H}^*)}\right)+\cos\left(\eta\sqrt{\boldsymbol{E}^{*2}-\boldsymbol{H}^{*2}-2i(\boldsymbol{E}^*\boldsymbol{H}^*)}\right)}{\cos\left(\eta\sqrt{\boldsymbol{E}^{*2}-\boldsymbol{H}^{*2}+2i(\boldsymbol{E}^*\boldsymbol{H}^*)}\right)-\cos\left(\eta\sqrt{\boldsymbol{E}^{*2}-\boldsymbol{H}^{*2}-2i(\boldsymbol{E}^*\boldsymbol{H}^*)}\right)}\right.$$
$$\left.+\frac{m^4}{e^2}+\frac{\eta^2}{3}(H^2-E^2)\right\}. \tag{54.47}$$

We now consider some limiting cases.

In the case of weak fields

$$L_1 = \frac{2}{45}\left(\frac{e^2}{4\pi}\right)^2\frac{1}{m^4}[(\boldsymbol{E}^2-\boldsymbol{H}^2)^2+7(\boldsymbol{EH})^2]+\dots . \tag{54.47'}$$

This expression coincides with the one obtained previously (cf. (54.27)).

We now consider strong fields. We first take $E^* = 0$ and $H^* \gg 1$. In this case we write $L_1$ in the form

$$L_1 = \int_0^\infty \frac{e^{-\varepsilon x}}{x} f(x)\, dx,$$

where

$$\varepsilon = \frac{1}{H^*} \ll 1$$

and

$$f(x) = -\frac{m^4}{8\pi^2}\frac{H^{*2}}{x^2}\left(x\coth x - 1 - \frac{1}{3}x^2\right).$$

---

[1] This expression was first obtained by Heisenberg and Euler (85). The derivation given here is due to Weisskopf (205); cf. also the work of Schwinger (171), (172).

The function $f(x)$ obviously tends to zero as $x \to 0$, and for $x \gg 1$ behaves like $f(x) = f(\infty) + 0(1/x)$, $x \gg 1$, where $f(\infty) = (e^2/24\pi^2)m^4H^{*2}$.

We now show that for functions having such behavior the following relation holds asymptotically for $\varepsilon \to 0$:

$$J(\varepsilon) \equiv \int_0^\infty \frac{e^{-\varepsilon x}}{x} f(x)\,dx \approx f(\infty) \ln \frac{1}{\varepsilon}. \tag{54.58}$$

In order to do this we utilize the identity

$$\int_0^\infty \frac{e^{-\varepsilon x}}{x} f(x)\,dx = \int_0^\infty \frac{e^{-x}}{x} f(x)\,dx$$
$$+ \int_0^\infty \frac{e^{-\varepsilon x} - e^{-x}}{x} [f(x) - f(\infty)]\,dx + f(\infty) \int_0^\infty \frac{e^{-\varepsilon x} - e^{-x}}{x}\,dx.$$

The first term in this expression represents a number which does not depend on $\varepsilon$; in the second term we can make the transition to the limit $\varepsilon \to 0$ as a result of which we again obtain a number independent of $\varepsilon$, while the third term is equal to

$$f(\infty) \int_0^\infty \frac{e^{-\varepsilon x} - e^{-x}}{x}\,dx = -f(\infty) \ln \varepsilon,$$

as can be verified by differentiation with respect to $\varepsilon$. This term is the principal term in the integral $J(\varepsilon)$ for $\varepsilon \to 0$.

By utilizing these results we obtain the asymptotic expression for $L_1$ in the case $H^* \gg 1$, $E^* = 0$

$$L_1 = \frac{m^4}{8\pi^2} e^2 \frac{1}{3} H^{*2} \ln H^*, \qquad H^* \gg 1. \tag{54.49}$$

Similarly it can be shown that in the case of strong electric fields $E^* \gg 1$, $H^* = 0$ the function $L_1$ has the form

$$L_1 = -\frac{m^4}{8\pi^2} e^2 \frac{1}{3} E^{*2} \ln E^*. \tag{54.49'}$$

Thus the ratio of the added term in the Lagrangian $L_1$ to its classical value $L_0 = \frac{1}{2}(E^2 - H^2)$ increases at high field intensities only logarithmically with the field

$$\frac{L_1}{L_0} \sim \frac{\alpha}{3\pi} \ln \frac{e|\boldsymbol{F}|}{m^2}, \tag{54.50}$$

where $\boldsymbol{F}$ is either the electric or the magnetic field. Therefore, the non-linearity of the equations of electrodynamics represents a small correction even in the case when the field is considerably greater than the "critical" value $m^2c^3/e\hbar$. If for $E$ we take the field at the "electron boundary" equal to

$$E = \frac{e}{4\pi r_0^2} = 4\pi \frac{m^2c^4}{e^3},$$

then the ratio $|L_1/L_0|$ will be equal to

$$\left|\frac{L_1}{L_0}\right| \sim \frac{1}{3\pi} \frac{\ln 137}{137},$$

i.e., also in this case we have

$$\left|\frac{L_1}{L_0}\right| \ll 1.$$

## § 55. Photon Scattering by the Coulomb Field of a Nucleus

### 55.1. *General Expression for the Cross Section for Photon Scattering by a Constant Electromagnetic Field*

Four electromagnetic field operators appear following the normal product symbol in the scattering matrix $S^{(4)}$ which describes non-linear interactions between electromagnetic fields (cf. formula (54.1)). If all these operators refer to photon states, then the matrix element of $S^{(4)}$ will describe the photon-photon scattering process discussed in the preceding section. However, we can also have two other cases, viz., when only two of the factors $A_\mu(x)$ refer to photon states, while the other two refer to the external electromagnetic field, or when three of the factors $A_\mu(x)$ refer to photon states, and one refers to the external field. In the former case the element of the matrix $S^{(4)}$ describes the scattering of a photon by an external electromagnetic field (82), (6), (105), while in the latter case it describes the splitting of one photon into two, or the coalescence of two photons into one in the external field. These effects (together with photon-photon scattering) exhaust all the possibilities for fourth order processes of interaction between electromagnetic fields.

We confine ourselves to a discussion of the scattering of a photon by an external electromagnetic field. If the field depends on the time, then photon scattering will occur with a change in frequency; however, if the field is constant, then scattering occurs without a change in photon frequency (coherent scattering).

Of particular interest is the coherent scattering of photons by the Coulomb field of a nucleus. The cross section for such scattering becomes comparable with the cross section for the scattering of photons by electrons at energies of the order of $10^{10}$ eV, but coherent scattering of photons by nuclei can be detected at much lower energies due to its characteristic angular distribution which has a sharp maximum in the region of small scattering angles; we shall see later that in the case of heavy nuclei and at a photon energy of $\omega = 300$ MeV the differential cross section for such scattering at angles of $\sim 0.01°$ exceeds by three orders of magnitude the corresponding cross section for Compton scattering.

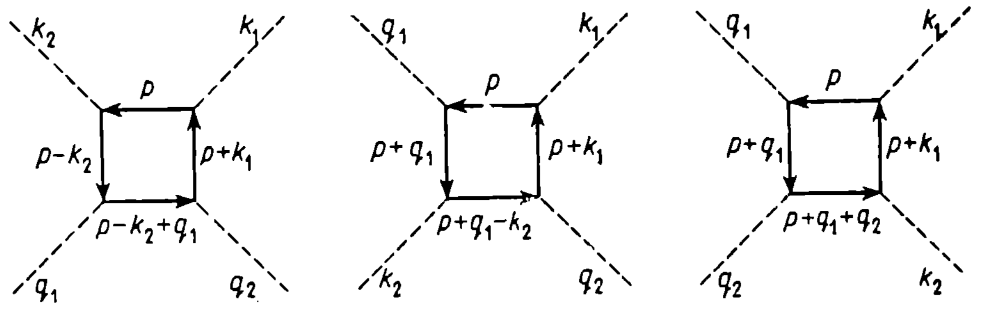


Fig. 115.

Figure 115 gives three diagrams representing the scattering of a photon by an external field. The matrix element which determines this process is given in accordance with the rules of § 25 by

$$
\begin{aligned}
S^{(4)}_{i\to f} = -\frac{e^4}{2(\omega_1\omega_2)^{1/2}} \int d^4q_1\, d^4q_2\, \delta(k_1-k_2+q_1+q_2) e_{1_\mu} e_{2_\nu} \\
\times \frac{A^{(e)}_\lambda(q_1)}{(2\pi)^4} \frac{A^{(e)}_\sigma(q_2)}{(2\pi)^4} \int d^4p \,\{ \mathrm{Sp}\big[\gamma_\mu (i\hat{p}+m)^{-1}\gamma_\nu \big(i(\hat{p}-\hat{k}_2)+m\big)^{-1}\gamma_\lambda \\
\times (i\hat{p}-i\hat{k}_2+i\hat{q}_1+m)^{-1}\gamma_\sigma (i\hat{p}+i\hat{k}_1+m)^{-1}\big] + \mathrm{Sp}[\gamma_\mu (i\hat{p}+m)^{-1}\gamma_\lambda \\
\times (i\hat{p}+i\hat{q}_1+m)^{-1}\gamma_\nu (i\hat{p}+i\hat{q}_1-i\hat{k}_2+m)^{-1}\gamma_\sigma (i\hat{p}+i\hat{k}_1+m)^{-1}] \\
+ \mathrm{Sp}[\gamma_\mu (i\hat{p}+m)^{-1}\gamma_\sigma (i\hat{p}+i\hat{q}_1+m)^{-1}\gamma_\lambda (i\hat{p}+i\hat{q}+i\hat{q}_1+m)^{-1}\gamma_\nu \\
\times (i\hat{p}+i\hat{k}_1+m)^{-1}] \},
\end{aligned}
$$

where $A^{(e)}_\mu(q)$ is the Fourier component of the external field

$$A^{(e)}_\mu(q) = \int A^{(e)}_\mu(x)e^{-iqx}\,d^4x,$$

while $k_1$ and $k_2$ are the photon momenta and $e_1$ and $e_2$ are the photon polarizations.

By utilizing the definition (54.5) of the photon-photon scattering tensor $I_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$ we can rewrite this expression (after regularization) in the form

$$S^{(4)}_{i\to f} = -\frac{i}{2}\left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\sqrt{\omega_1\omega_2}} \int d^4q_1\,d^4q_2(2\pi)^4\,\delta(k_1-k_2+q_1+q_2)$$
$$\times e_{1\mu}e_{2\nu}\frac{A^{(e)}_\lambda(q_1)}{(2\pi)^4}\,\frac{A^{(e)}_\sigma(q_2)}{(2\pi)^4}\,I_{\mu\nu\lambda\sigma}(k_1,-k_2,q_1,q_2). \qquad (55.1)$$

If the field is constant then $A^{(e)}_\mu(q) = A^{(e)}_\mu(\mathbf{q})\,(2\pi)\,\delta(q_0)$, the frequencies of the incident and the scattered photons are equal, $\omega_1 = \omega_2 = \omega$, and

$$S^{(4)}_{i\to f} = -\frac{i}{2}\left(\frac{e^2}{4\pi}\right)^2 \frac{1}{\omega}\int d\mathbf{q}\,\frac{A^{(e)}_\lambda(\mathbf{q})}{(2\pi)^3}\,\frac{A^{(e)}_\sigma(\mathbf{k}_2-\mathbf{k}_1-\mathbf{q})}{(2\pi)^3}(2\pi)^4\delta(\omega_1-\omega_2)$$
$$\times e_{1\mu}e_{2\nu}I_{\mu\nu\lambda\sigma}(\mathbf{k}_1,-\mathbf{k}_2,\mathbf{q},\mathbf{k}_2-\mathbf{k}_1-\mathbf{q}). \qquad (55.2)$$

On multiplying $|S^{(4)}_{i\to f}|^2$ by the number of final states of the scattered photon $\omega_2^2 d\omega_2 do_2/(2\pi)^3$, where $do_2$ is the element of solid angle containing $\mathbf{k}_2$, and on dividing this by the flux density of incident photons $J = 1$, we obtain the differential cross section for the coherent scattering of a photon by a constant field

$$d\sigma = \frac{1}{4}\left(\frac{e^2}{4\pi}\right)^4 (2\pi)^4 do_2 \left|\int d\mathbf{q}\,\frac{A^{(e)}_\lambda(\mathbf{q})}{(2\pi)^3}\,\frac{A^{(e)}_\sigma(\mathbf{k}_2-\mathbf{k}_1-\mathbf{q})}{(2\pi)^3}\,e_{1\mu}e_{2\nu}\right.$$
$$\left.\times I_{\mu\nu\lambda\sigma}(\mathbf{k}_1,-\mathbf{k}_2,\mathbf{q},\mathbf{k}_2-\mathbf{k}_1-\mathbf{q})\right|^2. \qquad (55.3)$$

### 55.2. *Relation between the Forward Scattering Amplitude for a Photon and Pair-Production by a Photon in the Field of a Nucleus*

We shall now show that the matrix element for the coherent forward scattering of a photon by a constant electromagnetic field is related by a simple expression to the total cross section for pair production by a photon in the same field.

We utilize the unitary nature of the scattering matrix

$$\begin{aligned}SS^{+} = {} & 1+e(S^{(1)}+S^{(1)+})+e^2(S^{(2)}+S^{(2)+}+S^{(1)}S^{(1)+})\\ & +e^3(S^{(3)}+S^{(3)+}+S^{(1)}S^{(2)+}+S^{(2)}S^{(1)+})\\ & +e^4(S^{(4)}+S^{(4)+}+S^{(1)}S^{(3)+}+S^{(3)}S^{(1)+}+S^{(2)}S^{(2)+})+\ldots = 1.\end{aligned}$$

From this it follows that each of the expressions in brackets is equal to zero, in particular

$$S^{(2)}S^{(2)+}+S^{(4)+}+S^{(4)}+S^{(1)}S^{(3)+}+S^{(3)}S^{(1)+} = 0. \tag{55.4}$$

We evaluate the matrix element of this expression corresponding to coherent forward scattering of a photon $k$ by an arbitrary constant field

$$\begin{aligned}(k|S^{(4)}+S^{(4)+}|k) = {} & -(k|S^{(2)}S^{(2)+}|k)-(k|S^{(1)}S^{(3)+}|k)\\ & -(k|S^{(3)}S^{(1)+}|k). \qquad (55.4')\end{aligned}$$

The terms appearing on the right hand side of this equation can be written in the form

$$(k|S^{(2)}S^{(2)+}|k) = \sum_f (k|S^{(2)}|f)(f|S^{(2)+}|k),$$

$$(k|S^{(1)}S^{(3)+}|k) = \sum_f (k|S^{(1)}|f)(f|S^{(3)+}|k),$$

$$(k|S^{(3)}S^{(1)+}|k) = \sum_f (k|S^{(3)}|f)(f|S^{(1)+}|k),$$

where $f$ denotes the final states of the matrix elements of $S^{(1)}$, $S^{(2)}$, $S^{(3)}$. Obviously, only electron-positron pair states can be used as such states; but the matrices $S^{(1)}$ and $S^{(3)}$ have no elements connecting single photon states with states of the field containing one pair, so that the second and the third terms in (55.4′) vanish and, consequently,

$$(k|S^{(2)}S^{(2)+}|k) \equiv \sum_f (k|S^{(2)}|f)(f|S^{(2)+}|k) = -2\mathrm{Re}(k|S^{(4)}|k). \tag{55.5}$$

It can be easily shown that the left hand side of this equation gives, up to a numerical factor, the total cross section $\sigma_{\pm}(\omega)$ for pair production by the photon $k$ in the field under discussion. Indeed, on writing explicitly the $\delta$-functions in the matrix elements, i.e., on setting

$$(a|S^{(i)}|b) = (a|M^{(i)}|b)\delta(\varepsilon_a-\varepsilon_b),$$

we can in accordance with § 26 write

$$\sigma_{\pm}(\omega) = \frac{1}{2\pi}\sum_f |(k|M^{(2)}|f)^2\delta(\omega-\varepsilon_f).$$

Therefore, it follows from (55.5) that

$$\sigma_{\pm}(\omega) = -\frac{1}{\pi}\operatorname{Re}(k|\mathrm{M}^{(4)}|k). \tag{55.6}$$

Thus, the real part of the matrix element for the forward scattering of a photon by a constant electromagnetic field is given, up to a numerical factor, by the cross section for pair production by the photon in the same field (98).

Equation (55.6) follows from the more general relation connecting the total cross section for the scattering of particles by a force field with the amplitude for their elastic forward scattering. This relation, known as the optical theorem, has in accordance with (5.13) the form

$$\sigma_t = 4\pi\lambda\,\operatorname{Im} f(\omega, 0), \tag{55.7}$$

where $\sigma_t$ is the total particle scattering cross section given by the sum of the elastic scattering cross section ($\sigma_e$) and the absorption cross section ($\sigma_a$), $\sigma_t = \sigma_e + \sigma_a$, $f(\omega, \theta)$ is the amplitude for elastic scattering by an angle $\theta$, $\lambda$ is the wavelength of the particles, and $\omega$ is their energy.

In the case of photons moving in a constant electromagnetic field absorption is associated with the conversion of photons into pairs, so that the cross section for photon absorption is equal to the cross section for pair production by photons. The elastic scattering cross section $\sigma_e$ for the photons is considerably smaller than the pair production cross section, and therefore

$$\sigma_t \approx \sigma_{\pm}(k). \tag{55.7'}$$

The amplitude $f(\omega, \theta)$ for the scattering of photons by a constant field is related to the matrix element for scattering $S^{(4)}_{i\to f} \equiv M(\omega, \theta)\,\delta(\omega - \omega_f)$ by the expression

$$f(\omega, \theta) = -\frac{i}{(2\pi)^2}\,\omega M(\omega, \theta). \tag{55.8}$$

Indeed, the differential scattering cross section has the form

$$d\sigma = \frac{1}{(2\pi)^4}\,|M(\omega, \theta)|^2\omega^2\,do. \tag{55.9}$$

On the other hand, in accordance with the definition of the scattering amplitude it is given by

$$d\sigma = |f(\omega, \theta)|^2\,do. \tag{55.9'}$$

A comparison of these expressions yields the expression (55.8) up to a phase factor. We define the scattering amplitude completely by choosing this factor to be equal to $-i$; in this case both formula (55.9) and expression (55.7) will be valid.

### 55.3. *Momentum Distribution of Recoil Nuclei Accompanying Pair Production by a Photon in the Field of a Nucleus*

We now obtain the matrix element for coherent forward scattering of photons and the total cross section for pair production related to it.

In the case of forward scattering the photon polarization is obviously unchanged, $e_1 = e_2$, and, therefore, the value of the matrix element averaged over the polarization states is in accordance with (55.2) equal to

$$M^{(4)}(\omega, 0) = -\frac{i}{2}\left(\frac{e^2}{4\pi}\right)^2 \frac{(2\pi)^4}{2\omega}\int d\boldsymbol{q}\, \frac{A^{(e)}_\lambda(\boldsymbol{q})}{(2\pi)^3}\, \frac{A^{(e)}_\sigma(-\boldsymbol{q})}{(2\pi)^3}\, I_{\mu\mu\lambda\sigma}(k, -k, \boldsymbol{q}, -\boldsymbol{q}). \tag{55.10}$$

In accordance with (55.8) the forward scattering amplitude is equal to

$$f(\omega, 0) = -\frac{1}{4}\left(\frac{e^2}{4\pi}\right)^2 (2\pi)^2 \int d\boldsymbol{q}\, \frac{A^{(e)}_\lambda(\boldsymbol{q})}{(2\pi)^3}\, \frac{A^{(e)}_\sigma(-\boldsymbol{q})}{(2\pi)^3}\, I_{\mu\mu\lambda\sigma}(k, -k, \boldsymbol{q}, -\boldsymbol{q}). \tag{55.11}$$

On the basis of this expression and of expression (55.7) we can write the general expression for the total cross section for the production of pairs by a photon in the field $A^{(e)}_\mu(\boldsymbol{q})$

$$\sigma_\pm(\omega) = 4\pi\bar{\lambda}\, \mathrm{Im} f(\omega, 0) = -\frac{1}{2}\left(\frac{e^2}{4\pi}\right)^2 (2\pi)^{-3} \frac{1}{\omega}\, \mathrm{Im} \int d\boldsymbol{q}\, A^{(e)}_\lambda(\boldsymbol{q}) A^{(e)}_\sigma(-\boldsymbol{q}) I_{\mu\mu\lambda\sigma}(k, -k, \boldsymbol{q}, -\boldsymbol{q})$$

or

$$\sigma_\pm(\omega) = \int A^{(e)}_\mu(\boldsymbol{q}) A^{(e)}_\nu(-\boldsymbol{q})\, T_{\mu\nu}(\boldsymbol{q}, k)\, d\boldsymbol{q}, \tag{55.12}$$

where

$$T_{\mu\nu}(\boldsymbol{q}, k) = (2\pi)^{-6} \frac{4\pi}{\omega}\left(\frac{e^2}{4\pi}\right)^2 \mathrm{Re}[A_{\mu\nu}(\boldsymbol{q}, k) + B_{\mu\nu}(\boldsymbol{q}, k) + B_{\mu\nu}(-\boldsymbol{q}, k)],$$

$$A_{\mu\nu}(\boldsymbol{q}, k) = \int d^4p$$

$$\times \frac{\mathrm{Sp}\left[\gamma_\mu(i\hat{p}-m)\gamma_\lambda\left(i(\hat{p}-\hat{k})-m\right)\gamma_\nu\left(i(\hat{p}-\hat{k}-\hat{q})-m\right)\gamma_\lambda\left(i(\hat{p}-\hat{q})-m\right)\right]}{(p^2+m^2)\,[(p-k)^2+m^2]\,[(p-k-q)^2+m^2]\,[(p-q)^2+m^2]},$$

$$B_{\mu\nu}(\boldsymbol{q}, k) = \int d^4p$$

$$\times \frac{\mathrm{Sp}[\gamma_\mu(i\hat{p}+i\hat{q}-m)\gamma_\nu(i\hat{p}-m)\gamma_\lambda(i\hat{p}+i\hat{k}-m)\gamma_\lambda(i\hat{p}-m)]}{[(p+q)^2+m^2]\,[(p+k)^2+m^2](p^2+m^2)^2}. \qquad (55.12')$$

Since $\boldsymbol{q}$ is the momentum transferred to the nucleus as a result of pair production, the integrand in formula (55.12) defines the momentum distribution of the recoil nuclei.

We now obtain the tensor $T_{\mu\nu}(\boldsymbol{q}, k)$. We first note that the cross section for pair production must be invariant under gauge transformations, i.e., it must vanish if $A_\mu^{(e)}(x) = \partial\chi(x)/\partial x_\mu$, where $\chi(x)$ is an arbitrary scalar function. From this it follows that

$$q_\mu T_{\mu\nu}(\boldsymbol{q}, k) = 0. \qquad (55.13)$$

On the other hand, it is clear that the tensor $T_{\mu\nu}(\boldsymbol{q}, k)$ must have the form

$$T_{\mu\nu}(\boldsymbol{q}, k) = (2\pi)^{-6}\frac{4\pi}{\omega}\left(\frac{e^2}{4\pi}\right)^2$$

$$\mathrm{Re}[I_1\delta_{\mu\nu}+I_2k_\mu k_\nu+I_3q_\mu q_\nu+I_4(k_\mu q_\nu+q_\mu k_\nu)], \qquad (55.14)$$

where $I_i$ are invariant functions of $\boldsymbol{q}$ and $k$.

It follows from (55.13) that

$$\begin{aligned} I_1+I_3q^2+I_4qk &= 0,\\ I_2qk+I_4q^2 &= 0, \end{aligned} \qquad (55.14')$$

so that in order to obtain $T_{\mu\nu}$ it is sufficient to determine only two of the four functions $I_i$.

We write $I_i$ in the form

$$I_i(\boldsymbol{q}, k) = A_i(\boldsymbol{q}, k)+B_i(\boldsymbol{q}, k)+B_i(-\boldsymbol{q}, k), \qquad (55.15)$$

where $A_i$ and $B_i$ are the contributions to $I_i$ made by $A_{\mu\nu}$ and $B_{\mu\nu}$. In accordance with (55.14) it is sufficient to obtain only the real parts of these quantities.

We first of all determine $B_i$.

On evaluating in (55.12′) the trace of the matrix product in accordance with the formulas of § 26, and on utilizing the first of relations

$$\frac{1}{abc^2} = 3!\int_0^1 dx \int_0^x dy \cdot y\,[a+(b-a)x+(c-b)y]^{-4},$$

$$\frac{1}{abcd} = 3!\int_0^1 dx \int_0^x dy \int_0^y dz[a+(b-c)x+(c-d)y+(d-c)z]^{-4}, \tag{55.16}$$

we obtain

$$B_2(q,k) = -48\int d^4p \int_0^1 dx \int_0^x dy(x-y)y$$

$$\times \frac{p^2\big((1+4(x-y)\big)+2[(x-y)(Q^2+3m^2-2Qk)+Q^2+m^2]}{[p^2+m^2+q^2(1-x)-Q^2]^4}, \tag{55.17}$$

$$B_3(q,k) = -48\int d^4p \int_0^1 dx \int_0^x dy(1-x)y$$

$$\times \frac{p^2(1-4x)-2x(Q^2+3m^2-2Qk)}{[p^2+m^2+q^2(1-x)-Q^2]^4},$$

where

$$Q = q(1-x)+k(x-y).$$

Further, on noting that

$$\int d^4p \frac{1}{(p^2+a^2)^4} = \frac{\pi^2 i}{6a^4},$$

$$\int d^4p \frac{p^2}{(p^2+a^2)^4} = \frac{\pi^2 i}{3a^2},$$

and on setting $x-y=\eta$, $1-x=\xi$ we rewrite (55.17) in the form

$$\begin{aligned}B_2(\mathbf{q},k) &= -16\pi^2 i\int_0^1 d\xi\int_0^{1-\xi} d\eta\,\eta(1-\xi-\eta)\left[\frac{1+4\eta}{m^2+q^2\xi-Q^2}\right.\\ &\qquad\left.+\frac{\eta(Q^2+3m^2-2qk\xi)+Q^2+m^2}{(m^2+q^2\xi-Q^2)^2}\right],\\ B_3(\mathbf{q},k) &= -16\pi^2 i\int_0^1 d\xi\int_0^{\xi-1} d\eta\,\xi(1-\xi-\eta)\left[\frac{4\xi-3}{m^2+q^2\xi-Q^2}\right.\\ &\qquad\left.-\frac{(1-\xi)(Q^2+3m^2-2qk\xi)}{(m^2+q^2\xi-Q^2)^2}\right],\end{aligned}\tag{55.17'}$$

where $Q=q\xi+k\eta$. Integration over $\eta$ yields

$$\begin{aligned}B_2(\mathbf{q},k) &= -16\pi^2 i\int_0^1 d\xi\left\{\left[-3(1-\xi)\frac{[m^2+q^2\xi(1-\xi)]^2}{\beta^3\xi^3}\right.\right.\\ &\quad+3\frac{[m^2+q^2\xi(1-\xi)]^3}{\beta^4\xi^4}+2(1-\xi)(4m^2+4r^2\xi)\\ &\quad\times\frac{m^2+q^2\xi(1-\xi)}{\beta^3\xi^3}-12(m^2+r^2\xi)\frac{[m^2+q^2\xi(1-\xi)]^2}{\beta^4\xi^4}\\ &\quad\left.+\frac{(1-\xi)}{\beta^2\xi^2}(2m^3+q^2\xi)-2(2m^2+q^2\xi)\frac{m^2+q^2\xi(1-\xi)}{\beta^3\xi^3}\right]\\ &\quad\left.\times\ln\left(m^2+4r^2\xi(1-\xi)\right)\right\}+\text{purely imaginary terms},\end{aligned}\tag{55.18}$$

$$\begin{aligned}B_3(\mathbf{q},k) &= -\frac{16\pi^2 i}{\beta^2}\int_0^1\frac{d\xi}{\xi}\{(3\xi-2)[m^2+4r^2\xi(1-\xi)]\\ &\quad+4(1-\xi)(m^2+r^2\xi)\}\{\ln[4r^2\xi(1-\xi)]-\ln[m^2+q^2\xi(1-\xi)]\},\end{aligned}$$

where $\beta=2qk$, $4r^2=(q-k)^2$.

We consider the quantity $B_3\ (\mathbf{q},k)$. Since in the case of a constant field $q^2\geqslant 0$, $\ln\,(m^2+4r^2\xi(1-\xi))$ will be real, and, therefore, the whole term proportional to this logarithm can be omitted.

The term containing $\ln(m^2+4r^2\xi(1-\xi))$ gives a contribution to $B_3(\boldsymbol{q}, k)$ different from zero only for $r^2 < -m^2$. In this case $m^2+4r^2\xi(1-\xi)$ has two real roots within the range of integration (0,1)

$$\xi_{1,2} = \frac{1}{2}\left(1 \mp \sqrt{1+\frac{m^2}{r^2}}\right),$$

and since $m^2$ contains an infinitesimal negative imaginary part, we have

$$\ln[m^2+4r^2\xi(1-\xi)] = \begin{cases} \ln|m^2+4r^2\xi(1-\xi)|, & 0 < \xi < \xi_1, \\ \ln|m^2+4r^2\xi(1-\xi)| - i\pi, & \xi_1 < \xi < \xi_2, \\ \ln|m^2+4r^2\xi(1-\xi)|, & \xi_2 < \xi < 1. \end{cases}$$

Therefore

$$\operatorname{Im} \int_0^1 \frac{d\xi}{\xi} f(\xi) \ln[m^2+4r^2\xi(1-\xi)] = -i\pi \int_{\xi_1}^{\xi_2} \frac{d\xi}{\xi} f(\xi),$$

where $f(\xi)$ has no singularities within the interval (0,1). By utilizing this formula we can easily show that

$$\operatorname{Re} B(\boldsymbol{q}, k) = \frac{32\pi^2 m^2}{\beta^2}\left[\ln \frac{1-\sqrt{1+\dfrac{m^2}{r^2}}}{1+\sqrt{1+\dfrac{m^2}{r^2}}} + \sqrt{1+\frac{m^2}{r^2}}\right]. \tag{55.19}$$

In a similar manner we can obtain $\operatorname{Re} B_2(\boldsymbol{q}, k)$ and $\operatorname{Re} A_i(\boldsymbol{q}, k)$;

$$\operatorname{Re} B_2(\boldsymbol{q}, k) = \frac{16\pi^3}{\beta^4}\left\{\boldsymbol{q}^2(12m^2\boldsymbol{q}^2 - 8m^2\beta - \beta^2) \ln \frac{1-\sqrt{1+\dfrac{m^2}{r^2}}}{1+\sqrt{1+\dfrac{m^2}{r^2}}}\right.$$
$$\left. + \sqrt{1+\frac{m^2}{r^2}}\,[\beta^3 + 2\beta^2(m^2-\boldsymbol{q}^2) + 2\beta\boldsymbol{q}^2(-8m^2+\boldsymbol{q}^2) + 2\boldsymbol{q}^4(8m^2-\boldsymbol{q}^2)]\right\}, \tag{55.19'}$$

$$\operatorname{Re} A_2(\boldsymbol{q}, k) = A_2'(\boldsymbol{q}, k) + A_2'(-\boldsymbol{q}, k),$$

$$\operatorname{Re} A_3(\boldsymbol{q}, k) = A_3'(\boldsymbol{q}, k) + A_3'(-\boldsymbol{q}, k),$$

$$A_2'(q,k)=\frac{16\pi^3}{\beta^2}[q^2-\beta+2m^2]\sqrt{1+\frac{m^2}{r^2}},$$

$$A_3'(q,k)=\frac{16\pi^3}{\beta^4}\left\{2(2m^2+q^2)q^2(\beta-q^2+2m^2)\ln\frac{1-\sqrt{1+\frac{m^2}{r^2}}}{1+\sqrt{1+\frac{m^2}{r^2}}}\right.$$

$$\left.+\sqrt{1+\frac{m^2}{r^2}}\,[\beta^3+\beta^2(2m^2-q^2)+6q^2(2m^2+q^2)(q^2-\beta)]\right\}. \qquad (55.20)$$

On substituting these expressions into (55.15), and on utilizing the fact that $T_{\mu\nu}$ is real, we finally obtain

$$\operatorname{Re} I_2=\frac{8\pi^3q^2}{\beta^4}\left\{[2(4m^4+6m^2q^2-q^4)+2(q^2-2m^2)\beta-\beta^2]\right.$$

$$\left.\times\ln\frac{1-\sqrt{1+\frac{m^2}{r^2}}}{1+\sqrt{1+\frac{m^2}{r^2}}}+\sqrt{1+\frac{m^2}{r^2}}\,[4(q^2-\beta)(m^2-2q^2)-\beta^2]\right\},$$

$$(55.21)$$

$$\operatorname{Re} I_3=\frac{32\pi^3}{\beta^2}\left[2m^2\ln\frac{1-\sqrt{1+\frac{m^2}{r^2}}}{1+\sqrt{1+\frac{m^2}{r^2}}}+(\beta-q^2)\sqrt{1+\frac{m^2}{r^2}}\right],$$

if $r^2\equiv\frac{1}{4}(q-k)^2<-m^2$ and $I_i=0$, if $r^2\geqslant -m^2$.

We now consider in greater detail the particularly interesting case when the field in which the pair is produced is a screened Coulomb field

$$A^{(e)}(q)=0,\qquad A_0^{(e)}(q)=\frac{Ze}{q^2}\big(1-F(q)\big),$$

where $F(\boldsymbol{q})$ is the atomic form-factor. In this case only one component of the tensor $T_{44}$ which is equal to

$$T_{44} = (2\pi)^{-6}\frac{4\pi}{\omega}\left(\frac{e^2}{4\pi}\right)^2 \mathrm{Re}(-I_1+I_2\omega^2)$$

differs from zero and

$$\sigma_\pm(\omega) = -\frac{Z^2e^6}{2\omega(2\pi)^7}\int d\boldsymbol{q}\frac{\left(1-F(\boldsymbol{q})\right)^2}{q^4}\,\mathrm{Re}(I_1-I_2\omega^2), \tag{55.22}$$

where

$$\begin{aligned}
&\mathrm{Re}(I_1-\omega^2 I_2)\\
&= \frac{8\pi^3}{\beta^2}\left\{-4q^2\left[2m^2\ln\frac{1-\sqrt{1+\dfrac{m^2}{r^2}}}{1+\sqrt{1+\dfrac{m^2}{r^2}}}+\sqrt{1+\frac{m^2}{r^2}}(\beta-q^2)\right]\right.\\
&\quad-\frac{\beta^2-4q^2\omega^2}{\beta^2}\left[\sqrt{1+\frac{m^2}{r^2}}\left(\beta^2-4(\boldsymbol{q}^2-\beta)(m^2-2\boldsymbol{q}^2)\right)\right.\\
&\quad\left.\left.+\ln\frac{1-\sqrt{1+\dfrac{m^2}{r^2}}}{1+\sqrt{1+\dfrac{m^2}{r^2}}}\left(\beta^2+2(\beta-\boldsymbol{q}^2)(2m^2-\boldsymbol{q}^2)-8m^2(\boldsymbol{q}^2+m^2)\right)\right]\right\},
\end{aligned} \tag{55.22'}$$

if $r^2 < -m^2$, and $\mathrm{Re}(I_1-\omega^2 I_2) = 0$, if $r^2 \geqslant -m^2$.

On introducing the new variables

$$2mQ = |\boldsymbol{q}|, \quad \mu = (\omega^2-|\boldsymbol{q}-\boldsymbol{k}|)\frac{1}{4m^2}, \quad \gamma = \frac{\omega}{2m},$$

we can write (55.22) in the form

$$\sigma_\pm(\omega) = \int\limits_{\gamma-\sqrt{\gamma^2-1}}^{\gamma+\sqrt{\gamma^2-1}} dQ\,P(Q,\omega), \tag{55.23}$$

where

$$P(Q,\omega)=\frac{Z^2e^6}{32\pi^3\omega^2}[1-F(Q^2)]\frac{I(Q,\gamma)}{Q^3},\tag{55.23'}$$

$$\begin{aligned}I(Q,\gamma)=J_0&+(1-2Q^2)J_1+(2Q^4-Q^2-4Q^2\gamma^2-\tfrac{1}{2})J_3+2Q^2\gamma^2\\&\times(1+6Q^2-4Q^4)J_4+K_0+(1-4Q^2)K_1+[(4Q^2-1)Q^2-4Q^2\gamma^2]K_2\\&+4Q^2\gamma^2(8Q^2-1)K_3+4Q^4\gamma^2(1-8Q^2)K_4\end{aligned}$$

and

$$J_n=\int\limits_1^{2\gamma Q-Q^2}\frac{d\mu}{(Q^2+\mu)^n}\ln\frac{1-\sqrt{1-\frac{1}{\mu}}}{1+\sqrt{1-\frac{1}{\mu}}},\qquad K_n=\int\limits_0^{2\gamma Q-Q^2}\frac{d\mu}{(Q^2+\mu)^n}.$$

The function $P(Q,\omega)$ determines the distribution of recoil nuclei with respect to the absolute values of the momenta (98).

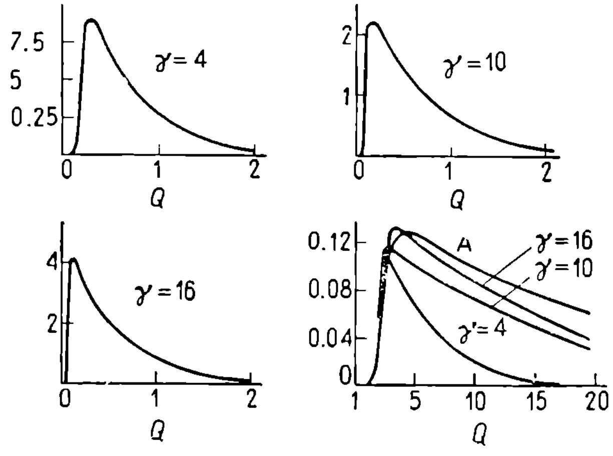


Fig. 116.

Figure 116 shows the momentum distribution of recoil nuclei for different values of the photon energy. The quantity $P/(Z^2\alpha^3/32\pi^3m^2)$ is plotted along the vertical axes of these graphs (the letter $A$ denotes the asymptotic curve).

Formulas (55.23) and (55.23′) are very complicated. Approximately we can assume (17) that the function $P(Q, \omega)$ is inversely proportional to $q$ between the limits

$$q_{\min} = \begin{cases} \dfrac{1}{a}, \\ \dfrac{m^2}{\omega}, \end{cases} \qquad q_{\max} = m$$

and falls off very rapidly outside this interval ($a$ is the atomic radius in the Thomas-Fermi model). In other words

$$\sigma_{\pm}(\omega) = A \int_{q_{\min}}^{q_{\max}} \frac{dq}{q},$$

where $A$ is a constant.

If screening can be neglected, then the lower limit of integration is equal to $m^2/\omega$ and

$$\sigma_{\pm}(\omega) = A \int_{\frac{m^2}{\omega}}^{m} \frac{dq}{q}.$$

A comparison of formula with formula (32.9) which determines the cross section for pair production by a photon in the Coulomb field of a nucleus shows that

$$A = \frac{28}{9}\left(\frac{e^2}{4\pi}\right)^3 \frac{Z^2}{m^2} = \frac{28}{9}\overline{\Phi}.$$

Thus, the momentum distribution of recoil nuclei in the absence of screening is given approximately by

$$d\sigma_{\pm} = \frac{28}{9}\overline{\Phi}\,\frac{dq}{q}, \qquad \frac{m^2}{\omega} \leqslant q \leqslant m. \qquad (55.24)$$

In the case of complete screening this formula is replaced by

$$d\sigma_{\pm} = \frac{28}{9}\overline{\Phi}\,\frac{dq}{q}, \qquad \frac{1}{a} \leqslant q \leqslant m. \qquad (55.24')$$

### 55.4. *Angular Distribution of Recoil Nuclei and Total Cross Section for Pair Production by a Photon in the Coulomb Field of a Nucleus*

It is well known that when the following condition is satisfied

$$R_{0\,\max} \ll a, \qquad R_0 \sim \frac{1}{|\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{k}|}$$

(cf. subsection 29.6) screening of the nuclear field plays no essential role, and the quantity $F(\boldsymbol{q})$ can be neglected in (55.23′). We determine the angular distribution of recoil nuclei in this case, i.e., their distribution with respect to the angle $\theta$ between the vectors $\boldsymbol{q}$ and $\boldsymbol{k}$.

On introducing the variables

$$\eta = \gamma\cos\theta = \gamma\frac{\boldsymbol{q}\boldsymbol{k}}{q\omega}, \qquad \text{where} \quad \gamma = \frac{\omega}{2m},$$

we obtain for $F(\boldsymbol{q}) = 0$

$$\sigma_\pm(\omega) = \frac{Z^2e^6}{16\pi^2\omega^2}\int\limits_1^\gamma d\eta \int\limits_{\eta-(\eta^2-1)^{1/2}}^{\eta+(\eta^2-1)^{1/2}} \frac{dQ}{Q^2}\left\{\ln\frac{1-\sqrt{1-\dfrac{1}{2\eta Q-Q^2}}}{1+\sqrt{1-\dfrac{1}{2\eta Q-Q^2}}}\left[\left(1-\frac{\gamma^2}{\eta^2}\right)\right.\right.$$
$$\times\left.\left(1-\frac{1}{4\eta^2}+\frac{1}{2\eta Q}-\frac{1}{8Q^2\eta^2}-\frac{Q}{\eta}+\frac{Q^2}{2\eta^2}\right)+\frac{\gamma^2}{2\eta^2}\right]$$
$$+\sqrt{1-\frac{1}{2\eta Q-Q^2}}\left[\left(1-\frac{\gamma^2}{\eta^2}\right)\left(1-\frac{1}{4\eta^2}+\frac{1}{2\eta Q}\right)\right.$$
$$\left.\left.+\frac{1}{\eta^2}\left(1-\frac{2\gamma^2}{\eta^2}\right)(Q^2-2\eta Q)\right]\right\}. \qquad (55.25)$$

This expression can be written in the form

$$\sigma_\pm(\omega) = \int\limits_0^{\arccos\frac{1}{\gamma}} d\theta P(\theta,\omega), \qquad (55.26)$$

where

$$P(\theta, \omega)=\frac{Z^2e^6}{16\pi^3\omega^2}\gamma\sin\theta\int\limits_0^{\eta^2} d\mu\,\frac{1}{(\eta^2-\mu^2)^{1/2}}\left\{\left(1-\frac{1}{\mu}\right)^{1/2}\left[\left(1-\frac{\gamma^2}{\eta^2}\right)\right.\right.$$

$$\left.\times\left(\frac{2\eta^2}{\mu^3}+\frac{2(\eta^2-1)}{\mu^2}+\frac{\eta^2-4}{4\mu}\right)+\left(1-2\frac{\gamma^2}{\eta^2}\right)\left(-\frac{2}{\mu}+\frac{1}{\eta^2}\right)\right]$$

$$+\ln\frac{1-\left(1-\dfrac{1}{\mu}\right)^{1/2}}{1+\left(1-\dfrac{1}{\mu}\right)^{1/2}}\left[\frac{\gamma^2}{\eta^2\mu^2}-\frac{\gamma^2}{2\eta^4\mu}+\left(1-\frac{2\gamma^2}{\eta^2}\right)\left(\frac{1}{2\eta^2}+\frac{1-8\eta^2}{4\eta^2\mu}\right.\right.$$

$$\left.\left.\left.+\frac{16\eta^4-16\eta^2-1}{\eta^2\mu^2}+\frac{2\eta^2+1}{\mu^3}-\frac{\eta^2}{\mu^4}\right)\right]\right\}. \quad (55.26')$$

This is the function which determines the angular distribution of recoil nuclei (98).

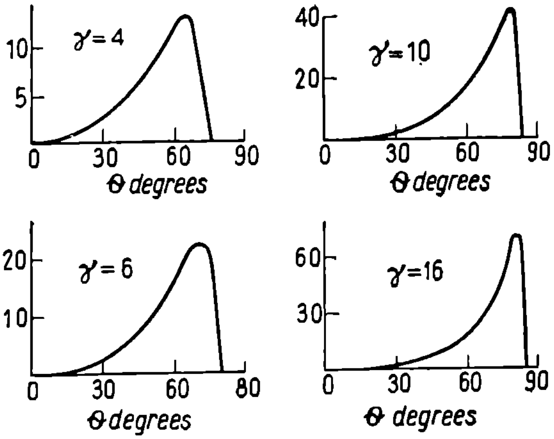


Fig. 117.

Figure 117 shows the angular distribution of recoil nuclei for different values of the photon energy.

The quantity $P(\theta)/(Z^2a^3/64\pi^3m^2)$ is plotted along the vertical axes of these graphs.

On integrating by parts the term containing $\ln\{1-(1-(1/\mu))^{1/2}\}/\{1+(1-(1/\mu))^{1/2}\}^{-1}$, and on introducing the elliptic integrals

$$F(x)=\int\limits_0^{\pi/2}(1-x\sin^2\varphi)^{-1/2}\,d\varphi,$$

$$E(x)=\int\limits_0^{\pi/2}(1-x\sin^2\varphi)^{1/2}\,d\varphi,$$

$$L(\eta)=\int\limits_0^{\eta}F\left[\left(1-\frac{1}{\xi^2}\right)^{1/2}\right]\frac{d\xi}{\xi},$$

we can write $\sigma_\pm(\omega)$ in the form (160)

$$\sigma_\pm(\omega)=\frac{Z^2e^6}{64\pi^3m^2\gamma^2}\left\{4\int\limits_1^{\gamma}\frac{L(\eta)}{\eta}\,d\eta+2L(\gamma)\right.$$
$$+\frac{2}{27}\left\{-(64+109\gamma^2)E\left[\left(1-\frac{1}{\gamma^2}\right)^{1/2}\right]\right.$$
$$\left.\left.+\left(125+\frac{6}{\gamma^2}+42\gamma^2\right)F\left[\left(1-\frac{1}{\gamma^2}\right)^{1/2}\right]\right\}\right\}. \qquad (55.27)$$

### 55.5. *Small Angle Coherent Scattering of Photons by the Field of a Nucleus* (23)

It follows from formulas (55.6), (55.8) that the imaginary part of the forward scattering amplitude for a photon in the Coulomb field of a nucleus is given by

$$a_2(\omega,0)=\frac{\omega}{4\pi}\sigma_\pm(\omega),\qquad \omega\geqslant 2m,$$

$$a_2(\omega,0)=0,\qquad \omega\leqslant 2m.$$

This expression can also be rewritten in the form

$$a_2(\omega,0)=0,\qquad \omega\leqslant 2m,$$

$$a_2(\omega,0)=\frac{Z^2}{m}\left(\frac{e^2}{4\pi}\right)^3\left\{\frac{y}{\pi}[2C_2(y)-D_2(y)]+\frac{1}{27y\pi}\left[-(109+64y^2)\right.\right.$$
$$\left.\left.\times E\left((1-y^2)^{1/2}\right)+(42+125y^2+64y^4)F\left((1-y^2)^{1/2}\right)\right]\right\},\qquad \omega\geqslant 2m,\qquad (55.28)$$

where $y = \frac{2m}{\omega}$ and

$$C_2(y) = \int_1^{1/y} \frac{\cosh^{-1} x}{x} \sinh^{-1}\left(\frac{1}{yx}\right) dx,$$

$$D_2(y) = \int_1^{1/y} \frac{\sinh^{-1}\left(\frac{1}{xy}\right)}{(x^2-1)^{1/2}} dx.$$

On utilizing relations (5.19) and the preceding expressions we can obtain the real part of the forward scattering amplitude for the photon:

$$a_1(\omega, 0) = \frac{Z^2}{m}\left(\frac{e^2}{4\pi}\right)^3 \left\{\frac{y}{\pi}\big(2C_1(y) - D_1(y)\big) + \frac{1}{27\pi y}\left[(109+64y^2)E_1(y)\right.\right.$$
$$\left.\left. -(67-6y^2)(1-y^2)F_1(y)\right] - \frac{y^2}{9} - \frac{9}{4}\right\}, \qquad (55.29)$$

where

$$C_1(y) = \mathrm{Re}\int_0^{1/y} \frac{\arcsin x}{x} \cosh^{-1}\left(\frac{1}{yx}\right) dx,$$

$$D_1(y) = \mathrm{Re}\int_0^{1/y} \frac{\cosh^{-1}\left(\frac{1}{yx}\right)}{(1-x^2)^{1/2}} dx,$$

$$E_1(y) = \mathrm{Re}\int_0^{1/y} \left(\frac{1-y^2x^2}{1-x^2}\right)^{1/2} dx,$$

$$F_1(y) = \mathrm{Re}\int_0^{1/y} \frac{dx}{[(1-x^2)(1-y^2x^2)]^{1/2}}.$$

We note that the functions $E_1(p)$, $F_1(p)$, $E(p)$ and $F(p)$ are interrelated by

$$\begin{aligned} &E_1(p) = E(p), \; p \leqslant 1, \\ &E_1(p) = pE\left(\frac{1}{p}\right) + \frac{1}{p}(1-p^2)F\left(\frac{1}{p}\right), \quad p \geqslant 1, \\ &F_1(p) = F(p), \; p \leqslant 1, \\ &F_1(p) = \frac{1}{p}F\left(\frac{1}{p}\right), \quad p \geqslant 1. \end{aligned} \tag{55.30}$$

We consider several limiting cases. If $\omega \leqslant 2m$, then the cross section for pair production is equal to zero and, consequently, $a_2(\omega, 0) = 0$. For $\omega \lll 2m$ we have

$$f(\omega, 0) = a_1(\omega, 0) = \frac{73}{72} \cdot \frac{1}{32} \cdot \frac{Z^2}{m}\left(\frac{e^2}{4\pi}\right)^3 \left(\frac{\omega}{m}\right)^2. \tag{55.31}$$

The differential photon scattering cross section for photon $\theta = 0$ and $\omega \lll 2m$ has the form

$$\frac{d\sigma}{do} = |a_1(\omega, 0)|^2 = \left(\frac{73}{72}\right)^2 \left(\frac{1}{32}\right)^2 Z^4 \left(\frac{e^2}{4\pi}\right)^6 \frac{\omega^4}{m^6}; \tag{55.31'}$$

it is proportional to $\omega^4$.

For $\omega \ggg 2m$ we have

$$\begin{aligned} a_1(\omega, 0) &= \frac{Z^2}{m}\left(\frac{e^2}{4\pi}\right)^3 \frac{7}{18}\left(\frac{\omega}{m}\right); \\ a_2(\omega, 0) &= \frac{Z^2}{m}\left(\frac{e^2}{4\pi}\right)^3 \frac{7}{9\pi}\left(\frac{\omega}{m}\right) \ln \frac{2\omega}{m}. \end{aligned} \tag{55.32}$$

In this case the imaginary part of the photon scattering amplitude exceeds the real part in order of magnitude by a factor of $\ln(\omega/m)$.

Figure 118 shows the dependence of the forward scattering amplitude on photon energy.

Since the scattering amplitude increases with $\omega$, the case of particular interest is the scattering of high frequency photons by a nuclear field. We therefore consider this case in greater detail.

In order to do this we expand the photon scattering amplitude

$$f(\omega, \theta) = a_1(\omega, \theta) + ia_2(\omega, \theta)$$

into a series in terms of amplitudes corresponding to different values of the angular momentum $l$ and parity $I$. In the high energy case the scattering amplitude for a given value of $l$ can be assumed to be independent of $l$. Therefore, the expansion of the photon scattering amplitude $f(\omega, \theta)$ has in the high energy case the same form as the expansion for the scattering amplitude for a scalar particle.

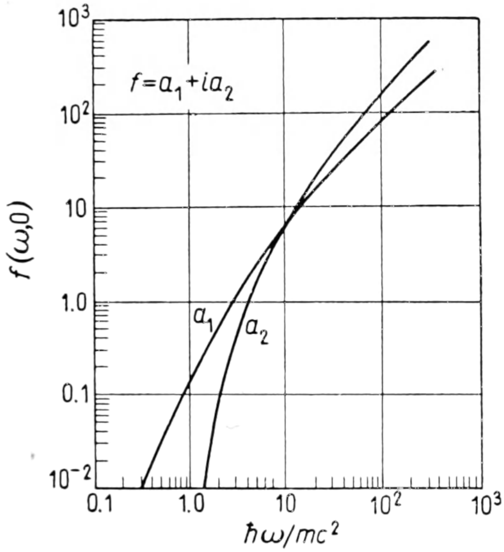


Fig. 118.

We write this expansion in the form

$$f(\omega, \theta) = -\frac{i}{2}\lambda\!\!\!^{-} \sum_l (2l+1)(C_l e^{2i\delta_l}-1) P_l(\cos\theta), \qquad (55.33)$$

where $\delta_l$ is the phase at infinity, $P_l(\cos\theta)$ are the Legendre polynomials, and $C_l$ is the absolute value of the amplitude of the outgoing wave of angular momentum $l$. In the case of purely elastic scattering $C_l = 1$; however, if absorption of particles is taking place, then $C_l < 1$, and the quantity $\gamma_l = 1 - C_l^2$ determines the probability of absorption of particles of angular momentum $l$.

The cross section for particle absorption is determined by the formula

$$\sigma_a = \pi\lambda\!\!\!^{-}{}^2 \sum_{l=0}^{\infty} (2l+1)\gamma_l, \qquad (55.34)$$

and the elastic scattering cross section is given by the formula

$$\sigma_e = \int_0^\pi |f(\theta)|^2 2\pi \sin\theta \, d\theta = \pi\lambda\!\!\!^{-2} \sum_{l=0}^{\infty} (2l+1)|C_l e^{2i\delta_l} - 1|^2. \qquad (55.35)$$

Therefore, the total scattering cross section is equal to

$$\sigma_t = \sigma_e + \sigma_a = 2\pi\lambda\!\!\!^{-2} \sum_{l=0}^{\infty} (2l+1)(1 - \operatorname{Re} C_l e^{2i\delta_l}). \qquad (55.36)$$

We note that a comparison of this formula with the forward scattering amplitude

$$f(\omega, 0) = -\frac{i}{2}\lambda\!\!\!^{-} \sum_l (2l+1)(C_l e^{2i\delta_l} - 1)$$

leads to relation (55.7).

For high energies small angle scattering predominates; in this case large values of $l$ are important, and the sum (55.33), which at these energies contains many terms, can be replaced by an integral over $l$ or, what amounts to the same thing, by an integral over the impact parameter $b = l\lambda\!\!\!^{-}$.

For $l \gg 1$ and $\theta \ll 1$ the following asymptotic expression holds:

$$P_l(\cos\theta) \approx J_0(bs), \quad \text{and } s = \frac{\theta}{\lambda\!\!\!^{-}} = \omega\theta,$$

where $J_0(x)$ is a Bessel function; on substituting this expression into (55.33), and on replacing the sum by an integral we obtain

$$\begin{aligned} a_1(\omega, s) &= \omega \int b \, db \, \alpha(b, \omega) J_0(bs), \\ a_2(\omega, s) &= \omega \int b \, db \, \beta(b, \omega) J_0(bs), \end{aligned} \qquad (55.37)$$

where $\alpha(b, \omega) \equiv \alpha_l$ and $\beta(b, \omega) \equiv \beta_l$ are related to $C_l$ and $\delta_l$ by the expressions

$$\alpha_l \approx 2\delta_l, \quad \text{and} \quad \beta_l \approx 1 - C_l \approx \tfrac{1}{2}\gamma_l. \qquad (55.37')$$

The quantity $\gamma(b, \omega) = 2\beta(b, \omega)$ represents the probability of pair production by a wave packet of photons of frequency $\omega$, passing at a distance $b$ from the nucleus. This quantity can be easily obtained by using expressions (55.23), (55.23') for the momentum distribution of recoil nuclei. Indeed, the momentum transferred to the nucleus is approximately equal to $q = 1/b$, so that the momentum distribution of recoil nuclei at the same time determines the pair production probability as a function of the impact parameters $b$.

On utilizing the approximate formula (55.24) for the momentum distribution of recoil nuclei, and on setting in it $q = 1/b$, we obtain

$$\sigma_{\pm}(\omega) = \frac{28}{9}\frac{Z^2}{m^2}\left(\frac{e^2}{4\pi}\right)^3 \int\limits_{\frac{1}{m}}^{\frac{1}{q_{\min}}} \frac{db}{b}.$$

On comparing this formula with (55.6) and (55.37) we obtain the desired quantity $\beta(b, \omega)$ in the absence of screening:

$$\beta(b, \omega) = \frac{28}{9}\frac{Z^2}{m^2}\left(\frac{e^2}{4\pi}\right)^3 \frac{1}{4\pi b^2} \equiv \frac{C}{b^2}, \quad b_0 \leqslant b \leqslant b_{\max} = \frac{\omega}{m^2}, \tag{55.38}$$

$$\beta(b, \omega) = 0, \quad b > \frac{\omega}{m^2},$$

where

$$C = \frac{7}{9}\frac{Z^2}{\pi m^2}\left(\frac{e^2}{4\pi}\right)^3.$$

In the case of complete screening the quantity $\omega/m^2$ must be replaced by the atomic radius $a$. If $b > \omega/m^2$, then (55.38) yields $\beta = 0$, but it is more correct to take $\beta = Cm^2$.

For a given value of $b$ the quantity $\beta(b, \omega)$ can be regarded in accordance with (55.38) as a definite function of $\omega$. If $1/m < b < a$, then we have

$$\beta(b, \omega) = \begin{cases} \dfrac{C}{b^2}, & \omega > bm^2, \\ 0, & \omega < bm^2. \end{cases} \tag{55.38'}$$

Further,

$$\beta(b, \omega) = \begin{cases} 0 & \text{for} \quad b > a, \ \omega \text{ arbitrary}, \\ Cm^2 & \text{for} \quad b < 1/m, \ \omega > m. \end{cases} \tag{55.38''}$$

If we know the quantity $\beta(b,\omega)$ as a function of the frequency, we are able to obtain $\alpha(b,\omega)$. Indeed, $\alpha(b,\omega)$ and $\beta(b,\omega)$ are interrelated for a given value of $b$ by the dispersion relations (cf. subsection 5.4)

$$\begin{aligned}\alpha(b,\omega) &= \frac{\omega}{\pi}\mathcal{P}\int\limits_{-\infty}^{\infty}\frac{\beta(b,\omega')}{(\omega'-\omega)\omega'}d\omega',\\ \beta(b,\omega) &= -\frac{\omega}{\pi}\mathcal{P}\int\limits_{-\infty}^{\infty}\frac{\alpha(b,\omega')}{(\omega'-\omega)\omega'}d\omega'.\end{aligned} \tag{55.39}$$

Therefore, on substituting into these expressions (55.38′), (55.38″), we obtain

$$\begin{aligned}&\alpha(b,\omega) = \frac{C}{\pi b^2}\ln\frac{\omega+bm^2}{|\omega-bm^2|}, \quad \frac{1}{m} < b < a,\\ &\alpha(b,\omega) = 0, \quad b > a.\end{aligned} \tag{55.40}$$

We now evaluate $a_2(\omega,\theta)$ and $a_1(\omega,\theta)$. On introducing the notation $t = bm$ and $x = s/m$ we obtain in accordance with (55.37), (55.38)

$$a_2(s,\omega) = C\omega\left[\int\limits_0^1 t\,dt\,J_0(xt)+\int\limits_1^{\infty}\frac{dt}{t}J_0(xt)\right], \tag{55.41}$$

where we have replaced the upper limit in the second integral by infinity, although in actual fact it is equal to

$t_{\max} = \omega/m$ in the absence of screening,

$t_{\max} = am$ in the case of complete screening.

(This substitution leads to an error only for very small values of $x$, smaller than $1/t_{\max}$, i.e., for angles considerably smaller than $(m/\omega)^2$ in the absence of screening, and considerably smaller than $1/\omega a$ in the case of complete screening).

On noting that

$$\int\limits_0^1 t\,dt\,J_0(xt) = \frac{J_1(x)}{x}$$

and

$$\int_{x}^{\infty} \frac{du}{u} J_0(u) = \ln\frac{2}{\gamma x} + \frac{x^2}{8} - \frac{x^4}{2^8} + \frac{x^6}{2^9 3^3} - \dots, \qquad x \ll 1,$$

where $\ln(2/\gamma) = 0.116$, we obtain

$$a_2(\omega, \theta) = C\omega\left(\ln\frac{2}{\gamma x} + \frac{1}{2} + \frac{x^2}{16} - \frac{x^4}{2^8 \cdot 3} + \frac{x^6}{2^{11} 3^3} - \dots\right) \equiv C\omega F_2(x). \tag{55.41'}$$

In accordance with (55.37), (55.40) the real part of the scattering amplitude is equal to

$$a_1(\omega, \theta) = \frac{C}{\pi} m\left[\int_0^1 t\,dt\,J_0(xt)\ln\frac{\omega + mt}{|\omega - mt|} + \int_1^{\infty} \frac{dt}{t} J_0(xt)\ln\frac{\omega + mt}{|\omega - mt|}\right]. \tag{55.42}$$

For $\omega \gg m$ we can replace the logarithm in this expression approximately by $2tm/\omega$, in consequence of which we obtain

$$a_1(\omega, \theta) = \frac{Cm}{\omega}\left[\frac{2}{x}\int_0^{\infty} J_0(u)\,du + 2\int_0^{x} J_0(u)\,du(u^2 x^{-3} - x^{-1})\right]$$

$$= \frac{Cm}{\pi}\left(\frac{2}{x} - \frac{4}{3} + \frac{x^2}{3 \cdot 5} - \frac{x^4}{2^4 \cdot 5 \cdot 7} + \dots\right) \equiv CmF_1(x). \tag{55.42'}$$

The total photon scattering cross section is obviously equal to

$$\sigma(\omega) = 2\pi \int_0^{\pi} |a_1(\omega, \theta) + ia_2(\omega, \theta)|^2 \sin\theta\, d\theta \equiv \sigma_1 + \sigma_2, \tag{55.43}$$

where

$$\sigma_1 = 2\pi(Cm)^2\left(\frac{m}{\omega}\right)^2 \int_0^{\infty} F_1^2(x) x\,dx,$$

$$\sigma_2 = 2\pi(Cm)^2 \int_0^{\infty} F_2^2(x) x\,dx \tag{55.43'}$$

and $x = \omega\theta/m$. Since $\omega \gg m$, the part of the cross section which is associated with the imaginary part of the scattering amplitude is considerably greater than that which is associated with the real part, $\sigma_2 \gg \sigma_1$.

In order to evaluate $\sigma_1$ and $\sigma_2$ we note that

$$\int J_0(xt) J_0(xt') x\, dx = t^{-1}\delta(t-t').$$

Therefore

$$\int F_2^2(x) x\, dx = \int_0^1 t\, dt + \int_1^\infty \frac{dt}{t^3} = 1.$$

Moreover,

$$\int F_1^2(x) x\, dx = \left(\frac{2}{\pi}\right)^2 \int_{x_1}^\infty \frac{x\, dx}{x^2} = 4\pi^{-2} \ln \frac{1}{x_1},$$

where $x_1$ is the smallest value of $x$ for which (55.41) is still valid, i.e., $x_1 = 1/am$ in the case of complete screening, and $x_1 = m/\omega$ in the absence of screening.

On utilizing these formulas and the definitions (55.43) we finally obtain (23)

$$\sigma_2 = 2\pi(Cm)^2 = \frac{98}{81\pi} \frac{Z^4}{m^2} \left(\frac{e^2}{4\pi}\right)^6, \tag{55.44}$$

$$\sigma_1 = \begin{cases} \dfrac{3}{\pi^2}\left(\dfrac{m}{\omega}\right)^2 \sigma_2 \ln \dfrac{\omega}{m} & \text{in absence of screening,} \\[2ex] \dfrac{3}{\pi^2}\left(\dfrac{m}{\omega}\right)^2 \sigma_2 \ln am & \text{in case of complete screening.} \end{cases} \tag{55.44'}$$

Within the range of applicability of these formulas, i.e., for $\omega \gg m$, the total scattering cross section is almost completely determined by the absorption of photons associated with pair production. This part of the cross section may be called the cross section for potential scattering in analogy with the scattering of fast neutrons by absorbing nuclei, while that part of the cross section which is associated with the real part of the scattering amplitude may be referred to as the cross section for dispersion scattering. The ratio of the cross section for potential scattering $\sigma_2$ to the cross section for pair production $\sigma_\pm(\omega)$ is equal to

$$\frac{\sigma_2}{\sigma_\pm(\omega)} = \frac{Cm^2}{2\ln\frac{\omega}{m}} = \frac{7}{18}\left(\frac{e^2}{4\pi}\right)^3 Z^2 \frac{1}{\ln\frac{\omega}{m}}.$$

Figure 119 shows the functions $F_1^2(x)$ and $F_2^2(x)$ which determine the angular dependence of the differential cross sections for potential and for dispersion scattering. For comparison we have also given the angular dependence of the diffraction scattering of particles by an opaque sphere of radius $r_0 = 7/m$.

The absolute value of the cross section $\sigma_2$ in the case of urinium is approximately 6 mbarn, while the ratio $\sigma_2/\sigma_\pm(\omega)$ is approximately equal to 1/8000.

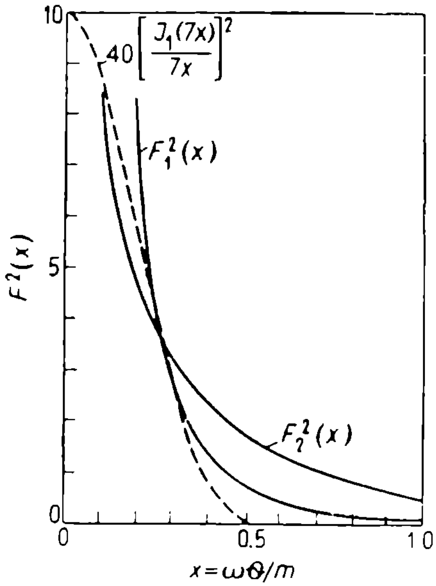


Fig. 119.

Up to energies of the order of $10^{10}$ eV the total cross section for coherent scattering of photons by nuclei remains smaller than the cross section for Compton scattering. However, the differential cross section for coherent small angle scattering of photons by nuclei may be considerably greater than the differential cross section for Compton scattering. Indeed, the differential cross section for potential scattering per unit solid angle is given in accordance with (55.43′) by the formula

$$\sigma_2(\theta) = \left(\frac{7}{9\pi}\right)^2 \frac{Z^4}{m^2} \alpha^6 \left(\frac{\omega}{m}\right)^2 F_2^2(x), \quad x = \frac{\omega}{m}\theta .$$

In the case of uranium we have

$$\sigma_2(\theta) \approx \left(\frac{\omega}{m}\right)^2 F_2^2(x) \frac{\text{mbarn}}{\text{steradian}} .$$

For $\omega = 300$ MeV and $x = 0.1$, which corresponds to $\theta = 0.01°$, this formula gives 3000 barn/steradian, while the differential cross section for Compton scattering under the same conditions gives only 7 barn/steradian. The Rayleigh scattering is also small in this case, and, therefore, the scattering of photons for large energies and small scattering angles is determined almost completely by potential scattering.

CHAPTER IX

# Electrodynamics of Particles of Zero Spin

## § 56. Field Equations for Scalar Particles

### 56.1. *First Order Equations*

In this chapter we shall give a brief review of the electromagnetic properties of charged particles of zero spin.

The electrodynamics of scalar particles is of interest because it is the only example of electrodynamics of particles of spin other than $\frac{1}{2}$ in which it is possible to eliminate all the occurring divergences with the aid of renormalization.

For the investigation of electromagnetic properties of scalar particles it is convenient to start not with the second order scalar wave equation, but with the first order equations derived in § 20 and formally of the same appearance as the Dirac equations (51), (104):

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi = 0. \tag{56.1}$$

Here $\psi$ is the wave function for the particle, $m$ is its mass and $\beta_\mu$ are Hermitian matrices satisfying the relations

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\lambda + \delta_{\lambda\nu} \beta_\mu . \tag{56.2}$$

The linear equations (56.1) describe the properties of free particles of either zero or unit spin, with the matrices $\beta_\mu$ satisfying the same relations (56.2) in both cases. The difference between the two values of spin consists of the fact that in the case of zero spin the wave function $\psi$ has five components, while in the case of unit spin it has ten components. In accordance with this in the former case the matrices $\beta_\mu$ have five rows, while in the latter case they have ten rows.

The five-component wave function which describes particles of zero spin comprises the scalar wave function and its four derivatives with respect to the coordinates and the time which transform like the components of a vector.

Since the matrices $\beta_\mu$ satisfy the same relations in the case of scalar particles and in the case of particles of unit spin, the general formulas for the matrix elements which determine the various processes of interaction for particles of spins 0 and 1 are of the same form. The probabilities of these processes are, of course, different for the two kinds of particles, since traces of products of the five-rowed and the ten-rowed matrices $\beta_\mu$ are different.

We give here the formula required for calculating probabilities of various processes which determines the trace of the product of $n$ matrices $\beta_\mu$:

$$\mathrm{Sp}\,\{\beta_\mu\beta_\nu\beta_\lambda\beta_\sigma\beta\,\ldots\,\beta_\varrho\beta_\tau\beta_\varkappa\}$$
$$=\begin{cases}\delta_{\mu\nu}\,\delta_{\lambda\sigma}\ldots\,\delta_{\tau\varkappa}+\delta_{\nu\lambda}\ldots\,\delta_{\varrho\tau}\,\delta_{\mu\varkappa}, & \text{if } n \text{ is an even number,}\\ 0, & \text{if } n \text{ is an odd number.}\end{cases} \tag{56.3}$$

We introduce the matrices $\eta_\mu = 2\beta_\mu^2 - \mathrm{I}$ where I is the five-rowed indentity matrix. It can then be easily shown, by utilizing (56.1) and the condition that $\beta_\mu$ are Hermitian, that the function $\overline{\psi} = \psi^*\eta_4$ satisfies the equation

$$\frac{\partial\overline{\psi}}{\partial x_\mu}\beta_\mu + m\overline{\psi} = 0. \tag{56.4}$$

It can be easily shown that the quantity $\overline{\psi}\psi$ is a scalar, while $\overline{\psi}\beta_\mu\psi$ is a four-vector.

With the aid of $\overline{\psi}$ we can construct the Lagrangian density $L_0$ for the free field corresponding to the scalar particles, and also the current density $j_\mu$ and the energy-momentum tensor $T_{\mu\nu}$

$$L_0 = -\frac{1}{2}\left(\overline{\psi}\beta_\mu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\overline{\psi}}{\partial x_\mu}\beta_\mu\psi\right) - m\overline{\psi}\psi, \tag{56.5}$$

$$j_\mu = ie(\overline{\psi}\beta_\mu\psi), \tag{56.6}$$

$$T_{\mu\nu} = \frac{1}{2}\left(\overline{\psi}\beta_\nu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\psi}{\partial x_\mu}\beta_\nu\psi\right)\cdot \tag{56.7}$$

We now consider the interaction between the charged scalar particle and the electromagnetic field. In order to obtain the equation describing this interaction we must, as in the case of the Dirac equation, replace in (56.1) $\partial/\partial x_\mu$ by $\partial/\partial x_\mu - ieA_\mu(x)$, where $A_\mu(x)$ is the potential of the electromagnetic field and $e$ is the charge of the particle. As a result of this we obtain

$$\left[\beta_\mu\left(\frac{\partial}{\partial x_\mu}-ieA_\mu\right)+m\right]\psi=0. \tag{56.8}$$

The analogous equation for $\bar{\psi}$ has the form

$$\bar{\psi}\left[\beta_\mu\left(\frac{\partial}{\partial x_\mu}+ieA_\mu\right)+m\right]=0. \tag{56.9}$$

The current density for charged scalar particles in the presence of an electromagnetic field is determined, as in the case of the free particles, by expression (56.6). Therefore, the equation which determines the electromagnetic field $A_\mu$ produced by charged scalar particles has the form

$$\frac{\partial^2 A_\nu}{\partial x_\mu^2}=-j_\nu. \tag{56.10}$$

The equations for the scalar particles (56.8) and (56.9), like the Dirac equations, are invariant under the transformation of charge conjugation. This transformation has the form

$$\psi'=\bar{\psi}C, \quad \bar{\psi}'=C^{-1}\psi, \tag{56.11}$$

where $C$ is a matrix with the following properties:

$$\begin{aligned} C^{-1}\beta_\mu C &= -\tilde{\beta}_\mu, \\ C &= C^{+}=\tilde{C}, \\ C^2 &= 1. \end{aligned} \tag{56.12}$$

If the representation of the matrices $\beta_\mu$ is such that the matrices $\beta_k$ are real for $k=1,2,3$, while $\beta_4$ is a purely imaginary matrix, then

$$C=\eta_4. \tag{56.13}$$

However, if all the matrices $\beta_\mu$ are real, then

$$C=\eta_1\eta_2\eta_3\eta_4. \tag{56.13'}$$

It can be easily shown that the quantities $\psi'$ and $\bar{\psi}'$ satisfy the equations

$$\begin{aligned} \left[\beta_\mu\left(\frac{\partial}{\partial x_\mu}+ieA_\mu\right)+m\right]\psi'&=0, \\ \bar{\psi}'\left[\beta_\mu\left(\frac{\partial}{\partial x_\mu}-ieA_\mu\right)+m\right]&=0. \end{aligned} \tag{56.14}$$

With the aid of the charge-conjugate functions $\psi'$ and $\bar{\psi}'$ we can write the current density $j_\mu$ in symmetric form

$$j_\mu = \frac{ie}{2}(\bar{\psi}\beta_\mu\psi - \bar{\psi}'\beta_\mu\psi'). \tag{56.15}$$

Finally, we give the expression for the Lagrangian density in the case of interacting scalar and electromagnetic fields

$$L = \frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu} - \frac{1}{2}\bar{\psi}\left(\beta_\mu\frac{\partial}{\partial x_\mu}+m\right)\psi - \frac{1}{2}\bar{\psi}'\left(\beta_\mu\frac{\partial}{\partial x_\mu}+m\right)\psi' + j_\mu A_\mu. \tag{56.16}$$

The last term describes, as in the case of electrodynamics for electrons, the interaction between the fields.

If we treat $\psi$, $\bar{\psi}$ and $A_\mu$ as operators, then equations (56.8),. (56.9) and (56.10) describe the interacting quantized scalar and electromagnetic fields in the Heisenberg picture.

We now demonstrate how the quantization of the scalar charged particle field is carried out.

## 56.2. *Quantization of the Free Scalar Field*

We consider the state of a free scalar particle of momentum $p(\mathbf{p}, ip_0)$

$$\psi(x) = \varphi(p)e^{ipx}.$$

From (56.1) it follows that $\varphi(p)$ satisfies the equation

$$(i\hat{p}+m)\varphi(p) = 0, \tag{56.17}$$

where $\hat{p} = \beta_\mu p_\mu$. It can be easily shown that for any vector $q$ the following relation holds:

$$\hat{q}(q^2-q^2) = 0. \tag{56.18}$$

On utilizing this relation and (56.17) we obtain $p^2+m^2 = 0$, whence it follows that $p_0 = \pm\sqrt{\mathbf{p}^2+m^2}$.

Thus, a given value of the momentum of the particle corresponds to two solutions $\varphi(p)$ which differ in the sign of $p_0$. These solutions may be referred to as solutions of positive and negative frequency. We denote the amplitudes $\varphi(p)$ corresponding to them by $\varphi^{(+)}(p)$ and $\varphi^{(-)}(p)$.

In order to determine the physical meaning of these solutions we define the energy and the charge of the field by utilizing the general formulas (56.6), (56.7):

$$\begin{aligned} E &= -\int T_{44} dr = \pm p_0\left(\bar{\varphi}^{(\pm)}(p)\beta_4\varphi^{(\pm)}(p)\right), \\ Q &= \frac{1}{i}\int j_4\, dr = e\left(\bar{\varphi}^{(\pm)}(p)\beta_4\varphi^{(\pm)}(p)\right) \end{aligned} \tag{56.19}$$

(we assume the normalizing volume both here and later to be equal to unity). Since the energy of the field $E$ must be positive, it is natural to normalize $\varphi\,(p)$ in the following manner:

$$\left(\bar{\varphi}^{(\pm)}(p)\beta_4\varphi^{(\pm)}(p)\right) = \pm 1. \tag{56.20}$$

In this case $E$ coincides with $|p_0| = \sqrt{p^2+m^2}$, while the charge of the field is equal to $+e$ in the state $\varphi^{(+)}(p)$ and to $-e$ in the state $\varphi^{(-)}(p)$. Thus, solutions of positive and negative frequencies describe particles of both signs of the charge.

We now proceed to quantize the scalar particle field. As in the case of the electron-positron field, we must regard $\psi$ and $\bar{\psi}$ as operators operating in particle number space and obeying definite commutation relations. In order to establish these relations we utilize expansions of $\psi$ and $\bar{\psi}$ in terms of plane waves:

$$\begin{aligned} \psi &= \sum_p \{\mathrm{a}_p\varphi^{(+)}(p)\,e^{ipx}+\mathrm{b}_p^+\varphi^{(-)}(p)\,e^{-ipx}\}, \\ \psi &= \sum_p \{\mathrm{a}_p^+\bar{\psi}^{(+)}(p)e^{-ipx}+\mathrm{b}_p\bar{\psi}^{(-)}(p)\,e^{ipx}\}, \end{aligned} \tag{56.21}$$

where $\varphi^{(\pm)}(p)$ satisfy the normalization conditions (56.20) while $\mathrm{a}_p$, $\mathrm{a}_p^+$, $\mathrm{b}_p$ and $\mathrm{b}_p^+$ are certain operators in particle number space. Substitution of (56.21) into (56.6),(56.7) leads to the following expressions for the energy $E$, momentum $\boldsymbol{P}$ and charge $Q$ of the field:

$$\begin{aligned} E &= \sum_p p_0(\mathrm{a}_p^+\mathrm{a}_p+\mathrm{b}_p\mathrm{b}_p^+), \\ \boldsymbol{P} &= \sum_p \boldsymbol{p}(\mathrm{a}_p^+\mathrm{a}_p+\mathrm{b}_p\mathrm{b}_p^+), \\ Q &= \sum_p e(\mathrm{a}_p^+\mathrm{a}_p-\mathrm{b}_p\mathrm{b}_p^+). \end{aligned} \tag{56.22}$$

Since these quantities must represent sums of the energies, momenta and charges of the individual particles the operators $\mathrm{a}_p$ and $\mathrm{b}_p$ must satisfy the following commutation relations:

$$\begin{aligned} &[\mathrm{a}_p, \mathrm{a}_q^+] = \delta_{pq}, && [\mathrm{b}_p, \mathrm{b}_q^+] = \delta_{pq}, && \\ &[\mathrm{a}_p, \mathrm{a}_q] = 0, && [\mathrm{b}_p, \mathrm{b}_q] = 0, && [\mathrm{a}_p, \mathrm{b}_q] = 0, \\ &[\mathrm{a}_p^+, \mathrm{a}_q^+] = 0, && [\mathrm{b}_p^+, \mathrm{b}_q^+] = 0, && [\mathrm{a}_p^+, \mathrm{b}_q^+] = 0. \end{aligned} \tag{56.23}$$

Moreover, the eigenvalues of the operators $\mathrm{a}_p^+\mathrm{a}_p$ and $\mathrm{b}_p^+\mathrm{b}_p$ will be integers, which we denote by $N_p^{(+)}$ and $N_p^{(-)}$ while $E$, $\boldsymbol{P}$, $Q$ assume the form

$$\begin{aligned} E &= \sum_p p_0(N_p^{(+)}+N_p^{(-)}-1), \\ \boldsymbol{P} &= \sum_p \boldsymbol{p}(N_p^{(+)}+N_p^{(-)}-1), \\ Q &= \sum_p e(N_p^{(+)}-N_p^{(-)}-1). \end{aligned} \tag{56.24}$$

The quantities $N_p^{(+)}$ and $N_p^{(-)}$ are the numbers of positively and negatively charged particles of momentum $\boldsymbol{p}$.

### 56.3 *Commutators of the Field. Vacuum Expectation Values of Products of Field Components*

We now obtain the commutation relations for the operators $\psi(x)$ and $\overline{\psi}(x)$. On utilizing expansions (56.21) and the commutation relations (56.23), we obtain

$$[\psi_\alpha(x), \overline{\psi}_\beta(x')] = \sum_p F_{\alpha\beta}^+(p)e^{ip(x-x')} - \sum_p F_{\alpha\beta}^-(p)e^{-ip(x-x')}, \tag{56.23'}$$

where

$$\begin{aligned} F_{\alpha\beta}^+(p) &= \varphi_\alpha^{(+)}(p)\overline{\varphi}_\beta^{(+)}(p), \\ F_{\alpha\beta}^-(p) &= \varphi_\alpha^{(-)}(p)\overline{\varphi}_\beta^{(-)}(p). \end{aligned} \tag{56.25}$$

It can be easily shown that $F_{\alpha\beta}^+(p)$ and $F_{\alpha\beta}^-(p)$ satisfy the equations

$$\begin{aligned} (i\hat{p}+m)F^+(p) &= 0, \\ (-i\hat{p}+m)F^-(p) &= 0. \end{aligned}$$

Therefore, the left hand side of these equations must coincide up to a numerical factor with the polynomial $\hat{p}(\hat{p}^2+m^2) = 0$:

$$\begin{aligned} F^+(p) &= a\hat{p}(i\hat{p}-m), \\ F^-(p) &= b\hat{p}(i\hat{p}+m). \end{aligned} \tag{56.25'}$$

The coefficients $a$ and $b$ can be evaluated from the normalization condition (56.20):

$$\big(\varphi^{(+)}(p)\beta_4\varphi^{(+)}(p)\big) = \mathrm{Sp}\,\big(\beta_4 F^{+}(p)\big) = 1,$$
$$\big(\varphi^{(-)}(p)\beta_4\varphi^{(-)}(p)\big) = \mathrm{Sp}\,\big(\beta_4 F^{-}(p)\big) = -1.$$

On substituting into these equations expressions (56.25′) we obtain

$$\frac{1}{a} = \mathrm{Sp}\,\big(\beta_4 \hat{p}(i\hat{p}-m)\big),$$
$$\frac{1}{b} = -\mathrm{Sp}\big(\beta_4 \hat{p}(i\hat{p}+m)\big).$$

Further, on utilizing formulas (56.3) which determine the traces of the products of the matrices $\beta_\mu$, we finally obtain

$$a = b = \frac{i}{2mp_0}$$

and

$$F^{+}(p) = \frac{i\hat{p}(i\hat{p}-m)}{2mp_0}, \quad F^{-}(p) = \frac{i\hat{p}(i\hat{p}+m)}{2mp_0}. \tag{56.26}$$

On substituting these expressions into (56.23′) we obtain the commutators for the field:

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')] = -\frac{i}{m}\left(\beta_\mu \frac{\partial}{\partial x_\mu}\left(\beta_\nu \frac{\partial}{\partial x_\nu} - m\right)\right)_{\alpha\beta} \Delta(x-x'). \tag{56.27}$$

To these relations we must add the obvious relations

$$\begin{aligned} [\psi_\alpha(x), \psi_\beta(x')] &= 0, \\ [\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')] &= 0. \end{aligned} \tag{56.27′}$$

We define the chronological product of the field operators

$$T\big(\bar{\psi}_\alpha(x)\psi_\beta(x')\big) = \tfrac{1}{2}\big(1+\varepsilon(t, t')\big)\bar{\psi}_\alpha(x)\psi_\beta(x') + \tfrac{1}{2}\big(1-\varepsilon(t, t')\big)\psi_\beta(x')\bar{\psi}_\alpha(x), \tag{56.28}$$

where

$$\varepsilon(t, t') = \begin{cases} 1, & t > t', \\ -1, & t < t', \end{cases}$$

and calculate its expectation value in the vacuum state of the scalar particle field $\langle T\big(\bar{\psi}_\alpha(x)\psi_\beta(x')\big)\rangle_0$.

The vacuum state is defined as the state characterized by the minimum energy of the field, i.e.,

$$\langle \mathrm{a}_p^+ \mathrm{a}_p \rangle_0 = \langle \mathrm{b}_p^+ \mathrm{b}_p \rangle_0 = 0. \tag{56.29}$$

On utilizing this definition and the definitions of the functions $F^+(p)$ and $F^-(p)$, we can, first of all, obtain the quantities

$$\langle \bar{\psi}_\alpha(x)\psi_\beta(x')\rangle_0 = \sum_p F^-_{\alpha\beta}(p)\, e^{ip(x-x')},$$

$$\langle \psi_\beta(x')\bar{\psi}_\alpha(x)\rangle_0 = \sum_p F^+_{\alpha\beta}(p)\, e^{-ip(x-x')}.$$

Further, in accordance with (56.28) we obtain the quantity of interest to us

$$\langle T(\psi_\alpha(x)\bar{\psi}_\beta(x'))\rangle_0 = \frac{1}{2m}\left[\beta_\mu \frac{\partial}{\partial x_\mu}\left(\beta_\nu \frac{\partial}{\partial x_\nu} - m\right)\right]_{\alpha\beta} \Delta^{(1)}(x-x')$$

$$-\frac{i\varepsilon(t,t')}{2m}\left[\beta_\mu \frac{\partial}{\partial x_\mu}\left(\beta_\nu \frac{\partial}{\partial x_\nu} - m\right)\right]_{\alpha\beta} \Delta(x-x'). \tag{56.30}$$

This expression can be rewritten in the form

$$\langle T(\bar{\psi}_\lambda(x)\psi_\beta(x'))\rangle_0 = T^c_{\alpha\beta}(x-x') + \frac{i}{m}(1-\beta_0^2)\,\delta(x-x'), \tag{56.31}$$

where $\beta_0 = -i\beta_4$ and

$$T^c(x) = \frac{1}{2m}\left[\beta_\mu \frac{\partial}{\partial x_\mu}\left(\beta_\nu \frac{\partial}{\partial x_\nu} - m\right) - \frac{\partial^2}{\partial x_\nu^2} + m^2\right]\Delta^c(x). \tag{56.32}$$

The function $T^c(x-x')$ is the Green's function for equation (56.1)

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + m\right) T^c(x) = -i\delta(x). \tag{56.33}$$

This function may also be called the scalar particle propagator. We see that it differs from the vacuum expectation value of the chronological product $\bar{\psi}_\alpha(x)\psi_\beta(x')$ by the quantity $-(i/m)(1-\beta_0^2)\,\delta(x-x')$.

We recall that in the case of the electron-positron field the propagator does not differ from the corresponding vacuum expectation value. This is associated with the fact that not all the components of the field $\psi(x)$ are dynamically independent (cf. subsection 57.1).

We also determine the difference between the $T$- and $N$-products of two field operators $\psi(x)$ and $\overline{\psi}(x')$. This quantity, called the pairing of the field operators, is obviously equal to

$$\psi^a(x)\,\overline{\psi}^a(x') \equiv T\big(\psi(x)\,\overline{\psi}(x')\big) - N\big(\psi(x)\,\overline{\psi}(x')\big)$$
$$= T^c(x-x') + \frac{i}{m}(1-\beta_0^2)\,\delta(x-x'). \qquad (56.34)$$

We also give here the expression for the Fourier component of the function $T^c(x)$ which will be needed for subsequent development:

$$T^c(x) = \frac{1}{(2\pi)^4 i}\int T^c(p)\,e^{ipx}\,d^4p,$$
$$T^c(p) = (i\hat{p}+m)^{-1} = \frac{i\hat{p}(i\hat{p}-m)+p^2+m^2}{m(p^2+m^2)}. \qquad (56.35)$$

## § 57. The Scattering Matrix in Scalar Electrodynamics

### 57.1. *The Interaction Picture*

Equations (56.8), (56.9), (56.10) in which $\psi(x)$, $\overline{\psi}(x)$ and $A_\mu(x)$ are regarded as field operators are the fundamental equations for the electrodynamics of scalar particles in the Heisenberg picture. For the investigation of the different processes of interaction between the scalar particle field and the electromagnetic field it is convenient to go over to the interaction picture. The state vector $\Phi(t)$ in this picture is related to the constant state vector $\Phi_0$ in the Heisenberg picture by the expression

$$\Phi(t) = \mathrm{S}(t)\Phi_0, \qquad (57.1)$$

where the Hermitian operator $\mathrm{S}(t)$ satisfies the equation

$$i\frac{\partial \mathrm{S}}{\partial t} = \mathrm{V}(t)\mathrm{S}(t), \qquad (57.2)$$

where

$$\mathrm{V}(t) = \int \mathrm{H}(x)\,d\mathbf{r}$$

and $\mathrm{H}(x)$ is the Hamiltonian for the interaction between the fields in the interaction picture. The relation between the operators in the Heisenberg picture and in the interaction picture $f$ and $\mathrm{f}$, is given by the expression

$$f = \mathrm{S}^{-1}(t)\,\mathrm{f}\,\mathrm{S}(t). \qquad (57.3)$$

In electron electrodynamics the field operators $\psi(x)$ and $A_\mu(x)$ in the interaction picture have the same form as the operators for the corresponding free fields. But although this situation remains the same in the electrodynamics of scalar particles in the case of $A_\mu(x)$, it no longer holds for the operators $\psi(x)$. The point is that not all the components of the field $\psi(x)$ are dynamically independent. Indeed, on multiplying equation (56.8) from the right by $1-\beta_0^2$ we obtain

$$(1-\beta_0^2)\psi = -\frac{1}{m}\beta_k\left(\frac{\partial}{\partial x_k}-ieA_k\right)\beta_0^2\psi, \quad (k=1,2,3). \quad (57.4)$$

On utilizing the following specific representation of the matrix $\beta_0^2$

$$\beta_0^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we see that $\beta_0^2\psi$ contains only two components, and correspondingly $(1-\beta_0^2)\psi$ contains only three components. Thus, the components of $(1-\beta_0^2)\psi$ can be expressed in terms of the two components of $\beta_0^2\psi$ which are dynamically independent. On taking this fact into account we can require that the dynamically independent components should correspond to operators which in the interaction picture have the same form as the free field operators, i.e.,

$$\beta_0^2\psi = \beta_0^2\psi^{(0)}, \quad (57.5)$$

where $\psi(x)$ are the operators for the scalar particle field in the interaction picture and $\psi^{(0)}(x)$ are the free field operators for the same particles.

Expression (57.5) enables us to obtain a direct relation between $\psi(x)$ and $\psi^{(0)}(x)$. With this in mind we note that the field operators $\psi(x)$ in the interaction picture, like the corresponding operators in the Heisenberg picture, must satisfy the subsidiary condition (57.4). On utilizing this condition, and also the analogous condition for free fields

$$(1-\beta_0^2)\psi^{(0)} = -\frac{1}{m}\beta_k\frac{\partial}{\partial x_k}\beta_0^2\psi^{(0)} \quad (k=1,2,3)$$

and, finally, condition (57.5), we can easily obtain the expression relating $\psi(x)$ to $\psi^{(0)}(x)$ and $A_\mu^{(0)}(x)$:

$$\psi(x) = \left(1+\frac{ie}{m}(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}(x)\right)\psi^{(0)}(x). \tag{57.6}$$

Similarly we can obtain

$$\bar{\psi}(x) = \bar{\psi}^{(0)}(x)\left(1-\frac{ie}{m}\hat{\mathrm{A}}^{(0)}(x)(1-\beta_0^2)\right). \tag{57.7}$$

We now obtain the form of the Hamiltonian density H$(x)$. In view of the subsidiary condition (57.4) this quantity will differ from the "natural" value $-ie\bar{\psi}^{(0)}(x)\hat{\mathrm{A}}^{(0)}(x)\psi^{(0)}(x)$.

In order to obtain H$(x)$ we operate with operator $\beta_\mu\,\partial/\partial x_\mu+m$ on the field operator $\psi(x)$ in the Heisenberg picture. On noting that in accordance with (57.3) and (57.6)

$$\psi(x) = \mathrm{S}^{-1}(t)\psi^{(0)}(x)\mathrm{S}(t)+\frac{ie}{m}\mathrm{S}^{-1}(t)\,(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}(x)\psi^{(0)}(x)\mathrm{S}(t),$$

we obtain, on utilizing the relation $(1-\beta_0^2)\beta_0 = 0$,

$$\begin{aligned}\left(\beta_\mu\frac{\partial}{\partial x_\mu}+m\right)\psi &= \left(\beta_\mu\frac{\partial}{\partial x_\mu}+m\right)\mathrm{S}^{-1}(t)\psi^{(0)}\,\mathrm{S}(t)\\ &\quad+\frac{ie}{m}\mathrm{S}^{-1}(t)\left(\beta_k\frac{\partial}{\partial x_k}+m\right)(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}\psi^{(0)}\mathrm{S}(t) \qquad (k=1,2,3).\end{aligned}$$

Further, on utilizing equation (57.2), we obtain

$$\begin{aligned}\left(\beta_\mu\frac{\partial}{\partial x_\mu}+m\right)\psi = \mathrm{S}^{-1}(t)\Bigg\{&i[\mathrm{V}(t),\beta_0\psi^{(0)}]\\ &+\frac{ie}{m}\left(\beta_k\frac{\partial}{\partial x_k}+m\right)(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}\psi^{(0)}\Bigg\}\mathrm{S}(t).\end{aligned} \tag{57.8}$$

On the other hand, it follows from equation (57.6):

$$\left(\beta_\mu\frac{\partial}{\partial x_\mu}+m\right)\psi = \mathrm{S}^{-1}(t)\left\{ie\hat{\mathrm{A}}^{(0)}\left(1+\frac{ie}{m}(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}\right)\psi^{(0)}\right\}\mathrm{S}(t). \tag{57.9}$$

A comparison of (57.8) with (57.9) shows that the operator $\mathrm{V}(t)$ must satisfy the relation

$$[\mathrm{V}(t), \beta_0 \psi^{(0)}(x)] = e\hat{\mathrm{A}}^{(0)}(x)\left\{1+\frac{ie}{m}(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}(x)\right\}\psi^{(0)}(x)$$

$$-\frac{e}{m}\left(\beta_k\frac{\partial}{\partial x_k}+m\right)(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}(x)\psi^{(0)}(x) \qquad (k = 1, 2, 3). \tag{57.10}$$

We can satisfy this equation and also the analogous equation for $\overline{\psi}(x)$ by choosing $\mathrm{H}(x)$ in the form (141)

$$\mathrm{H}(x) = -ie\overline{\psi}^{(0)}(x)\hat{\mathrm{A}}^{(0)}(x)\left[1+\frac{ie}{m}(1-\beta_0^2)\hat{\mathrm{A}}^{(0)}(x)\right]\psi^{(0)}(x). \tag{57.11}$$

This can be easily shown if we utilize the commutation relations for $\psi^{(0)}(x)$ and $\overline{\psi}^{(0)}(x)$ at the same instant of time:

$$[\psi_\mu(\boldsymbol{r}, t), \overline{\psi}\,(\boldsymbol{r}', t)] = -i(\beta_0)_{\mu\nu}\,\delta(\boldsymbol{r}-\boldsymbol{r}')+\frac{i}{m}(\beta_0\beta_k+\beta_k\beta_0)_{\mu\nu}\frac{\partial}{\partial x_k}\delta(\boldsymbol{r}-\boldsymbol{r}').$$

### 57.2. *Rules for Calculating Elements of the Scattering Matrix*

We proceed to construct the scattering matrix S. This matrix, just as the scattering matrix in electron electrodynamics, is given by the value of $\mathrm{S}(t)$ for $t = \infty$, and can evidently be written in the form

$$\mathrm{S} = T\left(e^{-i\int \mathrm{H}(x)d^4x}\right), \tag{57.12}$$

where $T$ is the symbol for time-ordering. In electron electrodynamics the expansion of the last equation in terms of $\mathrm{H}(x)$ is at the same time an expansion in powers of the charge $e$. In scalar electrodynamics the situation is made more complicated by the fact that $\mathrm{H}(x)$ contains the charge $e$ not only as a common factor, but also inside the square brackets of (57.11). Therefore an expansion in powers of $\mathrm{H}(x)$ will not be an expansion in powers of $e$.

In order to avoid this difficulty we go over in the S-matrix from $T$-ordering to $N$-ordering without utilizing the expansion of the S-matrix. This can be done by means of the technique described in subsection 24.4.

By proceeding in this manner it can be shown that (7)

$$\mathrm{S} = N\left(e^{\Sigma'}e^{\Delta}e^{-i\int \mathrm{H}(x)\,d^4x}\right) \tag{57.12'}$$

where

$$\Sigma' = \Sigma - \eta,$$

$$\Sigma = \int d^4x_1 \int dx_2 \frac{\delta}{\delta\bar{\psi}(x_1)} T^c(x_1 - x_2) \frac{\delta}{\delta\psi(x_2)},$$

$$\eta = -\int d^4x \frac{\delta}{\delta\bar{\psi}(x)} \frac{i}{m} (1 - \beta_0^2) \frac{\delta}{\delta\psi(x)},$$

$$\Delta = \int d^4x_1 \int d^4x_2 \, D^c(x_1 - x_2) \frac{\delta^2}{\delta A_\mu(x_1)\, \delta A_\mu(x_2)}.$$

On the other hand, it can be easily shown that

$$e^{-i\int \mathrm{H}(x)\,d^4x} = e^{\eta} e^{-e\int \bar{\psi}(x)\,\hat{\mathrm{A}}(x)\,\psi(x)\,d^4x}.$$

On substituting this expression into (57.13) we obtain:

$$\mathrm{S} = N\left(e^{\Sigma} e^{\Delta} e^{-e\int \bar{\psi}(x)\,\hat{\mathrm{A}}(x)\,\psi(x)\,d^4x}\right). \tag{57.14}$$

(Here and later we omit the index zero from the field operators.)

This expression is formally of the same form as the analogous expression for the scattering matrix in electron electrodynamics. It corresponds to the interaction Hamiltonian (106)

$$\mathrm{H}(x)_{\mathrm{eff}} = -ie\bar{\psi}(x)\,\hat{\mathrm{A}}(x)\psi(x), \tag{57.15}$$

and to effective pairings between the field operators given by

$$\psi_\alpha^a(x)\bar{\psi}_\beta^a(x') = T_{\alpha\beta}^c(x - x'). \tag{57.16}$$

We thus see that for the evaluation of the elements of the scattering matrix we can start from the effective Hamiltonian (57.15), which differs from (57.11) by the absence of the last term, and that we can utilize the effective pairing between the field operators which differs from (56.34) also by the absence of the last term. From this it can be seen that the evaluation of the matrix elements of S can be carried out with the aid of Feynman diagrams and rules of the same kind as in electron electrodynamics, the only difference consisting of the fact that now in the diagrams the internal lines corresponding to scalar particles must be made to correspond to $T^c(p)$, and not $S^c(p)$; moreover, the vertices of the diagrams must be made to correspond to the matrices

$\beta_\mu$, and not $\gamma_\mu$. Finally, we must keep in mind in evaluating the cross sections for the different processes that scalar particles have no spin, and, therefore, the cross section need not be averaged over spins.

### 57.3. *Divergences of the Scattering Matrix*

We now proceed to investigate the problem of the divergences of the scattering matrix. In order to do this we consider an irreducible $n$th order diagram containing $N_m$ external scalar particle lines, and $N_p$ external photon lines. If the number of internal lines of the diagram is equal to $F$, then the number of independent variables of integration in the integral which determines the matrix element corresponding to the diagram will be $4(F-n+1)$. In the integrand of the matrix element we have a rational function the degree of whose denominator is equal to $2F$. Whether the integral is convergent or not is determined by the difference between the degrees of the numerator and the denominator. The integral will converge if $K = 2F-n'-4(F-n+1) \geqslant 1$, where $n'$ is the degree of the numerator, and will diverge if $K < 1$.

Thus the problem reduces to the determination of $n'$.

In order to determine $n'$ we consider in our diagram an internal scalar particle line which contains $s$ vertices, $s \leqslant n$. This line corresponds to the following expression in the integrand of the matrix element:

$$A_s = \prod_{r=1}^{s-1} \left( \beta_{\mu_r} \frac{i\hat{p}_r - m + \frac{1}{m}(\hat{p}_r^2 - p_r^2)}{p_r^2 + m^2} \right) \beta_{\mu_s}.$$

We determine the highest power of $p$ in the numerator of this expression. It might seem that this power is equal to $2(s-1)$, and coincides with the degree of the denominator. In actual fact, however, because of the properties of the matrices $\beta_\mu$ this is not so, and the degree of the numerator turns out to be equal to $s$.

For example, take $s = 3$. Then $A_3$ has the form

$$A_3 = \beta_\mu \left[ i\hat{p}_1 - m + \frac{1}{m}(\hat{p}_1^2 - p_1^2) \right] \beta_\nu \left[ i\hat{p}_2 - m + \frac{1}{m}(\hat{p}_2^2 - p_2^2) \right] \beta_\lambda$$
$$\times \frac{1}{(p_1^2 + m^2)(p_2^2 + m^2)}.$$

The highest power of $p$ in the numerator corresponds to the term

$$X_1 = (\hat{p}_1^2 - p_1^2)\beta_\nu(\hat{p}_2^2 - p_2^2),$$

which vanishes for all $p_1$ and $p_2$. Therefore, the degree of the numerator is equal to 3, and not to 4.

The vanishing of $X_1$ is a consequence of the more general relation

$$X_n = (\hat{p}^2 - p^2)\beta_{\mu_1}\beta_{\mu_2}\beta_{\mu_3}\cdots\beta_{\mu_n}(\hat{q}^2 - q^2) = 0,$$

which holds if $n$ is an odd number.

In order to show the validity of this last relation we introduce the matrices

$$\mathrm{P} = \beta_1^2\beta_2^2\beta_3^2\beta_4^2, \qquad \bar{\mathrm{P}} = \sum \beta_\mu \mathrm{P} \beta_\mu, \tag{57.17}$$

which differ from zero in the case of the five-rowed matrices $\beta_\mu$, and which are equal to zero in the case of the ten-rowed matrices $\beta_\mu$.

It can be shown that the matrices P and $\bar{\mathrm{P}}$ satisfy the relations

$$\begin{aligned} \beta_\mu\beta_\nu \mathrm{P} &= \delta_{\mu\nu}\mathrm{P}, \\ \mathrm{P}\beta_\mu\beta_\nu &= \delta_{\mu\nu}\mathrm{P}, \\ \beta_\mu \mathrm{P} &= \mathrm{P}\beta_\mu, \\ \mathrm{P}\beta_\mu &= \beta_\mu \mathrm{P}, \\ (\mathrm{P}+\bar{\mathrm{P}})\beta_\mu &= \beta_\mu(\mathrm{P}+\bar{\mathrm{P}}). \end{aligned} \tag{57.17'}$$

The last relation shows that the matrix $\mathrm{P}+\bar{\mathrm{P}}$ is proportional to the identity matrix. By taking its trace we can easily show that

$$\mathrm{P}+\bar{\mathrm{P}} = \mathrm{I}. \tag{57.18}$$

Finally the following relations hold:

$$\begin{aligned} \mathrm{P}\beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n} &= \beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n}\mathrm{P}, \quad \text{if } n \text{ is an even number,} \\ \mathrm{P}\beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n} &= \beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n}\bar{\mathrm{P}}, \quad \text{if } n \text{ is an odd number.} \end{aligned} \tag{57.19}$$

We now consider the quantity $X_n$. By utilizing (57.17) we can rewrite $X_n$ in the form

$$\begin{aligned} X_n &= (\hat{p}^2 - p^2)\beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n}(\hat{q}^2 - q^2)(P+P) \\ &= (\hat{p}^2 - p^2)\beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n}(\hat{q}^2 - q^2)\mathrm{P}. \end{aligned}$$

On utilizing the second of formulas (57.19) we obtain in the case of odd $n$:

$$X_n = (\hat{p}^2 - p^2)\,\mathrm{P}\beta_{\mu_1}\beta_{\mu_2}\cdots\beta_{\mu_n}(\hat{q}^2 - q^2);$$

but in accordance with (57.17) this expression vanishes.

By utilizing the vanishing of $X_n$ for odd $n$ we can easily show that the highest degree of the numerator in the expression for $A_s$ does not exceed $s$. From this we can conclude that the degree of the numerator of the rational fraction in the integrand of the matrix element cannot exceed the order of the diagram, i.e., the number $n$. On setting $n' = n$ we obtain for $K$ the value

$$K = 2F - n - 4(F - n + 1) = 3n - 2F - 4.$$

On the other hand, $2F + N_m + N_p = 3n$. Therefore

$$K = N_p + N_m - 4. \tag{57.20}$$

If $K \geqslant 1$, i.e., $N_p + N_m \geqslant 5$, the integral will be convergent. The integral will diverge if the following inequality is satisfied (131):

$$N_m + N_p < 5. \tag{57.21}$$

This inequality shows that the number of divergences which can be encountered in the electrodynamics of scalar particles is finite.

We enumerate the different types of divergences corresponding to the different values of $N_m$ and $N_p$:

1) $N_m = 2,\ N_p = 0,\ K = -2$ quadratic divergence,
2) $N_m = 0,\ N_p = 2,\ K = -2$ quadratic divergence,
3) $N_m = 2,\ N_p = 1,\ K = -1$ linear divergence,
4) $N_m = 0,\ N_p = 2,\ K = 0$ logarithmic divergence,
5) $N_m = 2,\ N_p = 2,\ K = 0$ logarithmic divergence,
6) $N_m = 4,\ N_p = 0,\ K = 0$ logarithmic divergence.

The cases $N_m = 0$, $N_p = 1$ and $N_m = 0$, $N_p = 3$ need not be considered since the corresponding matrix elements vanish. This follows from Furry's theorem which is valid in virtue of the invariance of the theory under the transformation of charge conjugation both in electron electrodynamics, and in the electrodynamics of scalar particles.

The types of divergences enumerated above correspond to the diagrams shown in Fig. 120. Diagrams *5* and *6* are new compared to ordinary electrodynamics. The former corresponds to a radiation correction to the Compton effect, while the latter corresponds to a correction to the scattering of a scalar particle by a scalar particle.

The elimination of divergences from the various matrix elements can be carried out by means of the same regularization methods which were employed in electron electrodynamics. It is important to note here that in actual matrix elements all the infinite constants, with the exception of two, cancel one another; while the elimination of the remaining two infinities requires use of the idea of renormalization of charge and mass of the particle.

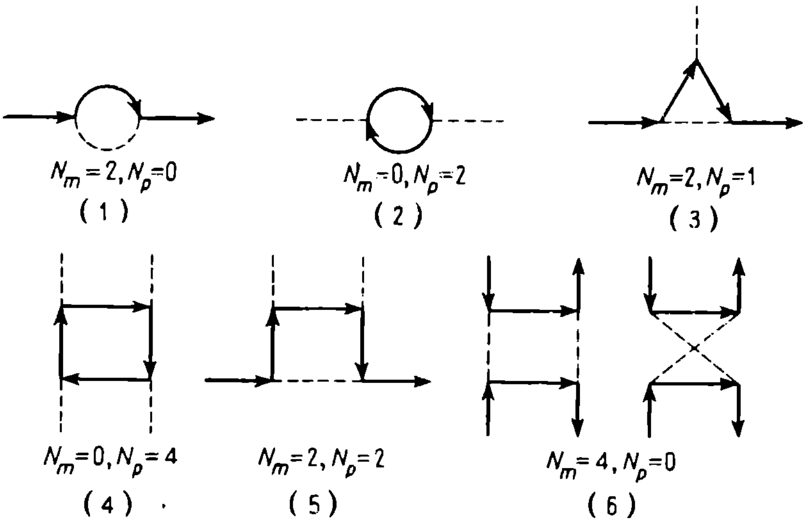


Fig. 120.

## § 58. Scattering of Scalar Particles

### 58.1. *Scattering of Scalar Particles by the Coulomb Field of a Nucleus*

We now proceed to investigate specific effects due to the interaction between scalar particles and electromagnetic fields. We begin with the scattering of scalar particles by the Coulomb field of a nucleus.

The probability of transition of a scalar particle from the state $\varphi(p_1)$ of momentum $p_1$ and energy $\varepsilon_1$ to the state $\varphi(p_2)$ of momentum $p_2$ and energy $\varepsilon_2$ in the field $A_\mu^{(e)}(x)$ is equal to

$$w_1 = 2\pi e^2 |\bar{\varphi}(p_2)\hat{A}^{(e)}(q)\varphi(p_1)|^2 \delta(\varepsilon_1 - \varepsilon_2), \quad q = p_2 - p_1, \qquad (58.1)$$

where

$$\hat{A}^{(e)}(\boldsymbol{q}) = \int \beta_\mu A_\mu^{(e)}(\boldsymbol{r}) e^{-i\boldsymbol{q}\boldsymbol{r}} d\boldsymbol{r}.$$

In the Coulomb field of a nucleus

$$\hat{A}^{(e)}(\boldsymbol{q}) = i\beta_4 \frac{Ze}{\boldsymbol{q}^2}$$

and

$$w_1 = 2\pi \frac{Z^2 e^4}{q^4} |\bar{\varphi}(p_2)\beta_4\varphi(p_1)|^2 \delta(\varepsilon_1 - \varepsilon_2).$$

The quantity $|\bar{\varphi}(p_2)\beta_4\varphi(p_1)|^2$ appearing here is a special case of the more general expression

$$|M|^2 = |\bar{\varphi}^{(\pm)}(p_f) Q \varphi^{(\pm)}(p_i)|^2, \tag{58.2}$$

where $Q$ is a certain matrix. This last expression can be evaluated in the following manner. First of all it is clear that

$$|M|^2 = \bar{\varphi}_\alpha^{(\pm)}(p_f) Q_{\alpha\beta} \varphi_\beta^{(\pm)}(p_i) \bar{\varphi}_\gamma^{(-)}(p_i) \bar{Q}_{\gamma\delta} \varphi_\delta^{(\pm)}(p_f),$$

where

$$\bar{Q} = \eta_4 Q^+ \eta_4.$$

Further, on recalling that in accordance with (56.26) we obtain

$$\varphi_\alpha^{(\pm)}(p)\bar{\varphi}_\beta^{(\pm)}(p) = F_{\alpha\beta}^{(\pm)}(p),$$

$$\begin{aligned} |M|^2 &= \operatorname{Sp}\{QF^{(\pm)}(p_i)\bar{Q}F^{(\pm)}(p_f)\} \\ &= \frac{1}{4m^2\varepsilon_i\varepsilon_f} \operatorname{Sp}\{Qi\hat{p}_i(i\hat{p}_i \mp m)\bar{Q}i\hat{p}_f(i\hat{p}_f \mp m)\}, \end{aligned} \tag{58.3}$$

where $\varepsilon_i$ and $\varepsilon_f$ are the energies of particles of momenta $p_i$ and $p_f$.

On setting $Q = \beta_4$ in this formula and on noting that $\beta_4 = \eta_4\beta_4^+\eta_4 = \beta_4$, we obtain

$$|\bar{\varphi}(p_2)\beta_4\varphi(p_1)|^2 = \frac{1}{4m^2\varepsilon_1\varepsilon_2} \operatorname{Sp}\{\beta_4 i\hat{p}_1(i\hat{p}_1 - m)\beta_4 i\hat{p}_2(i\hat{p}_2 - m)\}.$$

On utilizing (58.3) we have

$$\operatorname{Sp}\{\beta_4 i\hat{p}_1(i\hat{p}_1 - m)\beta_4 i\hat{p}_2(i\hat{p}_2 - m)\} = m^2(\varepsilon_1 + \varepsilon_2)^2.$$

Therefore

$$w_1 = \frac{Z^2e^4}{|\mathbf{p}_1-\mathbf{p}_2|^4}\frac{(\varepsilon_1+\varepsilon_2)^2}{4\varepsilon_1\varepsilon_2}2\pi\delta(\varepsilon_1-\varepsilon_2). \tag{58.1'}$$

On multiplying $w_1$ by $d\mathbf{p}_2/(2\pi)^3$ and on dividing by the flux of incident particles $J = p_1/\varepsilon_1$ we finally obtain the following formula for the differential cross section for the scattering of a scalar particle by the Coulomb field of a nucleus:

$$d\sigma = \left(\frac{Ze^2}{2pv}\right)^2 \frac{do}{\sin^4\frac{\vartheta}{2}}, \tag{58.4}$$

where $\theta$ is the scattering angle. This formula differs from the corresponding formula for the scattering of particles of spin $\frac{1}{2}$ by the absence of the factor $(1-v^2 \sin^2 \vartheta)$.

### 58.2. *Scattering of a Charged Scalar Particle by a Scalar Particle*

We now determine the cross section for the scattering of a charged scalar particle by a similar scalar particle.

The matrix element for this process has the form

$$S^{(2)}_{i\to f} = ie^2M(2\pi)^4\delta(p_1+p_1'-p_2-p_2'),$$
$$M = \frac{(\bar\varphi(p_2)\beta_\mu\varphi(p_1))(\bar\varphi(p_2')\beta_\mu\varphi(p_1'))}{(p_2-p_1)^2} + \frac{(\bar\varphi(p_2')\beta_\mu\varphi(p_1))(\bar\varphi(p_2)\beta_\mu\varphi(p_1'))}{(p_2'-p_1)^2}, \tag{58.5}$$

where $p_1$, $p_1'$ and $p_2$, $p_2'$ are the momenta of the particles before and after scattering and $\varphi(p)$ are the corresponding amplitudes of the wave functions of the particles.

We note that in contrast to the case of electron-electron scattering when the two terms in $M$ have opposite signs, in the present case both terms have the same sign. This difference is due to the different statistics in the case of electrons and of particles of spin zero.

The differential scattering cross section is equal to

$$d\sigma = \frac{e^4}{(2\pi)^2}|M|^2\frac{|\mathbf{p}_2|\,\varepsilon_2 d\varepsilon_2 do_2}{J}\delta(\varepsilon_1+\varepsilon_1'-\varepsilon_2-\varepsilon_2'), \tag{58.5'}$$

where $J$ is the flux density of the incident particles.

On utilizing (58.3) we can write $|M|^2$ in the form

$$
\begin{aligned}
|M|^2 = \frac{1}{16\varepsilon_1\varepsilon_1'\varepsilon_2\varepsilon_2' m^4}\biggl\{ & \frac{1}{(p_2-p_1)^4}\mathrm{Sp}[\beta_\mu i\hat{p}_1(i\hat{p}_1-m)\bar{\beta}_\nu i\hat{p}_2(i\hat{p}_2-m)] \\
& \times \mathrm{Sp}[\beta_\mu i\hat{p}_1'(i\hat{p}_1'-m)\bar{\beta}_\nu i\hat{p}_2'(i\hat{p}_2'-m)] \\
& + \frac{1}{(p_2'-p_1)^4}\mathrm{Sp}[\beta_\mu i\hat{p}_1(i\hat{p}_1-m)\bar{\beta}_\nu i\hat{p}_2'(i\hat{p}_2'-m)] \\
& \times \mathrm{Sp}[\beta_\mu i\hat{p}_1'(i\hat{p}_1-m)\bar{\beta}_\nu i\hat{p}_2(i\hat{p}_2-m)] \\
& + \frac{2}{(p_2-p_1)^2(p_2'-p_1)^2}\mathrm{Sp}[\beta_\mu i\hat{p}_1(i\hat{p}_1-m)\bar{\beta}_\nu i\hat{p}_2'(i\hat{p}_2'-m)\beta_\mu \\
& \qquad \times i\hat{p}_1'(i\hat{p}_1'-m)\bar{\beta}_\nu i\hat{p}_2(i\hat{p}_2-m)]\biggr\}.
\end{aligned}
$$

Evaluation of the traces of the matrices with the aid of formulas (56.3) yields

$$|M|^2 = \frac{1}{16\varepsilon_1\varepsilon_1'\varepsilon_2\varepsilon_2'}\left(\frac{(p_1+p_2)(p_1'+p_2')}{(p_2-p_1)^2}+\frac{(p_1+p_2')(p_1'+p_2)}{(p_2'-p_1)^2}\right)^2.$$

In the center of mass system this expression can be considerably simplified:

$$|M|^2 = \frac{1}{4\varepsilon^2}\left(1-\frac{2\varepsilon^2-m^2}{\varepsilon^2-m^2}\,\frac{2}{\sin^2\vartheta}\right)^2,$$

where $\vartheta$ is the scattering angle and $\varepsilon$ is the particle energy. On substituting this expression into (58.5′), and on noting that in the center of mass system $J = 2p/\varepsilon$, we finally obtain the following expression for the scattering of a scalar particle by a scalar particle (7):

$$d\sigma = \frac{\alpha^2}{4m^2v^4}\,\frac{\left(2\dfrac{\varepsilon^2}{m^2}-1\right)^2}{\left(\dfrac{\varepsilon}{m}\right)^6}\left(\frac{2}{\sin^2\vartheta}-\frac{\varepsilon^2-m^2}{2\varepsilon^2-m^2}\right)^2 do. \qquad (58.6)$$

## § 59. Scattering of a Photon by a Scalar Particle. Bremsstrahlung Photons from a Scalar Particle

### 59.1. *Scattering of a Photon by a Scalar Particle*

We now consider the interaction between scalar particles and photons. First of all we determine the cross section for the Compton scattering of photons by scalar particles (32). By utilizing the diagrams for this process shown in Fig. 21, and the general rules for writing matrix elements we obtain the following expression for the matrix element for the scattering of a photon by a free scalar particle:

$$S^{(2)}_{i\to f} = -\frac{ie^2}{2\sqrt{\omega_1\omega_2}}\bar{\varphi}(p_2)\left\{\hat{e}_2\frac{i\hat{f}_1(i\hat{f}_1-m)+f_1^2+m^2}{m(f_1^2+m^2)}\hat{e}_1\right.$$
$$\left.+\hat{e}_1\frac{i\hat{f}_2(i\hat{f}_2-m)+f_2^2+m^2}{m(f_2^2+m^2)}\hat{e}_2\right\}\varphi(p_1)(2\pi)^4\delta(p_1+k_1-p_2-k_2). \qquad (59.1)$$

Here $\varphi(p_1)$ and $\varphi(p_2)$ are the amplitudes of the wave functions of the particle in the initial and final states of momentum $p_1$ and $p_2$; $\omega_1$ and $\omega_2$ are the frequencies and $e_1$ and $e_2$ are the photon polarizations; and $f_1 = p_1+k_1 = p_2+k_2$, $f_2 = p_1-k_2 = p_2-k_1$ ($k_1$ and $k_2$ are the photon momenta).

The cross section for this process has the form

$$d\sigma = \frac{e^4}{4\omega_1\omega_2}|\bar{\varphi}(p_2)Q\varphi(p_1)|^2\frac{dp_2\,dk_2}{J(2\pi)^2}$$
$$\times\,\delta(\mathbf{p}_1+\mathbf{k}_1-\mathbf{p}_2-\mathbf{k}_2)\,\delta(\varepsilon_1+\omega_1-\varepsilon_2-\omega_2), \qquad (59.2)$$

where

$$Q = \frac{1}{m^3\varkappa_1}\hat{e}_2\left(i\hat{f}_1(i\hat{f}_1-m)+f_1^2+m^2\right)\hat{e}_1+\frac{1}{m^3\varkappa_2}\hat{e}_1\left(i\hat{f}_2(i\hat{f}_2-m)+f_2^2+m^2\right)\hat{e}_2,$$
$$m^2\varkappa_1 = f_1^2+m^2, \qquad m^2\varkappa_2 = f_2^2+m^2,$$

and $J = 1$ is the flux density of the incident photons.

On utilizing formula (58.3) we write $d\sigma$ in the form

$$d\sigma = r_0^2\frac{\omega_2^2}{m^4}\frac{do_2}{\varkappa_1^2}\,\mathrm{Sp}\,\{Qi\hat{p}_1(i\hat{p}_1-m)\bar{Q}i\hat{p}_2(i\hat{p}_2-m)\}, \qquad (59.3)$$

where $r_0 = e^2/4\pi m$ and $do_2$ is the element of solid angle containing the momentum of the particle after scattering.

The traces of the matrix products can be evaluated with the aid of the following formula:

$$\mathrm{Sp}\left\{\prod_{r=1}^{4}[i\hat{f}_r(i\hat{f}_r-m)+f_r^2+m^2]\hat{e}_r\right\}$$
$$= m^4[(e_1, f_1+f_2)(e_2, f_2+f_3)(e_3, f_3+f_4)(e_4, f_4+f_1)$$
$$-(e_1e_2)(f_2^2+m^2)(e_3, f_3+f_4)(e_4, f_4+f_1) -$$

$$\begin{aligned}&-(e_2e_3)(f_3^2+m^2)(e_1, f_1+f_2)(e_4, f_4+f_1)\\&-(e_3e_4)(f_4^2+m^2)(e_1, f_1+f_2)(e_2, f_2+f_3)\\&-(e_1e_4)(f_1^2+m^2)(e_2, f_2+f_3)(e_3, f_3+f_4)\\&+(e_1e_2)(e_3e_4)(f_2^2+m^2)(f_4^2+m^2)+(e_1e_4)(e_2e_3)(f_1^2+m^2)(f_3^2+m^2)].\end{aligned} \tag{59.4}$$

(This formula can be easily generalized to the case of an arbitrary number of factors).

On utilizing formula (59.4) we obtain

$$\begin{aligned}&\mathrm{Sp}\,\{Qi\hat{p}_1(i\hat{p}_1-m)\bar{Q}i\hat{p}_2(i\hat{p}_2-m)\}\\&=4m^2\left\{(e_1e_2)-\frac{1}{2m^2\varkappa_1}(e_1, f_1+p_1)(e_2, f_1+p_2)\right.\\&\qquad\left.-\frac{1}{2m^2\varkappa_2}(e_2, f_2+p_1)(e_1, f_2+p_2)\right\}^2\end{aligned}$$

and

$$d\sigma=4r_0^2\frac{\omega_2^2}{m^2}\frac{do_2}{\varkappa_1^2}\left[(e_1e_2)-\frac{2}{m^2\varkappa_2}(e_1p_1)(e_2p_2)-\frac{2}{m^2\varkappa_2}(e_1p_2)(e_2p_1)\right]^2. \tag{59.5}$$

If the particle was at rest before scattering, then $p_1=0$ and

$$d\sigma=r_0^2\left(\frac{\omega_2}{\omega_1}\right)^2(e_1e_2)^2\,do_2. \tag{59.6}$$

In the case of unpolarized photons this formula yields

$$d\sigma=\frac{1}{2}r_0^2\left(\frac{\omega_2}{\omega_1}\right)^2(1+\cos^2\vartheta)\,do_2, \tag{59.7}$$

where $\vartheta$ is the scattering angle.

### 59.2. *Bremsstrahlung from Scalar Particles*

We now proceed to determine the probability for the emission of a photon by a scalar particle in an external field $A_\mu^{(e)}(x)$ (39).

The matrix element corresponding to the bremsstrahlung process can be written in the form

$$\begin{aligned}S_{i\to f}^{(2)}=-\frac{ie^2}{\sqrt{2\omega}}\bar{\varphi}(p_2)&\left\{\hat{e}\,\frac{i\hat{f}_1(i\hat{f}_1-m)+f_1^2+m^2}{m^3\varkappa_1}\,\hat{A}^{(e)}(q)\right.\\&\left.+\hat{A}^{(e)}(q)\,\frac{i\hat{f}_2(i\hat{f}_2-m)+f_2^2+m^2}{m^3\varkappa_2}\,\hat{e}\right\}\varphi(p_1),\end{aligned} \tag{59.8}$$

where $p_1$ and $p_2$ are the four-momenta of the particle in the initial and final states, $\varphi(p_1)$ and $\varphi(p_2)$ are the corresponding amplitudes of the wave functions, $k$ and $e$ are the momentum and the polarization of the emitted photon, $A_\mu^{(e)}(q)$ is the Fourier component of the external field and

$$f_1 = p_2+k, \quad f_2 = p_1-k, \quad q = p_2-p_2+k,$$

$$m^2\varkappa_1 = f_1^2+m^2 = 2p_2k, \quad m^2\varkappa_2 = f_2^2 = -2p_1k.$$

If the external field is static, then

$$A_\mu^{(e)}(\boldsymbol{q}) = \delta_{\mu 4} A_4^{(e)}(\boldsymbol{q}) 2\pi\, \delta(\varepsilon_2-\varepsilon_1+\omega),$$

where $\varepsilon_1$ and $\varepsilon_2$ are the values of the particle energy and $\omega$ is the photon frequency. The differential cross section for bremsstrahlung is determined by the following formula:

$$d\sigma = \frac{e^4}{8\omega m^2} |A_4^{(e)}(\boldsymbol{q})|^2 \frac{|\boldsymbol{p}_2|}{|\boldsymbol{p}_1|} \operatorname{Sp} F \frac{d\boldsymbol{k}\, do_2}{(2\pi)^5}, \tag{59.9}$$

where

$$F = Qi\hat{p}_1(i\hat{p}_1-m)\bar{Q}i\hat{p}_2(i\hat{p}_2-m),$$

$$Q = \hat{e}\, \frac{i\hat{f}_1(i\hat{f}_1^2-m)+f_1^2+m^2}{m^3\varkappa_1} \beta_4 + \beta_4 \frac{i\hat{f}_2(i\hat{f}_2-m)+f_2^2+m^2}{m^3\varkappa_2}\, \hat{e}.$$

The evaluation of traces of products of the matrices $\beta_\mu$ can be carried out with the aid of formula (56.3). As a result of this we obtain

$$\operatorname{Sp} F = 16m^2 \left( \boldsymbol{e}, \frac{\varepsilon_1}{m^2\varkappa_1} \boldsymbol{p}_2 + \frac{\varepsilon_2}{m^2\varkappa_2} \boldsymbol{p}_1 \right)^2. \tag{59.9'}$$

If the emission of radiation occurs in the Coulomb field of a nucleus, we have $A_4(\boldsymbol{q}) = -i(Ze/\boldsymbol{q}^2)$ and the bremsstrahlung cross section assumes the form

$$d\sigma = \frac{4Z^2\alpha^3}{\pi^2} \frac{|\boldsymbol{p}_2|}{|\boldsymbol{p}_1|} \frac{1}{m^4\boldsymbol{q}^4} \left( \boldsymbol{e}, \frac{\varepsilon_1}{\varkappa_1} \boldsymbol{p}_2 + \frac{\varepsilon_2}{\varkappa_2} \boldsymbol{p}_1 \right)^2 \omega\, d\omega\, do\, do_2, \tag{59.10}$$

where $do$ and $do_2$ are the elements of solid angle containing the momenta $\boldsymbol{k}$ and $\boldsymbol{p}_2$.

The cross section for the emission of an unpolarized photon can be obtained by summing (59.10) over the two photon polarizations:

$$d\sigma = \frac{4Z^2\alpha^3}{\pi^2}\frac{|p_2|}{|p_1|}\frac{1}{m^4q^4}\left\{\omega^2\left(\frac{\varepsilon_1}{\varkappa_1}p_2+\frac{\varepsilon_2}{\varkappa_2}p_1\right)^2 - \left(k, \frac{\varepsilon_1}{\varkappa_1}p_2+\frac{\varepsilon_2}{\varkappa_2}p_1\right)^2\right\}\omega\, d\omega\, do\, do_2. \tag{59.11}$$

## § 60. Production and Annihilation of Pairs of Scalar Particles

### 60.1. *Production of Pairs of Scalar Particles by a Photon in the Coulomb Field of a Nucleus*

We now determine the cross section for the production of a pair of scalar particles by a photon in the Coulomb field of a nucleus (39). The matrix element for such a process can be written in the form

$$S^{(2)}_{i\to f} = -\frac{ie^2}{\sqrt{2\omega}}\varphi^{(+)}(p_+)\left\{\hat{A}^{(e)}(q)\frac{i\hat{f}_1(i\hat{f}_1-m)+f_1^2+m^2}{m(f_1^2+m^2)}\hat{e} + \hat{e}\frac{i\hat{f}_2(i\hat{f}_2-m)+f_2^2+m^2}{m(f_2^2+m^2)}\hat{A}^{(e)}(q)\right\}\varphi^{(-)}(p_-), \tag{60.1}$$

where $p_+$ and $p_-$ are the momenta of the two components of the pair, $\varphi^{(+)}(p_+)$ and $\varphi^{(-)}(p_-)$ are their wave functions and $q = p_+ + p_- - k$.

This expression can be obtained from the matrix element for bremsstrahlung if we carry out the following substitution in the latter: $p_1 \to -p_-$, $p_2 \to p_+$, and $k \to -k$. Therefore, the cross section for this process can be immediately obtained from formula (59.10) by means of such a substitution. As a result of this we obtain

$$do = 8\frac{Z^2\alpha^3}{(2\pi)^2}\frac{|p_+||p_-|}{q^4\omega^3}\frac{1}{m^4}\left\{\omega^2\left(\frac{\varepsilon_-}{\varkappa_1}p_+ + \frac{\varepsilon_+}{\varkappa_2}p_-\right)^2 - \left(k, \frac{\varepsilon_-p_+}{\varkappa_1} + \frac{\varepsilon_+p_-}{\varkappa_2}\right)^2\right\}d\varepsilon_+\, do_+\, do_-, \tag{60.2}$$

where

$$\varkappa_1 = -2\frac{kp_-}{m^2}, \quad \varkappa_2 = -2\frac{kp_+}{m^2}$$

and $do_+$ and $do_-$ are the elements of solid angle containing $p_+$ and $p_-$.

The total cross section for pair production is equal to (4)

$$\sigma = \frac{2}{27}\frac{Z^2\alpha^3}{m^2}\frac{1}{\gamma^2}\left\{(5-13\gamma^2)E\left(\sqrt{1-\frac{1}{\gamma^2}}\right) + \left(5+6\gamma^2-\frac{3}{\gamma^2}\right)F\left(\sqrt{1-\frac{1}{\gamma^2}}\right)\right\}, \tag{60.3}$$

where $\gamma = \omega/2m$ and $E(x)$ and $F(x)$ are the same functions which appear in the total cross section for the production of electron-positron pairs (cf. § 32).

### 60.2. *Production of a Pair of Scalar Particles by Two Photons*

The matrix element for the production of a pair of scalar particles by two photons has the form

$$S^{(2)}_{i\to f} = -\frac{ie^2}{2\sqrt{\omega\omega'}}\bar{\varphi}^{(-)}(p_-)\left\{\hat{e}\,\frac{i\hat{f}_1(i\hat{f}_1-m)+f_1^2+m^2}{m(f_2^2+m^2)}\,\hat{e}' + \hat{e}'\,\frac{i\hat{f}_2(i\hat{f}_2-m)+f_2^2+m^2}{m(f_2^2+m^2)}\,\hat{e}\right\}\varphi^+(p_+)(2\pi)^4\delta(p_+ + p_- - k -- k'), \tag{60.4}$$

where $k, k'$ and $e, e'$ are the momenta and the polarizations of the two photons, $\omega$ and $\omega'$ are their frequencies and $f_1 = p_+ - k'$ and $f_2 = p_+ - k$.

We give here only the final result of the calculation of the pair production cross section (7). In the c.m. system this cross section is equal to

$$d\sigma = \frac{1}{16}r_0^2\left(\frac{m}{\omega}\right)^2\sqrt{1-\left(\frac{m}{\omega}\right)^2}\left\{ee' - \frac{2(e'p_+)(ep_+)}{\omega^2\left[1-\left(1-\frac{m^2}{\omega^2}\right)\cos^2\theta\right]}\right\}do, \tag{60.5}$$

where $r_0 = \alpha^2/m$ and $\theta$ is the angle between $k$ and $p_+$.

The cross section averaged over the photon polarizations has the form

$$d\sigma = \frac{r_0^2}{8}\left(\frac{m}{\omega}\right)^2\sqrt{1-\left(\frac{m}{\omega}\right)^2}\left[1-2\frac{m^2}{\omega^2}\,\frac{1}{1-\left(1-\frac{m^2}{\omega^2}\right)\cos^2\theta} + 2\frac{m^4}{\omega^4}\,\frac{1}{\left[1-\left(1-\frac{m^2}{\omega^2}\right)\cos^2\theta\right]^2}\right]do. \tag{60.6}$$

Finally, the total pair production cross section is given by

$$\sigma = \pi r_0^2 \frac{1}{x^6}\{(x^2+1)x\sqrt{x^2-1} - (2x^2-1)\ln(x+\sqrt{x^2-1})\}, \qquad (60.7)$$

where $x = \omega/m$.

### 60.3. *Two-Photon Annihilation of a Pair of Scalar Particles*

The process which is inverse to the one just considered is the conversion of a pair of scalar particles into two photons. The calculation of the cross section for this process is analogous to the calculation of the cross section for pair production by two photons. We give here only the final results.

In the c. m. the differential cross section for two-photon annihilation of a pair of scalar particles has the form

$$d\sigma = \frac{1}{2} r_0^2 \frac{m^2}{\varepsilon^2} \frac{do}{\sqrt{1-\frac{m^2}{\varepsilon^2}}} \left\{1 - 2\frac{m^2}{\varepsilon^2} \frac{1}{1-\left(1-\frac{m^2}{\varepsilon^2}\right)\cos^2\theta} + 2\frac{m^1}{\varepsilon^4} \frac{1}{\left[1-\left(1-\frac{m^2}{\varepsilon^2}\right)\cos^2\theta\right]^2}\right\}, \qquad (60.8)$$

where $\theta$ is the angle between the momentum of one of the components of the pair and the momentum of one of the photons, $\varepsilon$ is the particle energy and $do$ is the element of solid angle containing $\boldsymbol{k}$.

The total cross section for two photon annihilation is equal to

$$\sigma = 4\pi r_0^2 \frac{1}{x^4}\left\{\frac{(x^2+1)x}{\sqrt{x^2-1}} - \frac{2x^2-1}{x^2-1}\ln(x-\sqrt{x^2-1})\right\}, \qquad (60.9)$$

where $x = \varepsilon/m$.

### 60.4. *Annihilation of Pairs of Scalar Particles into Electron-Positron Pairs and the Inverse Process*

In concluding this section we investigate the process of the conversion of an electron-positron pair into a pair of scalar particles, and also the inverse process of the conversion of a pair of scalar particles into an electron-positron pair (7). These processes correspond to the diagram shown in Fig. 121; $p_-^e$, $p_+^e$ denote the momenta of the components of the electron-positron pair, while $p_-^m$, $p_+^m$ are the momenta of the components of the pair of scalar particles.

The matrix element for both processes can be written in the form

$$S^{(2)}_{i\to f} = ie^2|M|^2(2\pi)^4(2\pi)^4\delta(p^e_+ + p^e_- - p^m_+ - p^m_-),$$
$$M = \frac{1}{(p^e_+ + p^e_-)^2}\left(\bar{v}(p^e_+)\gamma_\mu u(p^e_-)\right)\left(\varphi^{(-)}(p^m_-)\beta_\mu\varphi^{(+)}(p^m_+)\right), \tag{60.10}$$

where $u(p^e_-)$ and $v(p^e_+)$ are the spinor amplitudes of the electron and the positron, and $\varphi^{(-)}(p^m_-)$ and $\varphi^{(+)}(p^m_+)$ are the amplitudes of the wave functions for the scalar particles.

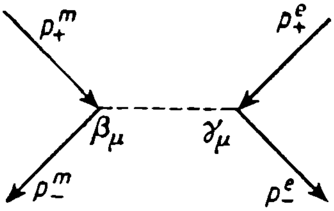


Fig. 121.

The square of the absolute value of $M$ which determines the cross sections for these processes summed over the electron and the positron spins can be written in the form

$$\overline{|M|^2} = \frac{1}{16\varepsilon^m_+\varepsilon^m_-\varepsilon^e_+\varepsilon^e_-}\,\frac{1}{(p^e_+ + p^e_-)^4}\,\mathrm{Sp}\,\{\gamma_\mu(i\hat{p}^e_- - m_e)\bar{\gamma}_\nu(i\hat{p}^e_+ + m_e)\}$$
$$\times\frac{1}{m^2}\,\mathrm{Sp}\{\beta_\mu i\hat{p}^m_+(i\hat{p}^m_+ - m)\beta_\nu i\hat{p}^m_-(i\hat{p}^m_- + m)\}, \tag{60.11}$$
$$\hat{p}^e = \gamma_\mu p^e_\mu,\ \hat{p}^m = \beta_\mu p^m_\mu,$$

where $m_e$ and $m$ are the masses of the electron and of the scalar particle. On evaluating the traces of the matrix products in accordance with formulas (56.3), (59.4), we obtain

$$\overline{|M|^2} = \frac{1}{2\varepsilon^m_+\varepsilon^m_-\varepsilon^e_+\varepsilon^e_-}\,\frac{1}{(p^e_+ + p^e_-)^4}\,\{(p^e_+ p^m_-)(p^e\, p^m_-)$$
$$+ (p^e_- p^m_+)(p^e_+ p^m_+) - (p^e_- p^m_-)(p^e_+ p^m_+) - (p^e_- p^m_+)(p^e_+ p^m_-)$$
$$- [(p^e_+ p^e_-) - m_e^2]\,[(p^m_+ p^m_-) + m^2]\}. \tag{60.12}$$

The cross section for the conversion of an electron-positron pair into a pair of scalar particles is equal to

$$d\sigma = e^4\,\overline{|M|^2}\,\frac{dp^m_+\,dp^m_-}{16\pi^2 J}\,\delta(p^e_+ + p^e_- - p^m_+ - p^m_-), \tag{60.13}$$

where $J$ is the flux density of the incident particles.

In the c. m. system we have

$$p_+^e + p_-^e = p_+^m + p_-^m = 0,$$

$$\varepsilon_+^e = \varepsilon_-^e = \varepsilon_+^m = \varepsilon_-^m = \varepsilon,$$

$$J = 2\sqrt{1-\left(\frac{m_e}{\varepsilon}\right)^2}$$

and

$$d\sigma = \frac{\alpha^2}{32\varepsilon^2}\,\frac{\left(1-\dfrac{m^2}{\varepsilon^2}\right)^{3/2}}{\left(1-\dfrac{m_e^2}{\varepsilon^2}\right)^{1/2}}\left\{1-\left(1-\frac{m_e^2}{\varepsilon^2}\right)\cos^2\theta\right\}do, \qquad (60.14)$$

where $\theta$ is the angle between the momenta of the positron and one of the scalar particles and $do$ is the element of solid angle containing the momentum of one of the scalar particles.

The total cross section for the production of a pair of scalar particles is equal to

$$\sigma = \frac{\pi\alpha^2}{16\varepsilon^2}\left(1-\frac{m^2}{\varepsilon^2}\right)^{3/2}\left(1+\frac{m_e^2}{\varepsilon^2}\right)\left(1-\frac{m_e^2}{\varepsilon^2}\right)^{-1/2} \qquad (60.15)$$

The cross section for the conversion of a pair of scalar particles into an electron-positron pair differs from the cross section for the production of a pair of scalar particles by the difference in the number of final states, and also by a different expression for the flux density which in this case is equal to

$$J = 2\sqrt{1-\frac{m^2}{\varepsilon^2}}.$$

The final expression for the cross section for the conversion of a pair of scalar particles into an electron-positron pair has the form

$$d\sigma = \frac{\alpha^2}{8\varepsilon^2}\left(1-\frac{m_e^2}{\varepsilon^2}\right)^{1/2}\left(1-\frac{m^2}{\varepsilon^2}\right)^{1/2}\left\{1-\left(1-\frac{m_e^2}{\varepsilon^2}\right)\cos^2\theta\right\}do. \qquad (60.16)$$

The total cross section for this process is equal to

$$\sigma = \frac{\pi\alpha^2}{4\varepsilon^2}\left(1-\frac{m_e^2}{\varepsilon^2}\right)^{1/2}\left(1-\frac{m^2}{\varepsilon^2}\right)^{1/2}\left(1+\frac{m_e^2}{\varepsilon^2}\right). \qquad (60.17)$$

## § 61. Polarization of the Vacuum in the Case of Charged Scalar Particles

### 61.1. *Vacuum Polarization Tensor for Scalar Particles*

In the preceding Chapter we have discussed the various effects due to the interaction of the electromagnetic field with the vacuum of the electron-positron field and of electrons with the vacuum of the electromagnetic field. In this section we shall discuss the analogous effects in scalar electrodynamics. We begin by determining the radiation corrections to the photon Green's functions due to the scalar particle vacuum. This problem reduces to the evaluation of the polarization tensor $\Pi_{\mu\nu}(k)$.

If we confine ourselves to the second order radiation corrections the polarization tensor $\Pi^{(2)}_{\mu\nu}(k)$ associated with the scalar particle vacuum will be given by the formula analogous to formula (47.28)

$$\Pi^{(2)}_{\mu\nu}(k) = -e^2 \int d^4p\, \mathrm{Sp}\{\beta_\mu (i\hat{p}+m)^{-1} \beta_\nu (i\hat{p}+i\hat{k}+m)^{-1}\}. \qquad (61.1)$$

The difference between this expression and (47.28) consists of the fact that this expression contains the matrices $\beta_\mu$ instead of $\gamma_\mu$ and that, moreover, this expression contains an additional minus sign associated with the different statistics for the scalar particles.

The tensor $\Pi^{(2)}_{\mu\nu}(k)$ diverges quadratically; the regularized value of the tensor can be found by means of the formula

$$\Pi^{(2)}_{\mu\nu R} = \Pi^{(2)}_{\mu\nu}(k) - \Pi^{(2)}_{\mu\nu}(0) - \left.\frac{\partial^2 \Pi^{(2)}_{\mu\nu}(k)}{\partial k_\sigma \partial k_\tau}\right|_{k=0} k_\sigma k_\tau. \qquad (61.2)$$

On evaluating the trace of the products of the matrices $\beta_\mu$ we obtain

$$\Pi^{(2)}_{\mu\nu}(k) = -2e^2[\delta_{\mu\nu}J_0 - (k^2\delta_{\mu\nu} - k_\mu k_\nu)J + \delta_{\mu\nu}k_\sigma J_\sigma - J_\mu k_\nu - k_\mu J_\nu - 2J_{\mu\nu}],$$

where

$$J_0 = \int \frac{d^4p}{(p+k)^2+m^2},$$

$$J = \int \frac{d^4p}{(p^2+m^2)\,[(p+k)^2+m^2]},$$

$$J_\mu = \int \frac{p_\mu\, d^4p}{(p^2+m^2)\,[(p+k^2)+m^2]},$$

$$J_{\mu\nu} = \int \frac{p_\mu p_\nu\, d^4p}{(p^2+m^2)\,[(p+k)^2+m^2]}.$$

With the aid of the identity

$$\frac{1}{ab}=\int\limits_0^1\frac{dx}{[ax+b(1-x)]^2}$$

the integrals can be written in the form

$$J=\int\limits_0^1 dx\int Fd^4p,$$

$$J_\mu=\int\limits_0^1 dx\int p_\mu Fd^4p,$$

$$J_{\mu\nu}=\int\limits_0^1 dx\int p_\mu p_\nu Fd^4p,$$

where

$$F=\frac{1}{(p^2+2pkx+m^2)^2}.$$

We shall not carry out here any detailed calculations, but shall directly give the final result for the regularized value of $\Pi^{(2)}_{\mu\nu R}(k)$:

$$\Pi^{(2)}_{\mu\nu R}(k)=-\frac{i\pi^2}{6}e^2(k_\mu k_\nu-k^2\delta_{\mu\nu})\frac{k^2}{m^2}\int\limits_0^1\frac{v^4\,dv}{1+\frac{k^2}{4m^2}(1-v^2)}, \tag{61.3}$$

where $v=1-2x$. On introducing the new variable $\theta$ by means of

$$\sin\Theta=-\frac{k^2}{4m^2},$$

we write $\Pi^{(2)}_{\mu\nu R}(k)$ in the form[1]

$$\Pi^{(2)}_{\mu\nu R}(k)=2\pi^2 ie^2(k_\mu k_\nu-\delta_{\mu\nu}k^2)\left\{\frac{4m^2+k^2}{3k^2}(1-\Theta\cot\Theta)+\frac{1}{9}\right\}. \tag{61.4}$$

When we have the expression for $\Pi^{(2)}_{\mu\nu R}(k)$ we can obtain the radiation correction to the external current $J^{(e)}_\mu(k)$ due to the interaction between $J_\mu^{(e)}(x)$ and the vacuum of the scalar field. The Fourier component of this additional term is equal to

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[1] Feynman (second of references (62)); (the factor 1/2 is missing in the formula for $\Pi^{(2)}_{\mu\nu R}$ given in this reference).

$$\delta A_\mu^{(e)}(q) = -\frac{ie^2}{(2\pi)^4}\Pi_{\mu\nu R}^{(2)}(q)A_\nu^{(e)}(q), \tag{61.5}$$

where $A_\mu^{(e)}(q)$ is the Fourier component of the external field.

In the case of slowly varying fields we can expand $\Pi_{\mu\nu}^{(2)}(k)$ into a power series in $k$. On restricting ourselves to fourth order terms we obtain

$$\Pi_{\mu\nu R}^{(2)}(k) = \frac{\pi^2 i}{30 m^2}e^2(k^2\delta_{\mu\nu}-k_\mu k_\nu)k^2. \tag{61.6}$$

In this case the additional term in the current is given by

$$\delta J_\mu^{(e)}(q) = \frac{\alpha}{30 m^2}q^2 J^{(e)}(q). \tag{61.7}$$

## 61.2. *Correction to Coulomb's Law*

If we know the polarization tensor $\Pi_{\mu\nu R}^{(2)}$ we can also obtain the radiaton corrections to Coulomb's law. The radiation correction to the external potential $A_\mu^{(e)}(k)$ is determined in momentum space by the following general formula:

$$\delta A_\mu^{(e)}(k) = \Pi_{\mu\nu R}^{(2)}(k)D_{\nu\lambda}^c(k)A_\lambda^{(e)}(k).$$

In coordinate space this formula yields for a static field

$$\delta A_\mu^{(e)}(r) = -\frac{i}{(2\pi)^7}\int \Pi_{\mu\nu R}^{(2)}(k)A_\nu^{(e)}(k)e^{ikr}\frac{dk}{k^2}.$$

In the case of the Coulomb field of a nucleus we obtain from this expression, on taking radiation correction into account and on utilizing expression (61.3),

$$\varphi(r)+\delta\varphi(r) = -\frac{Ze}{4\pi r}\left(1+\frac{e^2}{(4\pi)^2}\frac{1}{12\pi^2}\int e^{ikr}\,dk\int_0^1\frac{v^4\,dv}{m^2+\frac{1}{4}k^2(1-v^2)}\right) \tag{61.8}$$

or

$$\varphi(r)+\delta\varphi(r) = \varphi(r)\left(1+\chi_0(r)\right), \tag{61.8'}$$

where

$$\chi_0(r) = \frac{\alpha}{12\pi^2}\int\limits_0^\infty k\sin kr\, dk \int\limits_0^1 \frac{v^4\, dv}{m^2+\frac{1}{4}k^2(1-v^2)}\,. \qquad (61.9)$$

On carrying out the integration with respect to $k$ and on introducing in place of $v$ the new variable $\xi = (1-v^2)^{-1/2}$ we write $\chi_0(r)$ in the form (3)

$$\chi_0(r) = \frac{\alpha}{6\pi}\int\limits_1^\infty e^{-2mr\xi}\frac{(\xi^2-1)^{3/2}}{\xi^4}\, d\xi. \qquad (61.9')$$

The corresponding expression for the correction due to the electron-positron vacuum has the form

$$\chi_{1/2}(r) = \frac{2\alpha}{3\pi}\int\limits_1^\infty e^{-2m_{\frac{1}{2}}r\xi}\left(1+\frac{1}{2\xi}\right)\frac{(\xi^2-1)^{1/2}}{\xi^2}\, d\xi,$$

where $m_{1/2}$ is the electron mass.

In the limiting case $mr \ll 1$ and $mr \gg 1$ the function $\chi_0(r)$ is determined by the expressions

$$\chi_0(r) = \begin{cases} \dfrac{\alpha}{6\pi}\left(\ln\dfrac{1}{mr\gamma} - \dfrac{4}{3}\right), & \text{where} \quad mr \ll 1, \\[2ex] \dfrac{\alpha}{16\sqrt{\pi}}(mr)^{-5/2}e^{-2mr}, & \text{where} \quad mr \gg 1, \end{cases} \qquad (61.10)$$

where $\ln\gamma$ is Euler's constant.

### 61.3. *Photon-Photon Scattering. Radiation Corrections to the Lagrangian of the Electromagnetic Field*

The scalar particle vacuum, just as the electron-positron vacuum, leads to a number of nonlinear effects in electrodynamics, such as photon-photon scattering, the scattering of a photon by the Coulomb field of a nucleus, etc.

All these effects can be described in the electrodynamics of scalar particles, as in electron electrodynamics, with the aid of the fourth order scattering matrix:

$$S^{(4)} = \frac{e^4}{4}\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 N\left(A_\mu(x_1)A_\nu(x_2)A_\lambda(x_3)A_\sigma(x_4)\right)$$
$$\times \operatorname{Sp}\left\{\beta_\mu T^c(x_1-x_2)\beta_\nu T^c(x_2-x_3)\beta_\lambda T^c(x_3-x_4)\beta_\sigma T^c(x_4-x_1)\right\}.$$

In momentum space this matrix has the form

$$S^{(4)} = \frac{\alpha^2}{12} \int d^4k_1\, d^4k_2\, d^4k_3\, d^4k_4\, (2\pi)^4\, \delta(k_1+k_2+k_3+k_4) \times N\big(A_\mu(k_1) A_\nu(k_2) A_\lambda(k_3) A_\sigma(k_4)\big)\, J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4), \tag{61.11}$$

where

$$J_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) + T_{\mu\nu\sigma\lambda}(k_1, k_2, k_4, k_3) + T_{\mu\lambda\nu\sigma}(k_1, k_3, k_2, k_4)$$

and

$$T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4) = \frac{1}{i\pi^2} \int d^4p\, \mathrm{Sp}\, \{\beta_\mu (i\hat{p}+m)^{-1} \beta_\nu (i\hat{p}-i\hat{k}_2+m)^{-1} \times \beta_\lambda (i\hat{p}-i\hat{k}_2-i\hat{k}_3+m)^{-1} \beta_\sigma (i\hat{p}-i\hat{k}_2-i\hat{k}_3-i\hat{k}_4+m)^{-1}\}.$$

The quantity $T_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, k_4)$, which is called the fourth rank polarization tensor, diverges logarithmically for large momenta, just as the corresponding quantity in ordinary electrodynamics.

Therefore, regularization can be carried out in the same way as in electron electrodynamics.

On repeating the arguments of § 54 we can calculate the photon-photon scattering cross section, and also obtain the radiation corrections to the Lagrangian density for the electromagnetic field due to the vacuum state of the scalar particle field. We shall not carry out here any detailed calculations, but shall confine ourselves to stating the final results.

The photon-photon scattering cross section (in the center of mass system) is given in the case of low frequencies by the formula (4)

$$d\sigma = \frac{\alpha^4}{(4\pi^2)} \frac{34}{(90)^2} \frac{1}{m^2} \left(\frac{\omega}{m}\right)^6 (3+\cos^2\theta)^2\, do, \quad \text{where} \quad \omega \ll \mathrm{m}, \tag{61.12}$$

where $m$ is the mass of the scalar particle and $\theta$ is the scattering angle.

The total scattering cross section is given by

$$\sigma = \frac{\alpha^4}{\pi} \frac{7}{10} \frac{136}{90^2} \frac{1}{m^2} \left(\frac{\omega}{m}\right)^6, \quad \text{where} \quad \omega \ll m. \tag{61.13}$$

The correction to the Lagrangian density for the electromagnetic field is determined by the following formula (176):

$$L'(x) = \frac{\alpha^2}{360m^4}\left(7(\boldsymbol{E}^2-\boldsymbol{H}^2)^2+4(\boldsymbol{E}\boldsymbol{H})^2\right). \tag{61.14}$$

**Concluding Remarks**

The concept of an elementary particle in modern physics has undergone a number of essential changes. At present it is already evident that elementary particles are by no means the primary components of matter, but are rather states of excitation of some dynamical system; in the case of electrodynamics such a system is the combination of the electromagnetic and the electron-positron fields which interact with each other. Even in the absence of particles, i.e., of excitations, the system which is in its ground state — the vacuum state — has definite physical properties which manifest themselves in a number of phenomena.

The interaction between particles and vacuum manifests itself in a remarkable manner in the radiation shift of atomic levels, in the anomalous magnetic moment of the electron, in nonlinear electrodynamic effects, and in other phenomena. These facts are of considerable interest not only from a physical point of view, but also from a general philosophical point of view. In them we find a new confirmation of the well known thesis of V. I. Lenin regarding the inexhaustibility of the properties of the electron and the infiniteness of nature.

In criticizing idealists, who in connection with changes in the concepts of the structure of matter drew conclusions with respect to its disappearance, Lenin wrote: "The 'essence' of things or 'substance' are also relative; they express only an increase in the depth of human perception of objects, and even though yesterday this increase in the depth of perception did not extend beyond the atom, and today beyond the electron and the ether, nevertheless, dialectical materialism insists on the temporary, relative, approximate character of all these *landmarks* in the perception of nature by man's progressive science. The electron is just as *inexhaustible* as the atom, nature is infinite, but it *exists* infinitely, and it is this uniquely categorical, uniquely unconditional recognition of its *existence* outside human consciousness and human senses, that distinguishes dialectical materialism from relativistic agnosticism and idealism" (Lenin, "Materialism and Empiriocriticism," Gospolitizdat, 1948, p. 245).

The successes of quantum electrodynamics have demonstrated the correctness of our basic physical concepts within a definite domain of phenomena. However, these successes are relative. Electrodynamics turns out not to be a logically closed theory, i.e., it cannot be developed absolutely consistently without introducing auxiliary ideas which, so far, are of a semiempirical nature. Attempts to carry over the methods of quantum electrodynamics to domains of other phenomena have so far not resulted in any serious successes. Apparently, the difficulties of the present theory can be removed only by means of a new change, and, moreover, perhaps a cardinal one, in the basic physical concepts. It is quite probable that even the fundamental space-time concepts of modern physics will undergo a change in this process.

Like any other science, quantum electrodynamics gives expression to definite objective natural laws. "But this,—as V. I. Lenin points out, — is not a simple, direct, complete expression, but a process involving a series of abstractions, the formulation and creation of concepts, laws, etc., and it is these concepts, laws, etc., (thought, science = "logical idea") that *encompass* conditionally, and approximately the universal regularity of eternally moving and developing nature" (V. I. Lenin, "Philosophical Notebooks", 1947, p. 156).

These words of V. I. Lenin may be used to give an excellent characterization of the modern state of development of theoretical physics.

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